AN ALTERNATIVE APPROACH FOR THE RESAMPLING IN THE BOOTSTRAP FILTER

Alexsandro M. Jacob, Takashi Yoneyama

Divisão de Engenharia Eletrônica
Instituto Tecnológico de Aeronáutica
São José dos Campos SP 12228-900, Brazil
e-mail: {ajacob, takashi}@ita.br

keywords: particle filters, resampling, restoration method

ABSTRACT

The resampling step in the bootstrap filter aims at reducing the well-known particle degeneracy phenomenon. In practical applications of particle filters, a large number of particles is used for computing the estimates of desired states. However, the parallelizability of the filter is affected by the resampling that increases the hardware complexity. This paper proposes an alternative method to replace this step by one that, in accordance to the so-called attractor function, moves the particles in the likelihood estimate to regions where the weights are higher. The implementation of this new method allows the parallelization of the algorithm and the particle filter performance is similar to that obtained by the traditional bootstrap method.

1. INTRODUCTION

Particle filters can be defined as sequential Monte Carlo (SMC) methods used to solve estimation problems where time-varying signals must be presented in real time. Based on the parametric structure of a probabilistic dynamic system [1], these problems are described by the estimation of non-observable states of the model and/or detection of events described by the observed signals.

More specifically, if a SMC method implements a recursive Bayesian filter by using Monte Carlo simulations, this is called sequential importance sampling (SIS) algorithm [2]. The estimates are computed from the representation of the required posterior density function (pdf) via a set of importance weighted random samples. As the number of samples becomes very large, this representation tends to an equivalent one of the usual functional description of the posterior pdf.

The problem encountered by SIS method is that, as time increases, the distribution of the weighted random samples becomes more and more skewed, the so-called *degeneracy phenomenon* [3]. In this way, a resampling step [4] [5] must be added to the algorithm aiming at minimizing this problem, what means that it crucially affects the overall particle filter performance. Specifically, if the resampling step is applied at every algorithm iteration, the so-called sequential importance resampling (SIR) method, or *bootstrap*, is obtained [6].

In practical applications of particle filters, a large number of particles needs to be used for computing the estimates of desired states. However, the parallelizability of the filter is affected by the resampling that has the following disadvantages from a parallel hardware implementation viewpoint: the sampling period and memory requirements are increased, and the data exchange in implementations with multiple processing elements becomes a bottleneck of the parallel design [7]. In this way, modified resampling algorithms must be proposed to have an efficient mechanism that reduces the hardware complexity and maintains the filter performance [8].

This paper aims at proposing a method to address the particle degeneracy phenomenon by using the restoration of particles. The idea is to replace the resampling by an alternative method that moves the particles in the likelihood estimate to regions where the weights are higher, in accordance to the so-called attractor function. The implementation of this new method allows the possibility of parallelization of the algorithm and the particle filter performance is similar to that obtained by the SIR method.

Next section shows the general bootstrap approach to present clearly the mathematical notation used in this paper. Section 3 presents the basic concepts of the restoration method and how it can replace the resampling method in the SIR method. Experimental results comparing the filtering performance between the traditional SIR method and the new one is presented is Section 4, and conclusions in Section 5.

The first author is supported by the State of São Paulo Research Foundation (FAPESP) under Grant 02/10632-0.

2. THE GENERIC BOOTSTRAP APPROACH

At every time instant k, let $\{x_{1:k}^{(m)}, w_k^{(m)}\}_{m=1}^{M}$ denote a random measure where $x_k^{(m)}$ is the mth particle of the signal at k, $x_{1:k}^{(m)}$ is the mth trajectory of the signal, and $w_k^{(m)}$ is the weight of the mth particle. It is supposed that this random measure characterizes the posterior

$$p(x_k|y_{1:k}) \approx \sum_{m=1}^{M} w_k^{(m)} \delta(x_k - x_k^{(m)}), \qquad (1)$$

where $y_{1:k}$ is the observation signal up to k and $w_k^{(m)}$ is normalized. Therefore, equation (1) is a discrete weighted approximation to the true posterior $p(x_k|y_{1:k})$ where the weights are chosen using the principle of *importance sampling* [9].

