AN EFFICIENT MODEL FOR THE CONVERGENCE BEHAVIOR OF THE FXLMS ALGORITHM WITH GAUSSIAN INPUTS

Leonardo S. Resende and José Carlos M. Bermudez

Federal University of Santa Catarina – BRAZIL
{leonardo,bermudez}@eel.ufsc.br

ABSTRACT

This paper presents a simple and efficient analytical model for the convergence behavior of the filtered-x LMS (FXLMS) algorithm with Gaussian input data. Deterministic recursions are obtained for the mean weight vector and the mean square error. The new model predicts the algorithm behavior for a wide range of practical applications. This model can be employed either when the adaptive filter lies after the secondary path filter or when their order is reversed in the cascade sequence.

Simulation results display excellent agreement with the behavior predicted by the theoretical model for transient and steady-state phases of adaptation. The new simple model should be instrumental in designs systems for active control of sound and vibration.

1. INTRODUCTION

Active control of sound and vibration is an important application area for adaptive signal processing [1], [2]. In general, the controller is a causal and finite-duration impulse response (FIR) filter, assuring the properties of stability and absence of limit cycles.

Owing to its simplicity and robustness, the most widely used adaptive filtering algorithm in such active control systems is the least-mean-square (LMS) algorithm with the reference signal processed by a linear filter. This filter seeks to compensate for the effects of the secondary path from the output of the controller to the cancellation point. Because of this special arrangement, the algorithm is termed filtered-x LMS (FXLMS) algorithm.

In order to develop an effective algorithm design, it is very important to understand the algorithm behavior for the desired operating conditions, and such knowledge arises from analytical models.

Recent works on the statistical analysis of the FXLMS algorithm have a common point [3], [4], [5]. The FXLMS algorithm analysis starts from a scheme in which the secondary path filter processes the output of the adaptive filter. This is a physical imposition of active noise (ANC) and vibration (AVC) control systems, and usually leads to complex statistical analyses [3], [4]. The resulting analytical models for the second moment of the weight behavior tend to include large expressions that are sometimes hard to implement and time consuming in numerical simulations.

This paper presents an alternative approach for the analysis of the FXLMS behavior. Using an initial structural assumption, the remaining analysis does not require the determination of cross-correlations between delayed input vectors, becoming quite straightforward. The resulting analytical model is simple to implement and able to accurately predict the algorithm behavior for a wide range of applications.

2. FILTERED-X TRANSVERSAL FILTERING

Figure 1 illustrates the filtered-x transversal filtering problem in block diagram form. The request is to find the coefficients of the filter labeled W that minimizes the estimation error $e(n)=d(n)-y(n)$ in the mean-square sense. This problem can be viewed as a classical Wiener filtering problem where the transversal filter is composed of the cascade of two FIR (finite impulse response) filters: the unknown filter W of order $(N-1)$ with impulse response $\{w_0, w_1, \ldots, w_{N-1}\}$, and a predefined filter S of order $(M-1)$ with impulse response $\{s_0, s_1, \ldots, s_{M-1}\}$. The resulting transversal filter H of order $(L-1)$ has an impulse response given by the convolution sum:

$$h_l = \sum_{n=0}^{N-1} w_n s_{l-n} = \sum_{m=0}^{M-1} s_m w_{l-m}\,, \quad l=0, 1, \ldots, L-1, \qquad (1)$$

where $L=M+N-1$. The order in which the filters are cascaded in Figure 1 does not matter since they are assumed to be LTI (linear time-invariant) systems.

The input signal $x(n)$ and the desired response $d(n)$ are modeled as wide-sense, zero-mean stationary Gaussian discrete-time stochastic processes. Furthermore, without loss of generality, all the parameters are assumed to be real-valued.

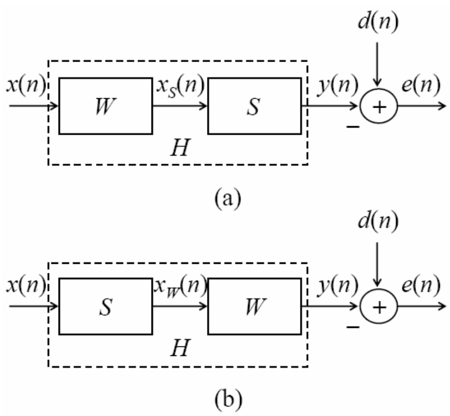


Figure 1: Filtered-x transversal filtering.

