A 2-LEVEL DOMAIN DECOMPOSITION ALGORITHM FOR INVERSE DIFFUSE OPTICAL TOMOGRAPHY

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ABSTRACT

In this paper[1], we explore domain decomposition algorithms for the inverse DOT problem in order to reduce the computational complexity and accelerate the convergence of the optical image reconstruction. We propose a combination of a two-level multigrid algorithm with a modified multiplicative Schwarz algorithm, where a conjugate gradient is used as an accelerator to solve each sub-problem formulated on each of the partitioned sub-domains. For our experiments, simulated phantom configuration with two rectangular inclusions is used as a testbed to measure the computational efficiency of our algorithms. No a priori information about the configuration is assumed except for the source and detector locations. For the application of our modified Schwarz algorithm alone, we observe an increase in efficiency of 100% as compared to the conjugate gradient solution obtained for the full domain. With the addition of the coarse grid, this efficiency rises to 400%. The coarse grid also serves to improve the overall appearance of the reconstructed image at the boundaries of the inclusions.

1. INTRODUCTION

Inverse diffuse optical tomography (DOT) problem involves estimation of the optical properties of biological tissues, which are pertinent to tissue's physiological and biochemical state. The most prominent applications of DOT are in detecting tumors in the breast, monitoring brain activity, and detecting brain tumors and hemorrhages.

DOT poses a computationally challenging inverse problem. Thereby, realization of real-time diffuse optical tomographic imaging requires computationally viable reconstruction algorithms that provide accurate quantitative results. In this work, we address the computational complexity of the inverse problem by proposing a two-level domain decomposition procedure. Domain decomposition methods convert the inverse problem into smaller-size problems that are easier to handle. Herein we apply a modified alternating Schwarz method with conjugate gradient (CG) algorithm as the iterative solver to accelerate the solution. We then extend the uni-level problem formulation to a 2-level problem to include a coarse grid correction in an attempt to improve quantitative accuracy and further accelerate the convergence.

The Schwarz algorithm begins by partitioning the domain into two or more overlapping sub-domains. The problem is then divided into subproblems on each of these sub-domains. Multiplicative Schwarz algorithm successively solves the localized problems in each sub-domain, every time using the latest solution available to initialize the next sub-problem. Therefore, at every iteration one sub-problem, initialized with the solution from the last iteration, is solved, whose solution is then used to initialize the next sub-domain and so on. We follow the multiplicative Schwarz algorithm using a fixed number of CG iteration to approximate the solution for each sub-domain. For full explanation of Schwarz algorithms see [1, 2].

Single level Schwarz algorithms are not well suited for problems exhibiting low frequency errors and suffer from low convergence rate. Multi-grid methods, on the other hand, use a hierarchy of grids at different scales to accelerate the convergence of standard iterative methods. The fundamental idea behind all multi-grid methods is to combine computations done on different grid scales in order to eliminate the error components of the finest grid, where the original problem has been formulated. This is achieved by approximating the *smoothed* fine grid error on a coarser grid where it can be accurately represented. The approximated error on the coarse grid appears to be more oscillatory, hence can be further eliminated by the iterations on the coarse grid. The solution obtained for the error on the coarse grid is then interpolated to the fine grid to correct the current fine grid solution estimate. There is no unique way of formulating an inverse problem in a multigrid framework. The formulation needs to be specific to the problem at hand, that is it must be tuned to address the requirements asserted in the problem. Note that, carefully designed multi-grid solvers have the potential of solving inverse problems with N unknowns within $O(N)$ work load [3], which makes them the most efficient solvers for many kinds of mathematical problems.

Multigrid methods have been applied for the DOT problem in the past, where the inverse problem is formulated on a hierarchy of *regular* rectangular grids [4]. Domain decomposition has also been previously proposed for Bayesian formulation of inverse DOT problem [5]. However, the nature of the decomposition was different in that work, where non-overlapping, and hence non-Schwarz type, "sliding window" decomposition was used. In our previous work, we have proposed Fast Adaptive Composite-grid (FAC) algorithms for Region-of-Interest DOT [6, 7], which implicitly pursued a domain decomposition with the aid of *a priori* information about the medium of interest. Note that FAC can be viewed as a Schwarz-like domain decomposition method in terms of fully overlapping sub-domains, hence achieving fast convergence with low cost by the use of coarse grid with substantially fewer points in the overlap region [8].

