SAR IMAGE FORMATION VIA INVERSION OF RADON TRANSFORMS

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ABSTRACT

In this paper review how image formation or reconstruction in synthetic aperture radar (SAR) in two and three dimensions can be viewed as the inversion of the circular and spherical Radon transforms, respectively. The advantage of viewing image formation in this way is that it could be used in situations where more standard methods could fail such as high squint and ultra-wideband SAR. The inversion formulae for the circular and spherical Radon transforms employ Hankel and Fourier transforms along with coordinate mappings. We show how the theory extends to include finite bandwidth signals and finite synthetic apertures.

1. INTRODUCTION

The Radon transform is an integral along a path, and in its simplest form this path is a straight line. We will refer to this form of the transform (of the function $f(x, y)$, $x, y \in \mathbb{R}$) as the normal Radon transform (NRT) because it is usually written using the normal equation for the line,

$$g(\rho,\theta) = (\mathcal{R}f)(\rho,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(\rho - x\cos\theta - y\sin\theta)\,dx\,dy, \qquad (1)$$

where $\delta(\cdot)$ is the Dirac delta function and (ρ, θ) are the parameters of a normal equation for the line of integration. Munson *et al.* [12] (extended by Jakowatz *et al.* [9]) showed the relationship between this line integral and spotlight SAR image formation using the polar-to-rectangular resampling process. This corresponds to the Fresnel approximation to the wavefront: the radiating energy from the radar is assumed to be planar. However, this approximation only holds in restricted physical geometries and limits the size and resolution of the image that can be formed [9].

In reality, the wavefront radiating from the radar's antenna is spherical, with the sector of the sphere determined by the antenna beam pattern and geometry. As a consequence, the received signal at a particular antenna position and time since transmission is an average (integral) of the surface reflectivity (at the radiation's frequency) at all points where the spherical wavefront impinged upon the ground at half that time (to account for the round trip). We will ignore the effect of shape and duration of the transmitted pulse by assuming that the received data has been range compressed by matched filtering the received signal against the transmitted pulse (see [14, 16]). The antenna position is usually a function of time along a straight line, and so this coordinate is referred to as *slow time*. In contrast, the time interval between transmission and reception of a pulse corresponding to a particular range is referred to as *fast time*.

If the geometry of the imaged terrain is planar, the locus of its intersection with the spherical wavefront will be the arc of a circle. Under this assumption, the received radar signal is an average of the ground reflectivity over a circular arc. Consequently, the radar performs a circular Radon transform (CRT) of the ground reflectivity. Therefore, forming an image of the ground reflectivity in synthetic aperture radar (SAR) is the process of inverting the circular averages, or in other words, inverting the circular Radon transform of the received radar signal. This approach to image formation provides an alternative view to understanding the SAR image formation process and can be used to develop algorithms that are free of the range curvature limitations of standard techniques. The approach used here may also be more intuitive for many readers unfamiliar with radar, because the concepts of doppler and phase are not required [1,9,11]. In addition, methods based on inverting the CRT could be used in situations where the more standard techniques may fail because the stationary phase approximation [16] is inappropriate, such as ultra-wideband or foliage penetration SAR [8], and high-squint SAR for tactical platforms. The CRT is the subject of a previous report [14]. Follow-on work is underway to determine the practical utility of the results presented there.

In this paper we summarise the previous work on the CRT [13, 14] and the spherical Radon transform (SRT) and its inverse for 3-dimensional SAR image formation [15]. Like [17], we assume that the radar is monostatic (co-located transmitter and receiver), and that the propagation medium is isotropic, homogeneous and non-dispersive so that the

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wave velocity is constant in space. Then using the Born approximation, the radar echo is assumed to be a sum of single-scattering objects. We also use the “start-stop” approximation because the sensor platform speed is much less than the wave velocity. Furthermore, we make two assumptions that can be removed at the cost of complexity. Firstly, the illumination of the ground by the antenna in both transmission and reception is assumed to be uniform. Secondly, we assume an aspect independent ground reflectivity.