Using vector notation, the N-by-1, M-by-1 and L-by-1 tap-weight vectors for the responses of the W, S and H filters in Figure 1 are respectively denoted by

$$\mathbf{w} = [w_0, w_1, \ldots, w_{N-1}]^{\mathrm{t}}, \tag{2}$$

$$\mathbf{s} = [s_0, s_1, \ldots, s_{M-1}]^{\mathrm{t}} \tag{3}$$

and

$$\mathbf{h} = [h_0, h_1, \ldots, h_{L-1}]^{\mathrm{t}}. \tag{4}$$

Now, the convolution sum in (1) can be described by

$$\mathbf{h}=\mathbf{S}\mathbf{w}, \tag{5}$$

where $\mathbf{S}$ is an L-by-N non-symmetric Toeplitz matrix having the vector $[\mathbf{s}^{\mathrm{t}}, \mathbf{0}^{\mathrm{t}}_{L-M}]^{\mathrm{t}}$ as its first column and the vector $[s_0, \mathbf{0}^{\mathrm{t}}_{N-1}]$ as its first row. For instance, $\mathbf{S}$ has the following form for M=3 and N=5:

$$\mathbf{S} = \begin{bmatrix} s_0 & 0 & 0 & 0 & 0 \\ s_1 & s_0 & 0 & 0 & 0 \\ s_2 & s_1 & s_0 & 0 & 0 \\ 0 & s_2 & s_1 & s_0 & 0 \\ 0 & 0 & s_2 & s_1 & s_0 \\ 0 & 0 & 0 & s_2 & s_1 \\ 0 & 0 & 0 & 0 & s_2 \end{bmatrix}. \tag{6}$$

Figure 2 shows the block diagram of the filtered-x transversal filtering scheme using vector notation.

The estimation error is given by

$$\begin{aligned} e(n) &= d(n) - y(n) \\ &= d(n) - \mathbf{w}^{\mathrm{t}}\mathbf{S}^{\mathrm{t}}\mathbf{x}(n), \end{aligned} \tag{7}$$

where

$$\mathbf{x}(n) = [x(n), x(n-1), \ldots, x(n-L+1)]^{\mathrm{t}} \tag{8}$$

denotes the L-by-1 tap-input vector of the filter $\mathbf{h}$. Note that the reference signal vector $\mathbf{x}_{\mathbf{w}}(n)$ of filter $\mathbf{w}$ is a filtered version of $\mathbf{x}(n)$ by filter $\mathbf{s}$:

$$\mathbf{x}_{\mathbf{w}}(n)=\mathbf{S}^{\mathrm{t}}\mathbf{x}(n), \tag{9}$$

giving rise to the name filtered-x transversal filtering for the scheme in Figure 2.

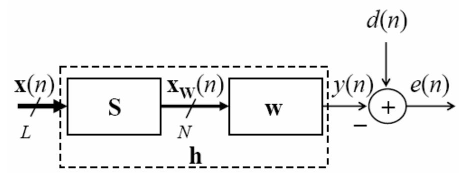


Figure 2: Filtered-x Wiener filtering using vector notation.

In the mean-square-error (MSE) sense, the vector $\mathbf{w}$ is chosen to minimize the following cost function:

$$\begin{aligned} J(\mathbf{w}) &= \mathrm{E}\{e^2(n)\} \\ &= \sigma_d^2 - 2\mathbf{w}^{\mathrm{t}}\mathbf{S}^{\mathrm{t}}\mathbf{p} + \mathbf{w}^{\mathrm{t}}\mathbf{S}^{\mathrm{t}}\mathbf{R}\mathbf{S}\mathbf{w}, \end{aligned} \tag{10}$$

where $\mathrm{E}\{\bullet\}$ stands for expected value, σ_d^2 is the variance of $d(n)$, $\mathbf{R}$ is the L-by-L autocorrelation matrix of $\mathbf{x}(n)$, and $\mathbf{p}$ is the L-by-1 cross-correlation vector between $\mathbf{x}(n)$ and $d(n)$. The optimum solution is given by

$$\mathbf{w}_{\mathrm{opt}} = (\mathbf{S}^{\mathrm{t}}\mathbf{R}\mathbf{S})^{-1}\mathbf{S}^{\mathrm{t}}\mathbf{p}. \tag{11}$$