According to the classical SIS [3], the weight updating for the optimal importance density function that minimizes the variance of the weights is given by

$$\begin{aligned} w_k^{(m)} &\propto w_{k-1}^{(m)} \frac{p(y_k|x_k^{(m)})p(x_k^{(m)}|x_{k-1}^{(m)})}{p(x_k^{(m)}|x_{k-1}^{(m)}, y_k)} \\ &= w_{k-1}^{(m)} p(y_k|x_{k-1}^{(m)}), \end{aligned} \qquad (2)$$

where $p(y_k|x_{k-1}^{(m)})$ does not have an analytical expression in the general case. To facilitate the filter implementation [4] [10], it is assumed that $p(x_k^{(m)}|x_{k-1}^{(m)})$ is a particular choice of importance density, what implies in a weight updating given by

$$w_k^{(m)} \propto w_{k-1}^{(m)} p(y_k|x_k^{(m)}). \qquad (3)$$

An important point is that this chosen importance sampling density is independent of y_k and, therefore, the state space is explored without any knowledge of the observations.

Due to the particle degeneracy phenomenon [3], a new step must be added to the algorithm in order to address this problem and also to improve the filter performance [11] [6] [1], as described in the following generic algorithm:

Generic SIS Filter
- For m=1:M
  - Sample $x_k^m \sim p(x_k^{m)}|x_{k-1}^{(m)})$
  - Evaluate $w_k^m = p(y_k|x_k^{(m)})$
- Evaluate $w_k^T = \sum_{m=1}^{M} w_k^{(m)}$
- For m=1:M
  - Evaluate $w_k^{(m)} = (w_k^{(m)})^{-1} w_k^T$
- Apply a method to minimize the degeneracy phenomenon if it exists
  - Obtain $\{\tilde{x}_k^{(m)}, \tilde{w}_k^{(m)}\}_{m=1}^{M}$

Based on the heuristic that particles having high weights must be duplicated and others must be discarded, the resampling method [4] [5] is used to minimize the particle degeneracy problem. This method aims at obtaining an unweighted empirical distribution approximation of $p(x_k|y_{1:k})$ presented in equation (1) [12] [13], in accordance to the *properly weighted sample* principle [1]. When the resampling procedure is applied at each time instant k, the so-called *bootstrap* filter is obtained.

3. THE RESTORATION METHOD

The generic SIS filter states that the weight associated to a particle $x_k^{(m)}$ sampled from $p(x_k^{(m)}|x_{k-1}^{(m)})$ is computed by $w_k^{(m)} = p(y_k|x_k^{(m)})$. The restoration method consists in moving the particles in the obtained likelihood estimate $\{x_k^{(m)}, w_k^{(m)}\}_{m=1}^{M}$ to regions where the importance weights are higher, in accordance to the so-called *attractor function* defined by $a(x_k^{(m)}, y_k)$. In other words, given $p(y_k|x_k^{(m)})$, to move the particles to regions where the importance random weights are higher, $a(x_k^{(m)}, y_k)$ must be chosen in order to guarantee the inequality $\tilde{w}_k^m \geq w_k^m$ for all m.

For each pair $\{x_k^{(m)}, w_k^{(m)}\}$, $m = 1, ..., M$, the main idea is to compute the corresponding pair $\{\tilde{x}_k^{(m)}, \tilde{w}_k^{(m)}\}$ by using the transformation $r(p(y_k|x_k^{(m)}), a(x_k^{(m)}, y_k); w_k^{(m)})$, the so-called *restoration function*, that depends on the structure of the importance sampling density $p(y_k|x_k^{(m)})$ and on $a(x_k^{(m)}, y_k)$.

In this way, by using the observed signal y_k, the restoration function is applied to obtain the restored observed values for each evolved particle $x_k^{(m)}$, $m = 1, ..., M$, that is,

$$\tilde{y}_k^{(m)} = r(p(y_k|x_k^{(m)}), a(x_k^{(m)}, y_k); w_k^{(m)}). \qquad (4)$$

Assuming that $h(\cdot)$ is the deterministic model of the observation signal, the new particle position is obtained by

$$\tilde{x}_k^{(m)} = h^{-1}(\tilde{y}_k^{(m)}). \qquad (5)$$

The restored weight is computed applying the original importance sampling density to the corresponding new particle position, or

$$\tilde{w}_k^{(m)} = p(y_k|\tilde{x}_k^{(m)}). \qquad (6)$$

In this case, the selection step occurs without the sorting among importance weights $\tilde{w}_k^{(m)}$, $m = 1, ..., M$, what facilitates the parallelization of the filter.

Figure 1 presents a basic visualization of the restoration method assuming that $p(y_k|x_k^{(m)})$ is Gaussian and that $a(x_k^{(m)}, y_k)$ is a composition of linear functions around y_k that guarantees $\tilde{w}_k^{(m)} \geq w_k^{(m)}$ for all m.