In the next section we present the theory for the CRT and SRT and their inverse.

2. RADON TRANSFORMS AND THEIR INVERSES

We refer the reader to the previous report [14] and the references contained therein for an introduction to the normal Radon transform.

2.1. Definition of the CRT and SRT

By analogy with (1), we define the circular Radon transform of a function to be the path integral of the function along a circle of radius t centred on the point $(u, 0)$ on the x-axis. This can be written as

$$g(u,t) = (\mathcal{R}_c f)(u,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \, \delta(t - \sqrt{(x-u)^2 + y^2}) \, dx \, dy. \qquad (2)$$

Again by analogy, we define the spherical Radon transform (SRT) of a function to be the integral of the function around a sphere of radius t centred on the point $(u_x, 0, u_z)$ in the x-z-plane:

$$g(u_x, u_z, t) = (\mathcal{R}_s f)(u_x, u_z, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t - \sqrt{(x-u_x)^2 + y^2 + (z-u_z)^2}) \, f(x,y,z) \, dx \, dy \, dz. \qquad (3)$$

The geometry of the spherical Radon transform is shown in Figure 1. There are many other possible families of CRT and SRT depending on the choice of centres for the circles and spheres: the particular definition presented here has been chosen because it best corresponds to the standard geometry used in SAR (it allows the sensor to move anywhere in a line or plane, respectively), and is tractable mathematically.

Focussing on the spherical case for the moment, we can transform the coordinate system to spherical coordinates by letting $x = u_x + t\cos\theta\sin\phi$, $y = t\sin\theta\sin\phi$, and $z = u_z + t\cos\phi$, where $(u_x, 0, u_z)$ is the coordinates of the sensor (*i.e.* it is in the x-z-plane) and θ is the azimuth and ϕ is the zenith. Then an equivalent definition of the SRT is

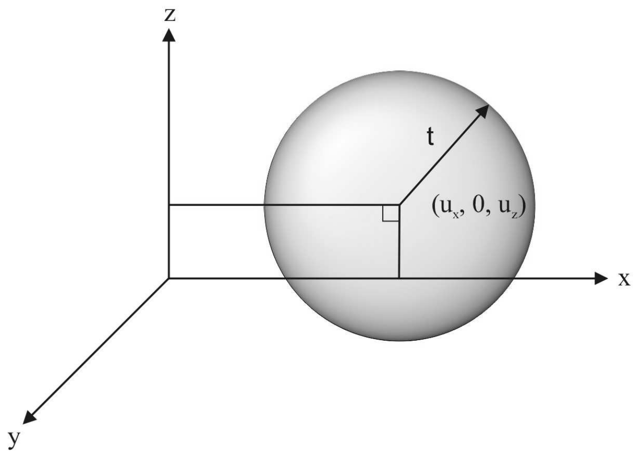

Fig. 1. The geometry of the spherical Radon transform. The origins of the spheres of radiation lie in the xz plane.

$$g(u_x, u_z, t) = (\mathcal{R}_s f)(u_x, u_z, t) = \int_0^{\pi}\int_0^{2\pi} f(u_x + t\cos\theta\sin\phi, t\sin\theta\sin\phi, u_z + t\cos\phi) \, t^2 \sin\phi \, d\phi \, d\theta. \qquad (4)$$

From (3), the dual operator for the SRT (changing the integration with respect to x, y and z to be with respect to u_x, u_z and t), called here the *spherical* backprojection operator, is defined to be

$$(\mathcal{R}_s^{\dagger} g)(x,y,z) = \int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t - \sqrt{(x-u_x)^2 + y^2 + (z-u_z)^2}) \, g(u_x, u_z, t) \, du_x \, du_z \, dt \qquad (5)$$

The circular backprojection operator is obviously the 2D equivalent. These operators are important because Radon transforms are commonly inverted by computing the corresponding backprojection operator in tomography [3]. In the NRT case, the backprojection operator is related to the inverse of the NRT by a scale factor [3, 14]. This simple relationship does not hold in the CRT or SRT cases as we will see later, and a more complicated relationship is required if the circular and spherical backprojection operators are used for inversion in each case. We proved that (5) is the adjoint of the SRT in [15].