Substituting (11) in (10) leads to the minimum mean-square error:

$$\begin{aligned} J_{\min} &= \sigma_d^2 - \mathbf{p}^{\mathrm{t}}\mathbf{S}(\mathbf{S}^{\mathrm{t}}\mathbf{R}\mathbf{S})^{-1}\mathbf{S}^{\mathrm{t}}\mathbf{p} \\ &= \sigma_d^2 - \mathbf{p}^{\mathrm{t}}\mathbf{S}\mathbf{w}_{\mathrm{opt}}. \end{aligned} \tag{12}$$

Observe that the N-by-N matrix $\mathbf{S}^{\mathrm{t}}\mathbf{R}\mathbf{S}$ in (10), (11) and (12) corresponds to the autocorrelation matrix of $\mathbf{x}_{\mathbf{w}}(n)$ (Figure 2), and the N-by-1 vector $\mathbf{S}^{\mathrm{t}}\mathbf{p}$ corresponds to the cross-correlation vector between $\mathbf{x}_{\mathbf{w}}(n)$ and $d(n)$:

$$\mathrm{E}\{\mathbf{x}_{\mathbf{w}}(n)\mathbf{x}^{\mathrm{t}}_{\mathbf{w}}(n)\}=\mathrm{E}\{\mathbf{S}^{\mathrm{t}}\mathbf{x}(n)\mathbf{x}^{\mathrm{t}}(n)\mathbf{S}\}=\mathbf{S}^{\mathrm{t}}\mathbf{R}\mathbf{S} \tag{13}$$

and

$$\mathrm{E}\{\mathbf{x}_{\mathbf{w}}(n)d(n)\}=\mathrm{E}\{\mathbf{S}^{\mathrm{t}}\mathbf{x}(n)d(n)\}=\mathbf{S}^{\mathrm{t}}\mathbf{p}. \tag{14}$$

In the next section, the FXLMS algorithm for updating the filter $\mathbf{w}$ in adaptive systems is derived using the structure in Figure 2.

3. FILTERED-X LMS ALGORITHMS

Like the standard LMS algorithm, the FXLMS algorithm can be derived using the steepest-descent optimization method [6]. The steepest-descent algorithm seeks the minimum of the error-performance surface in (10) by adjusting the weight vector in the direction opposite to its gradient vector. Thus,

$$\begin{aligned} \mathbf{w}(n) &= \mathbf{w}(n-1) - \tfrac{1}{2}\mu\,\nabla J[\mathbf{w}(n-1)] \\ &= \mathbf{w}(n-1) + \mu[\mathbf{S}^{\mathrm{t}}\mathbf{p} - \mathbf{S}^{\mathrm{t}}\mathbf{R}\mathbf{S}\mathbf{w}(n-1)], \end{aligned} \tag{15}$$

where $\nabla J[\mathbf{w}(n-1)]$ is the gradient of $J(\mathbf{w})$ in (10) at iteration n-1 and μ is the step-size parameter.

The algorithm described by (15) can be viewed as the deterministic filtered-x gradient-descent algorithm, since it requires prior knowledge of the deterministic quantities $\mathbf{p}$ and $\mathbf{R}$. These quantities are not available and the simplest strategy is to use instantaneous estimates given by:

$$\hat{\mathbf{p}}(n) = \mathbf{x}(n)d(n) \tag{16}$$

and

$$\hat{\mathbf{R}}(n) = \mathbf{x}(n)\mathbf{x}^{\mathrm{t}}(n) . \quad (17)$$

Substituting (16) and (17) in (15) leads to the stochastic gradient recursion

$$\begin{aligned}\mathbf{w}(n) &= \mathbf{w}(n-1) + \mu[\mathbf{S}^{\mathrm{t}}\mathbf{x}(n)d(n) - \mathbf{S}^{\mathrm{t}}\mathbf{x}(n)\mathbf{x}^{\mathrm{t}}(n)\mathbf{S}\mathbf{w}(n-1)] \\ &= \mathbf{w}(n-1) + \mu\,\mathbf{S}^{\mathrm{t}}\mathbf{x}(n)[d(n) - \mathbf{x}^{\mathrm{t}}(n)\mathbf{S}\mathbf{w}(n-1)] \\ &= \mathbf{w}(n-1) + \mu\,\mathbf{S}^{\mathrm{t}}\mathbf{x}(n)e(n) , \quad (18)\end{aligned}$$

where

$$\begin{aligned}e(n) &= d(n) - \mathbf{x}^{\mathrm{t}}(n)\mathbf{S}\mathbf{w}(n-1) \\ &= d(n) - y(n) \quad (19)\end{aligned}$$

is the a priori estimation error.