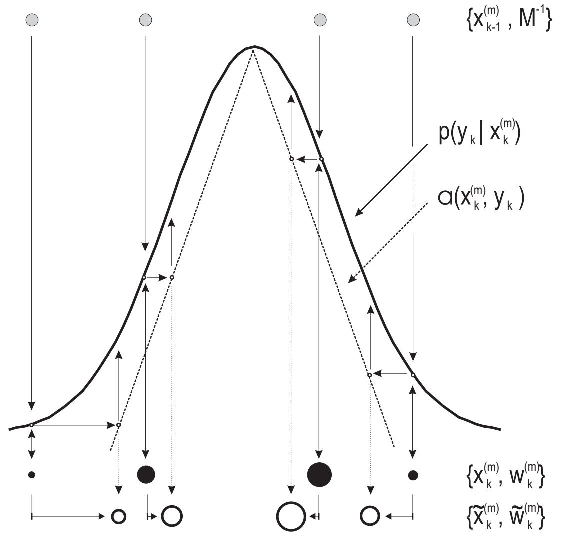

Fig. 1. Simple representation of the restoration method.

The obtained $r(p(y_k|x_k^{(m)}), a(x_k^{(m)}, y_k); w_k^{(m)})$ exemplified in Figure 1 makes a very significative correction in the positions of the particles located at the tail of $p(y_k|x_k^{(m)})$. Additionally, it is easily seen from the picture that, given $p(y_k|x_k^{(m)})$, the new distribution $p(y_k|\tilde{x}_k^{(m)})$ of particles depends exclusively on the joint properties shared by the functions $p(y_k|x_k^{(m)})$ and $a(x_k^{(m)}, y_k)$.

The following algorithm shows the implementation procedure of the restoration method:

`Restoration Method`

- `For m=1:M`
  - `Compute` $\tilde{y}_k^{(m)}$ `via (4)`
  - `Compute` $\tilde{x}_k^{(m)}$ `via (5)`
  - `Compute` $\tilde{w}_k^{(m)}$ `via (6)`

---

4. EXPERIMENTAL RESULTS

For demonstration purposes, the generic bootstrap algorithm with the restoration method will be applied to data artificially generated by the nonlinear, non-Gaussian model [4] [10]

$$x_k = \frac{1}{2}x_{k-1} + 25\frac{x_{k-1}}{1+x_{k-1}^2} + 8\,cos(1.2k) + v_k \quad (7)$$

$$y_k = \frac{x_k^2}{20} + u_k \quad (8)$$

for an initialization $x_0 \sim N(0,1)$, and with $v_k \sim N(0,10)$ and $u_k \sim N(0,1)$ being mutually independent white Gaussian noises.

Based on (7), (8) and Figure 1, a suggested attractor function $a(x_k^{(m)}, y_k) = a$ is the composition of linear functions around y_k, that is,

$$a = \begin{cases} N\delta^{-1}[x_k^{(m)} - \beta_1] & \text{if } \beta_1 \le x_k^{(m)} < y_k \\ -N\delta^{-1}[x_k^{(m)} - \beta_2] & \text{if } y_k \le x_k^{(m)} < \beta_2 \end{cases}, \quad (9)$$

where

$$\beta_1 = y_k - \delta \quad \text{and} \quad \beta_2 = y_k + \delta, \quad (10)$$

N is the normalization factor of $p(y_k|x_k^{(m)})$, and $\delta = 2.0$ is a *linear coefficient* that satisfies the relation $\tilde{w}_k^{(m)} \ge w_k^{(m)}$ for all m. Therefore, the restoration function is defined as

$$r(w_k^{(m)}) = \begin{cases} N^{-1}\delta w_k^{(m)} + \beta_1 & \text{if } \beta_1 \le x_k^{(m)} < y_k \\ -N^{-1}\delta w_k^{(m)} + \beta_2 & \text{if } y_k \le x_k^{(m)} < \beta_2 \end{cases}. \quad (11)$$

Moreover, based on (8),

$$\tilde{x}_k^{(m)} = \begin{cases} -\sqrt{20|\tilde{y}_k^{(m)}|} & \text{if } x_k^{(m)} < 0 \\ \sqrt{20|\tilde{y}_k^{(m)}|} & \text{if } x_k^{(m)} \ge 0 \end{cases}. \quad (12)$$

A comparison between the effects of the position corrections made by the traditional resampling and restoration methods can be seen in Figure 2. At time instant k, the initial random measure $\{x_k^{(m)}, w_k^{(m)}\}_{m=1}^{100}$ represented by $\bullet$ is transformed to the new pair $\{\tilde{x}_k^{(m)}, \tilde{w}_k^{(m)}\}_{m=1}^{100}$ represented by $\circ$. In both cases, the *effective sample size* [5] [14] increased. The evolution of the measures is described at time instant $k+1$ and, in this case, the results showed that the restoration method was able to make a better reconstruction of the likelihood function than in the resampling.