2.2. The Problem

The 3D SAR image reconstruction problem, given the data $g(u_x, u_z, t)$ consisting of integrals of $f(x, y, z)$ over spheres in three-dimensional space centred on the $y = 0$ plane, is to find a closed form solution for f. In SAR terms, g represents the range-compressed baseband received signal in fast-time t and slow-time u_x, u_z coordinates and is represented by (4). The equivalent in 2D is $g(u, t)$.

When g is data collected from a SAR system, then (4) should more properly be [6],

$$g(u_x, u_z, t) = \int_0^{\pi} \int_0^{2\pi} h(t)\, \mu(\theta, \phi)\, f(u_x + t\cos\theta\sin\phi, t\sin\theta\sin\phi, u_z + t\cos\phi)\, t^2 \sin\phi\, d\phi\, d\theta, \quad (6)$$

where $h(t)$ is the range attenuation factor under the omnidirectional isotropic scattering model, and $\mu(\theta, \phi)$ represents the antenna diagram assuming no frequency dependence. We will assume that the range attenuation factor has been compensated for in the processing to form $g(u_x, u_z, t)$ and so can be neglected here. We will also assume that the target is in the main beam so that the antenna diagram can also be neglected.

We may assume that $f(x, y, z) = 0$ for $y < 0$ or that $f(x, y, z) = f(x, -y, z)$ as convenient because either the SAR points to one side of the platform or for low frequency SAR, two antennas are used to distinguish the left- and right-hand sides [7]. Note here that the SRT annihilates functions $f(x, y, z)$ that are odd in y (*i.e.* $f(x, y, z) = -f(x, -y, z)$), and this property has important consequences. In particular, the inversion formulae typically give reconstructions f that are even, so if we know $f(x, y, z) = 0$ for $y < 0$, we will need to multiply reconstructions by a factor of two to give the correct answer in the upper half space.

2.3. Inverting the Spherical Radon Transform

In this section we present the extension to previously understood methods for inverting the CRT [13,14] to the spherical three-dimensional case [15]. Three approaches have been developed: the Square Fourier Transform approach [14], a Hankel transform method and a backprojection method. For brevity only the Hankel and backprojection approaches are presented here [15].

2.3.1. Hankel Transform Method

The inverse of the CRT can be expressed analytically in terms of a Hankel transform. This was discovered independently by Fawcett [4] and Andersson [1], and rediscovered more recently by Soumekh [16] and Milman [10, 11] as a statement of the ω-k migration (or range migration) algorithm [2]. In the report [14], we derived the inverse of the CRT in terms of the Hankel transform following Nilsson [13]. We now present the extension to invert the SRT.

From Hankel's Theorem, the Hankel transform pair of order ν can be written as

$$f^{(H_\nu)}(\gamma) = \mathcal{H}^t_\nu[f(t)](\gamma) = \int_0^\infty f(t)\sqrt{\gamma t}\, J_\nu(\gamma t)\, dt$$

$$f(t) = \mathcal{H}^\gamma_\nu[f(\gamma)](t) = \int_0^\infty f^{(H_\nu)}(\gamma)\sqrt{\gamma t}\, J_\nu(\gamma t)\, d\gamma,$$

and then from [15]

$$f^{(F,F,F)}(v_x, \rho, v_z) = \sqrt{\frac{\pi}{2}}\, |\rho|\; \mathcal{H}^t_{1/2}\left[g_0^{(F,F,I)}(v_x, v_z, t)\right](2\pi\sqrt{\rho^2 + v_x^2 + v_z^2}), \quad (7)$$

where $g_0^{(F,F,I)}$ denotes the Fourier transform with respect to the first and second variables, and the third variable is left intact. Therefore we see that the Fourier-Fourier-Hankel transform of $g_0 = g/t$, after a coordinate transformation and scaling, gives the 3-dimensional Fourier transform of f. The similarity to the 2D case [13, 14] can be seen:

$$f^{(F,F)}(v, \rho) = \frac{|\rho|}{2}\mathcal{H}^t_0\left[g_0^{(F,I)}(v, t)\right](v, \sqrt{\rho^2 + v^2}).$$