In this work, we assume that no a priori information is available about the unknown image. We have used the location of

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sources and detectors to determine the overlapping sub-regions. We formulate the inverse problem on the fine grid as two smaller size inverse problems on overlapping sub-regions. CG accelerator is used to approximate the solution on one sub-region. This solution is used to initialize the optimization in the other sub-region. The fine grid iterations are followed by a coarse grid correction scheme, where the inverse problem aims to solve the residual equation formulated on the global coarse grid, which is of relatively smaller size. A number of 2-grid cycles are run to further improve the accuracy of the reconstruction.

2. PROBLEM FORMULATION

2.1. Forward Model

The forward model for DOT is based on simplifying assumptions applied to radiative transport equation which results in a form of photon diffusion equation. In frequency domain, the diffusion equation is given by:

$$-D\nabla^2\Phi(r) - \frac{j\omega}{c}\Phi(r) + \mu_a(r)\Phi(r) = S \tag{1}$$

where D is the diffusion coefficient, c is the speed of light and $\mu_a(r)$ on the entire domain $\Omega \subset R^2$ is the spatially varying absorption coefficient. S stands for the point source located at $r = r_s$. In this work, we focus on the reconstruction of absorption coefficients μ_a of the medium, hence we assume that the diffusion coefficient D has a spatially uniform distribution. We have employed the perturbation approach [9] around a spatially invariant optical background with a first order Rytov approximation to solve the inverse problem. The cell-centered discretization on the grid Ω^h, yields a system of linear equations relating the differential absorption coefficients $\delta\mu_a(r_l)$ to the measurements:

$$\begin{bmatrix} y_{11}^{f_1} \\ \vdots \\ y_{1m}^{f_1} \\ \vdots \\ y_{11}^{f_2} \\ \vdots \\ y_{nm}^{f_p} \end{bmatrix} = \begin{bmatrix} W_{111}^{f_1} & \cdots & W_{11N}^{f_1} \\ \vdots & \ddots & \vdots \\ W_{1m1}^{f_1} & \cdots & W_{1mN}^{f_1} \\ \vdots & \ddots & \vdots \\ W_{111}^{f_2} & \cdots & W_{11N}^{f_1} \\ \vdots & \ddots & \vdots \\ W_{nm1}^{f_p} & \cdots & W_{nmN}^{f_p} \end{bmatrix} \times \begin{bmatrix} \delta\mu_a(r_1) \\ \\ \vdots \\ \\ \vdots \\ \\ \delta\mu_a(r_N) \end{bmatrix} \tag{2}$$

where $y_{ij}^{f_k}$ denote the real part of the measurement at i^{th} source and j^{th} detector at frequency f_k, $W_{ijh}^{f_k}$ is the weight for the l^{th} pixel for ij source-detector pair, and $\delta\mu_a(r_l)$ is the differential absorption coefficient for l_{th} pixel. We can denote this model system succinctly as:

$$y = W^h x^h \tag{3}$$

where W^h is the weight matrix and $y \in \mathbb{R}^M$, $x \in \mathbb{R}^N$, $W^h \in \mathbb{R}^{M\times N}$. N is the number of grid points on Ω^h and M is the total number of measurements.

2.2. Inverse Problem

We formulate the discrete inverse problem to yield a minimum least squares solution for the differential absorption coefficients x^h on Ω^h.

$$\hat{x}_{LS}^h = \arg\min_{x^h} \Psi(x^h) = \arg\min_{x^h} \|y - W^h x^h\|^2 \tag{4}$$

where $||\cdot||$ denotes the Euclidean norm.