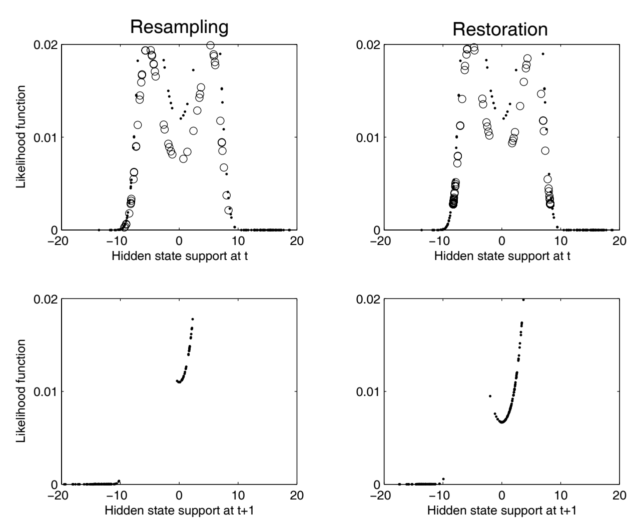

Fig. 2. Comparison of the corrections and evolving effects for the traditional resampling and the restoration method.

To compute the updating of the importance weights, the suggestion is that, at the end of the algorithm iteration $k-1$, there exists an initialization such that $w_k^{(m)} = M^{-1}$ for all m, as presented in Figure 1. Thus, after evolving the random measure $\{x_k^{(m)}, w_k^{(m)}\}_{m=1}^{M}$, the weighted state estimate is assigned according to equation (1). The restoration method is then applied in order to obtain $\{\tilde{x}_k^{(m)}, \tilde{w}_k^{(m)}\}_{m=1}^{M}$, the set with increased effective sample size. This procedure is called *Sampling Importance Restoration* (SIRe) method. Assuming $M = 250$, Figure 3 presents a comparison between the real state and the typical results obtained by SIRe method for the cases where the estimates are made before (SIRe - bef) and after (SIRe - aft) the correction of the weights.

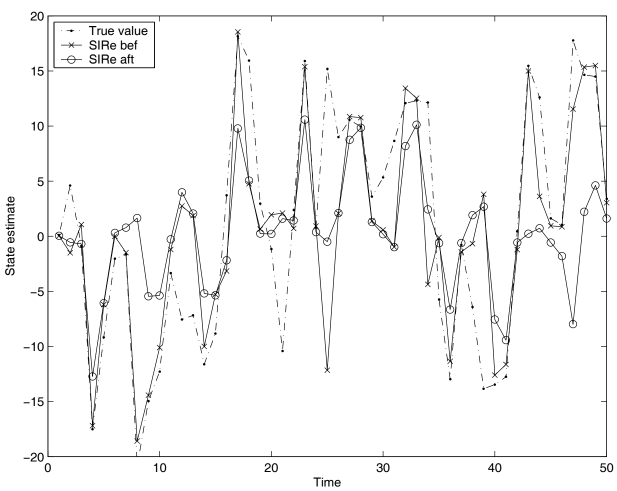

Fig. 3. Typical state estimation of the generic bootstrap filter with the restoration method.

The idea now is to compare de filter performance of SIRe methods with the results obtained by the bootstrap filter. Given $x_0 = 0.10$ and $M = 250$, the comparison was made for the time range $k = 1, 2, ..., 50$ with 100 different initializations of the noises. The estimation performance of the filters are given by the square root of the mean-square errors (RMSE) at each k, as presented in Figure 4.

The results showed that the SIRe - bef has practically the same estimation performance as that obtained by the bootstrap filter. However, as presented in Figure 3, the SIRe - aft smooths the estimates, what improved the filter performance in the RMSE viewpoint in this specific case. Details about theoretical aspects of the restoration method and the use of different attractor functions to improve the filter performance are being described in [15].