From $J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z$, we can see that the half-order Hankel transform is the special case of the Fourier sine transform,

$$\mathcal{H}^t_{1/2}\left[g_0^{(F,F,I)}(v_x, v_z, \sigma)\right] = \int_0^\infty g_0^{(F,F,I)}(v_x, v_z, t)\, \sin\sigma t\, dt.$$

2.3.2. Backprojection Method

Andersson [1] showed that the backprojection operator for the CRT can be used to implement $g^{(F,H_0)}$, the Fourier- and zeroth-order-Hankel transformed circular-Radon transformed data. We presented this in [14] following the derivation of Nilsson [13]. Here, we present the equivalent expression for the SRT case.

We take the backprojection operator (5) of $g_0(u_x, u_z, t) = \frac{g(u_x, u_z, t)}{t}$ rather than g,

$$(\mathcal{R}^\dagger_s g_0)(x, y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_0(u_x, u_z, \sqrt{(x-u_x)^2 + y^2 + (z-u_z)^2})\, du_x\, du_z. \quad (8)$$

Then we have previously shown [15] that

$$f^{(F,F,F)}(v_x, \rho, v_z) = \sqrt{\frac{\pi}{2}}\, |\rho|\, \sqrt{v_x^2 + \rho^2 + v_z^2}\, (\mathcal{R}^\dagger_s g_0)^{(F,F,F)}(v_x, \rho, v_z) \quad (9)$$

and in the 2D case [13, 14]

$$f^{(F,F)}(v,\rho) = \pi|\rho|(\mathcal{R}_c^{\dagger} g_0)^{(F,F)}(v,\rho).$$

3. FINITE APERTURE AND BANDWIDTH INVERSE SPHERICAL RADON TRANSFORM

An expression which represents a realistic SAR collection with finite bandwidth and synthetic aperture can be formed from (7) [15]. The result is

$$f_{\text{fb,fa}}^{(F,F,F)}(v_x,\rho,v_z) = \sqrt{\frac{\pi}{2}}|\rho| \mathcal{H}_{1/2}^t\left[g_{\text{fb,fa}}^{(F,F,I)}(v_x,v_z,t)\right]\left(2\pi\sqrt{\rho^2+v_x^2+v_z^2}\right).$$

where

$$g_{\text{fb,fa}}^{(F,F,I)}(v_x,v_z,t) = g_0^{(F,F,I)}(v_x,v_z,t) *_{v_x} L_x \operatorname{sinc}(L_x v_x) *_{v_z} L_z \operatorname{sinc}(L_z v_z) *_t v_b e^{-2\pi i v_0 t}\operatorname{sinc}(v_b t) \quad (10)$$

where $*_{v_x}$, $*_{v_z}$, and $*_t$ indicate convolution with respect to the variables v_x, v_z, and t, respectively, and L_x, L_z are the lengths of the x and z apertures. We observe that the effect of finite apertures in x and z is a convolution with a sinc function in the Fourier domains of v_x and v_z respectively, and the effect of finite bandwidth is a convolution with a shifted sinc function in the spatial domain of t.

4. CONCLUSION

In this paper we have reviewed a formulation for 2D and 3D SAR image formation that is not subject to the problems of range curvature because it exactly treats the circular and spherical nature of the radar's propagating wavefront. This is achieved by inverting the spherical Radon transform, which is an integral of a 3D function around a spherical surface, or the CRT in the 2D case. Inverting the CRT or SRT is shown to involve Fourier transforms and a Hankel transform along with coordinate transformations. We have shown the effect that finite bandwidth and synthetic aperture will have upon the reconstructed function in the 3D case.

5. REFERENCES

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