2.2.1. *Uni-Level Domain Decomposition Algorithm*

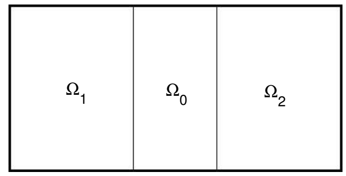

Fig. 1. Decomposition of Ω

In this work, we decompose the domain Ω into two overlapping sub-domains Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$ and Ω_0 is the overlapping region, that is $\Omega_0 = \Omega_1 \cap \Omega_2$ (Figure 1). We discard those source-detector pairings that would produce a coupling between the two sub-domains outside the overlapping region. In other words we remove any measurement that is due to a source in $\Omega_1\backslash\Omega_2$ and to a detector in $\Omega_2\backslash\Omega_1$ or vice versa. As a result, the measurement vector $\tilde{y} \in R^{\bar{M}}$ for $\bar{M} \leq M$ becomes

$$\tilde{y} = \begin{pmatrix} y_{\Omega_1\backslash\Omega_2} \\ y_{\Omega_0} \\ y_{\Omega_2\backslash\Omega_1} \end{pmatrix} \tag{5}$$

Regrouping the measurements in two vectors y_1 and y_2 yields $y_1 = (y_{\Omega_1\backslash\Omega_2} |\, y_{\Omega_0})^T$ and $y_2 = (y_{\Omega_0} |\, y_{\Omega_2\backslash\Omega_1})^T$. Similarly the differential absorption coefficients on the sub-domains are grouped as x_1^h and x_2^h, such that x_1^h is a finite dimensional approximation of differential absorption coefficients on Ω_1^h and x_2^h on Ω_2^h. As a result we can formulate two sub-domain problems as follows

$$\begin{aligned} \hat{x}_{1,LS}^h &= \arg\min_{x_1^h} \|y_1 - W_1^h x_1^h\|^2 \\ \hat{x}_{2,LS}^h &= \arg\min_{x_2^h} \|y_2 - W_2^h x_2^h\|^2 \end{aligned} \tag{6}$$

Instead of estimating x^h using the formulation given in equation 4, we propose a procedure that follows minimization of the two objective functionals formulated on the two sub-domains, with reduced number of measurements. This results in significant reduction in the size of the inverse problem and consequently the computational complexity of the overall inverse DOT problem.

A conjugate gradient (CG) algorithm is utilized to accelerate the solution on each sub-grid. The solution update obtained after a sweep of CG iterations on one domain is restricted to the overlapping region on the grid Ω_0^h by the discrete operator $\mathcal{I} : \Omega_1^h \rightarrow \Omega_0^h$. The restricted estimates are then used to update the x_2^h estimates on Ω_0^h, which is followed by iterations on Ω_2^h to yield a solution on entire discrete domain Ω_2^h. A υ number of cycles is applied until a desired level of convergence is achieved.

This approach takes advantage of the reduced size of inverse problem formulation by decomposing the domain and the associated measurements. Initialization in one sub-problem solution by the current estimates in the other one and use of CG algorithm are other important factors facilitating the acceleration of the solution.

2.2.2. *Integrating Coarse Grid Correction*

Single level Schwarz algorithms are not well suited for problems exhibiting low frequency errors. CG iterations tend to *smooth* the error in the solution. Even though CG algorithm accelerates the convergence to the solution x^h on Ω^h, it is unable to further eliminate the error with smooth content. This necessitates the use of a coarse grid correction scheme, which enables elimination of smooth error by restricting it onto a coarser grid and correcting the fine grid solution by interpolating the error in the coarse grid.

After the end of CG cycles on the sub-problems, the current estimates x_1^h and x_2^h are concatenated to yield the overall solution estimate x^h on Ω^h. The error between the actual solution x^* and the current solution estimate $\hat{x}^h$ is given by $e^h = x^* - \hat{x}^h$. Assuming that the error e^h on Ω^h is smoothed well enough by CG iterations, we can write a coarse grid approximation for this error, such that