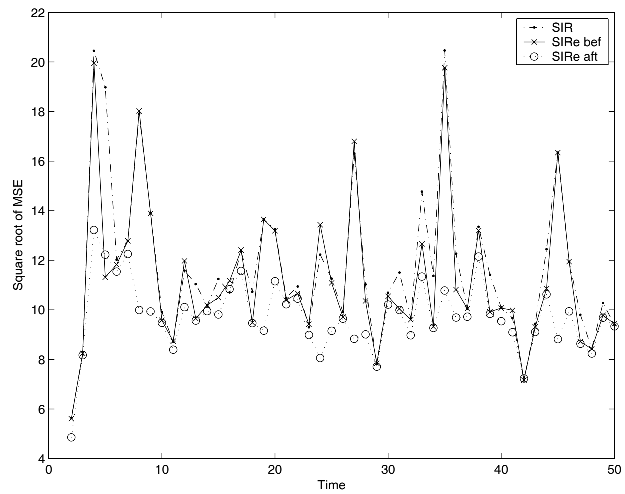

Fig. 4. Comparison of the estimation performance of the bootstrap with the SIRe methods.

5. CONCLUSIONS

This paper presented the basic concepts and preliminary results of the particle restoration method. Results showed that it can be an alternative to replace the traditional resampling step in SMC methods with real-time applications.

6. REFERENCES

[1] J. S. Liu, R. Chen, and T. Logvinenko, "A theoretical framework for sequential importance sampling with resampling," in *Sequential Monte Carlo Methods in Practice*, A. Doucet, N. de Freitas, and N. Gordon, Eds., chapter 11, pp. 225–246. Springer-Verlag, New York, 2001.

[2] M. S. Arulampalam, S. Maskell, N. Gordon, and T. Clapp, "A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian tracking," *IEEE Transactions on Signal Processing*, vol. 50, no. 2, pp. 174–188, 2002.

[3] A. Doucet, S. Godsill, and C. Andrieu, "On sequential Monte Carlo sampling methods for Bayesian filtering," *Statistics and Computing*, vol. 10, pp. 197–208, 2000.

[4] N. Gordon, D. Salmond, and F. M. Smith, "Novel approach to non-linear and to non-Gaussian Bayesian state estimation," *Proc. Inst. Elect. Eng., F*, vol. 140, pp. 107–113, 1993.

[5] J. S. Liu and R. Chen, "Blind deconvolution via sequential imputations," *Annals of Statistics*, vol. 90, no. 430, pp. 567–576, 1995.

[6] S. Godsill and T. Clapp, "Improvement strategies for Monte Carlo particle filters," in *Sequential Monte Carlo Methods in Practice*, A. Doucet, N. de Freitas, and N. Gordon, Eds., chapter 7, pp. 139–158. Springer-Verlag, New York, 2001.

[7] Miodrag Bolic, *Architectures for efficient implementation of particle filters*, Ph.D. thesis, Stony Brook University, Stony Brook, 2004.

[8] S. Hong, M. Bolic, and P. M. Djuric, "An efficient fixed-point implementation of residual resampling scheme for high-speed particle filters," *IEEE Signal Processing Letters*, vol. 11, no. 5, pp. 482–485, 2004.

[9] A. Doucet, "On sequential Monte Carlo methods for Bayesian filtering," Tech. Rep., Dept. Eng, Univ. Cambridge, UK, 1998.

[10] G. Kitagawa, "Monte Carlo filter and smoother for non-Gaussian nonlinear state-space approaches," *Journal of Computational and Graphical Statistics*, vol. 5, no. 1, pp. 1–25, 1996.

[11] C. Andrieu, A. Doucet, and E. Punskaya, "Sequential Monte Carlo methods for optimal filtering," in *Sequential Monte Carlo Methods in Practice*, A. Doucet, N. de Freitas, and N. Gordon, Eds., chapter 4, pp. 79–95. Springer-Verlag, New York, 2001.

[12] D. Crisan and A. Doucet, "A survey of convergence results on particle filtering methods for practitioners," *IEEE Transactions on Signal Processing*, vol. 50, no. 2, pp. 736–746, 2002.

[13] A. Doucet, N. de Freitas, and N. Gordon, "An introduction to sequential Monte Carlo methods," in *Sequential Monte Carlo Methods in Practice*, A. Doucet, N. de Freitas, and N. Gordon, Eds., chapter 1, pp. 3–14. Springer-Verlag, New York, 2001.

[14] J. S. Liu, "Metropolized independent sampling with comparison to rejection sampling and importance sampling," *Statistics and Computing*, vol. 6, pp. 113–119, 1996.

[15] A. M. Jacob and T. Yoneyama, "An approach to parallelize sequential Monte Carlo filters," to appear.