Equations (18) and (19) describe the well-known FXLMS algorithm. Figure 3 shows its block diagram representation using the structure in Figure 2. Observe that it can be viewed as the standard LMS algorithm, having $\mathbf{x}_{\mathbf{w}}(n)=\mathbf{S}^{\mathrm{t}}\mathbf{x}(n)$ as the reference signal vector for updating the filter. This is the reason for preceding its designation with the word "filtered-x".

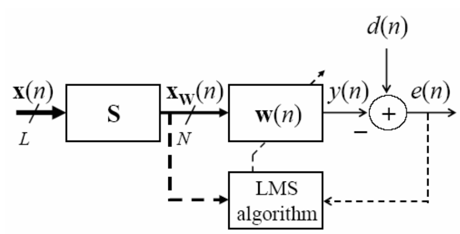


Figure 3: Filtered-x LMS algorithm.

In some applications of the FXLMS algorithm, such as active control of sound and vibration, the filter S in Figure 1 is unknown and must be estimated. Let

$$\hat{\mathbf{s}} = [\hat{s}_0, \hat{s}_1, \ldots, \hat{s}_{\hat{M}-1}]^{\mathrm{t}} \quad (20)$$

denote the $\hat{M}$-by-1 tap-weight vector of such an estimate of order ($\hat{M}-1$). For generalization, the order of the filter S is not necessarily equal to the order of its estimate ($\hat{M} \neq M$). The $\hat{L}$-by-N ($\hat{L} = \hat{M} + N - 1$) estimated matrix $\hat{\mathbf{S}}$ built from $\hat{\mathbf{s}}$ has the same form as in (6), i.e., a non-symmetric Toeplitz matrix having the vector $[\hat{\mathbf{s}}^{\mathrm{t}}, \mathbf{0}^{\mathrm{t}}{}_{\hat{L}-\hat{M}}]^{\mathrm{t}}$ as its first column and the vector $[\hat{s}_0, \mathbf{0}^{\mathrm{t}}{}_{N-1}]$ as its first row.

Using the estimation of $\mathbf{x}_{\mathbf{w}}(n)$ in (18), the recursive weight update equation becomes

$$\mathbf{w}(n) = \mathbf{w}(n-1) + \mu\,\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{x}_{\hat{L}}(n)e(n) , \quad (21)$$

where

$$\mathbf{x}_{\hat{L}}(n) = [x(n), x(n-1), \ldots, x(n-\hat{L}+1)]^{\mathrm{t}} \quad (22)$$

denotes the $\hat{L}$-by-1 input data vector at time n, and

$$\begin{aligned}e(n) &= d(n) + z(n) - y(n) \\ &= d(n) + z(n) - \mathbf{x}^{\mathrm{t}}(n)\mathbf{S}\mathbf{w}(n-1) . \quad (23)\end{aligned}$$

In (23), a measurement noise $z(n)$, which is assumed to be white and independent of any other signal, has also been considered. Figure 4 shows a more appropriate block diagram representation for the FXLMS algorithm in this case.

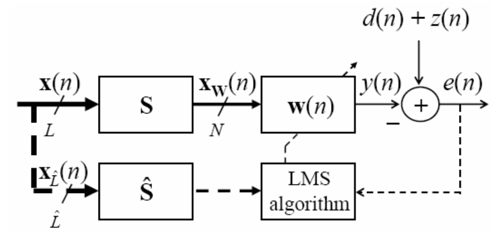


Figure 4: FXLMS algorithm employing an estimate of S.

A statistical analysis of the filtered-x LMS algorithm in (21) is presented in the following.