$$e^h = I_{2h}^h e^{2h}$$

where I_{2h}^h is the interpolation operator. We have selected $I_{2h}^h : \mathbb{R}^{N/4} \longrightarrow \mathbb{R}^N$ as $4(I_h^{2h})^T$ where $I_h^{2h} : \mathbb{R}^N \longrightarrow \mathbb{R}^{N/4}$ is the full weighting operator in 2D [10]. As a result, the objective functional given in equation 4 can be re-written as

$$\begin{aligned} \|y - W^h x^h\|^2 &= \|y - W^h(\hat{x}^h + e^h)\|^2 \\ &= \|y - W^h(\hat{x}^h + I_{2h}^h e^{2h})\|^2 \\ &= \|y - W^h\hat{x}^h - W^h I_{2h}^h e^{2h})\|^2 \\ &= \|r - W^{2h} e^{2h}\|^2 \end{aligned} \tag{7}$$

which is the coarse grid objective functional defined on the coarse grid Ω^{2h} with grid size of $2h$. $r = y - W^h\hat{x}^h$ is called the residual, e^{2h} is the coarse grid error, and $W^{2h} = W^h I_{2h}^h$ is the coarse grid operator. CG iterations on the coarse grid error e^{2h}, which is initially assigned to zero, eliminate the high-frequency components of e^{2h} which appeared to be smooth on Ω^h. This results from the fact that low frequency components appear more oscillatory on coarse grids as compared to fine grids.

The error estimate e^{2h} on Ω^{2h} can then be interpolated to the fine grid to correct the fine grid solution estimate x^h

$$x^h \leftarrow x^h + I_{2h}^h e^{2h} \tag{8}$$

A predefined number of 2-grid cycles are run to further improve the solution accuracy. A pseudo-code of the overall 2-level domain decomposition algorithm is given below.

Algorithm 1 Two-Grid Domain Decomposition Algorithm

```
W_1^h, W_2^h, y_1, y_2 ← partition(W^h, y)
W^{2h} ← restrict(W^h) {We only need to generate Ω^{2h} once}
x^h ← initialize(x^h)
repeat
    x_1^h ← CG(W_1, y_1, x_1) {Smoothing on Ω_1^h}
    x_2^h ← CG(W_2, y_2, x_2) {Smoothing on Ω_2^h}
    r ← y − W^h x^h {Calculate residual.}
    e^{2h} ← CG(W^{2h}, r, e^{2h}) {Solve for coarse grid error}
    e^h ← interpolate(e^{2h})
    x^h ← x^h + e^h
until convergence
```

3. EXPERIMENTAL RESULTS

We performed test cases against a simple simulated phantom configuration consisting of two rectangular inclusions as seen in Fig. 3. The resolution of our test image was 20 pixels by 40 pixels and the number of sources and detectors used was 17 and 33 respectively resulting in a 561 by 800 weight matrix W^h. We compared four test cases: (1) least squares solution using conjugate gradient on Ω^h for the full inverse problem, (2) least squares solution using conjugate gradient on Ω^h with the reduced source and detector configurations, (3) uni-level domain decomposition on the reduced source and detector configuration, and (4) two-level domain decomposition with modified multiplicative smoother on the reduced source and detector configurations. By discarding those measurements that couple source on one sub-domain and the detector on the other, we effectively reduce the dimension of the weight matrix W^h.

All algorithms were implemented in MATLAB. The results were compared using the same conjugate gradient and decomposition codes. Figure 2 shows a plot of square error between the actual and estimated image versus floating point operations (flops) required for reconstruction. The square error was calculated by taking the pixel by pixel difference between the true image (see Figure 3) and the reconstructed image then taking the sum of the squares of those differences. As shown in Figure 2, there is up to an average of 100% increase in efficiency using the uni-level domain decomposition algorithm compared with case (1). With the addition of coarse grid correction, there is approximately 400% increase in efficiency. We observed that the error curve tends to settle faster for single level methods. This is to be expected since the motivation for coarse grid correction is to compensate for the smoother's inability to properly handle globalized (i.e. low frequency) errors.

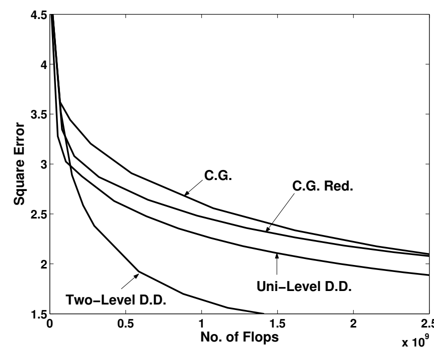

Fig. 2. Square Error vs. No. of Flops

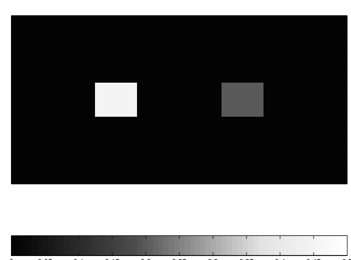

Fig. 3. Simulated Phantom Configuration

Figures 4, 5, 6, and 7 shows the reconstructed images for each method after 1200 iterations. As can be inferred from Figures 5

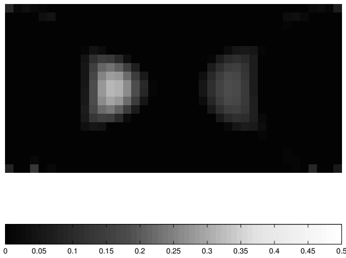

Fig. 4. Direct Method: 1200 Iterations

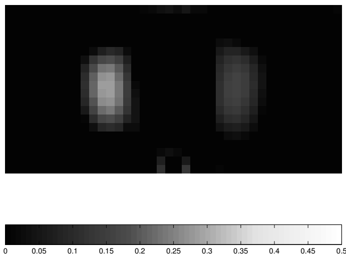

Fig. 5. Direct Method w/ Reduced S-D Pairs: 1200 Iterations

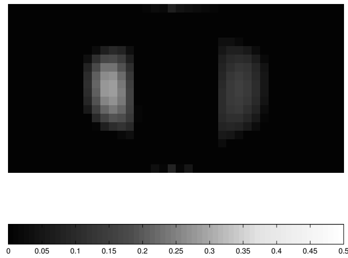

Fig. 6. One Grid D.D.: 1200 Iterations

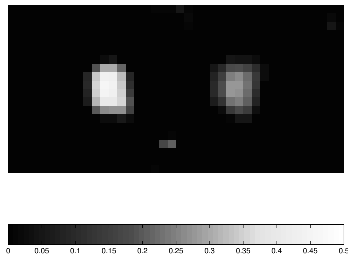

Fig. 7. Two Grid D.D.: 1200 Iterations

and 6, the uni-level domain decomposition method and the direct CG on the reduced source-detector configurations had nearly equal error after 1200 iterations, but the computational time was cut by half for the domain decomposition case. The effect of coarse grid correction on the domain decomposition algorithm is clear from Figure 7. By incorporating coarse grid correction, the algorithm is able to handle the smooth errors around the boundaries of the inclusions. In effect, we end up with a much sharper picture and better results around these boundaries.

4. CONCLUSION

In this paper, we investigated the effectiveness, in terms of computational efficiency, of applying Schwarz type domain decomposition method in the solution of linearized DOT inverse problem. We modified the classic Schwarz algorithm where CG was used to accelerate the convergence to the least squares solution for each sub-problem in the place of the usual Gauss-Siedel iterations. In our two grid domain decomposition algorithm, CG was applied to each sub-domain for several iterations (to get a good smoothing effect). The residual from the entire domain was then restricted to the coarse grid to find the coarse grid error which was then interpolated back to the fine grid for error correction. The addition of the domain decomposition reduced the computational burden while addition of the coarse grid allowed for correction of global (low frequency) errors. In terms of computational efficiency as compared with CG solution for the full problem, uni-level domain decomposition method showed 100% increase while two-level method saw nearly 400%. As a result of these preliminary findings, we feel that domain decomposition coupled with multi-grid methods is a viable option for decreasing the computation complexity of the DOT inverse problem even when there is no parallelization of the problem. As such, the algorithm proposed is well-suited for real-time DOT image reconstruction.

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