4. STATISTICAL ANALYSIS

4.1. Mean Weight Behavior

Using (23), the expected value of the weight adaptation equation in (21) leads to the recursion:

$$\begin{aligned}\mathrm{E}\{\mathbf{w}(n)\} &= \mathrm{E}\{\mathbf{w}(n-1)\} + \mu\,\hat{\mathbf{S}}^{\mathrm{t}}\mathrm{E}\{\mathbf{x}_{\hat{L}}(n)d(n)\} + \\ &\quad - \mu\,\mathrm{E}\{\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{x}_{\hat{L}}(n)\mathbf{x}^{\mathrm{t}}(n)\mathbf{S}\mathbf{w}(n-1)\} . \quad (24)\end{aligned}$$

Neglecting the statistical dependence between the tap-weight and signal vectors [4], the last expectation in the above equation can be approximated by

$$\begin{aligned}\mathrm{E}\{\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{x}_{\hat{L}}(n)\mathbf{x}^{\mathrm{t}}(n)\mathbf{S}\mathbf{w}(n-1)\} &\approx \mathrm{E}\{\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{x}_{\hat{L}}(n)\mathbf{x}^{\mathrm{t}}(n)\mathbf{S}\}\mathrm{E}\{\mathbf{w}(n-1)\} \\ &= \hat{\mathbf{S}}^{\mathrm{t}}\mathrm{E}\{\mathbf{x}_{\hat{L}}(n)\mathbf{x}^{\mathrm{t}}(n)\}\mathbf{S}\mathrm{E}\{\mathbf{w}(n-1)\} . \quad (25)\end{aligned}$$

Using (25) in (24) yields

$$\begin{aligned}\mathrm{E}\{\mathbf{w}(n)\} &= \mathrm{E}\{\mathbf{w}(n-1)\} + \mu\,\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{p}_{\hat{L}} - \mu\,\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{R}_{\hat{L}xL}\mathbf{S}\mathrm{E}\{\mathbf{w}(n-1)\} \\ &= (I - \mu\,\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{R}_{\hat{L}xL}\mathbf{S})\,\mathrm{E}\{\mathbf{w}(n-1)\} + \mu\,\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{p}_{\hat{L}} , \quad (26)\end{aligned}$$

where

$$\mathbf{p}_{\hat{L}} = \mathrm{E}\{\mathbf{x}_{\hat{L}}(n)d(n)\} \quad (27)$$

is the $\hat{L}$-by-1 cross-correlation vector between $\mathbf{x}_{\hat{L}}(n)$ and $d(n)$, and

$$\mathbf{R}_{\hat{L}xL} = \mathrm{E}\{\mathbf{x}_{\hat{L}}(n)\mathbf{x}^{\mathrm{t}}(n)\} \quad (28)$$

is the $\hat{L}$-by-L autocorrelation matrix of the input signal $x(n)$.

Equation (26) is a deterministic recursion for the mean weight behavior of the FXLMS algorithm. Assuming that the algorithm converges as $n \to \infty$ and defining

$$\lim_{n\to\infty}\mathrm{E}\{\mathbf{w}(n)\} = \mathbf{w}_{\infty}, \quad (29)$$

the steady-state mean weight vector can be easily derived. Substituting $\mathbf{w}_{\infty}$ for $\mathrm{E}\{\mathbf{w}(n)\}$ and $\mathrm{E}\{\mathbf{w}(n-1)\}$ in (26) yields

$$\mathbf{w}_{\infty} = (\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{R}_{\hat{L}xL}\mathbf{S})^{-1}\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{p}_{\hat{L}} . \quad (30)$$

In the particular case of perfect estimation of the filter S response, $\hat{L}=L$, $\hat{\mathbf{S}}$=$\mathbf{S}$, and (30) becomes

$$\mathbf{w}_\infty=(\mathbf{S}^{\mathrm{t}}\mathbf{R}\mathbf{S})^{-1}\mathbf{S}^{\mathrm{t}}\mathbf{p}\,, \qquad (31)$$

which corresponds to the optimum solution in (11).

4.2. Mean Square Error Behavior

Squaring the estimation error in (23) and taking the expected value neglecting the statistical dependences of $d(n)$ and $\mathbf{x}(n)$ with $\mathbf{w}(n–1)$ yields:

$$\mathrm{E}\{e^2(n)\}=\sigma_d^2+\sigma_z^2-2\mathrm{E}\{\mathbf{w}^{\mathrm{t}}(n-1)\}\,\mathbf{S}^{\mathrm{t}}\mathbf{p}+ \\ +tr[\mathbf{S}^{\mathrm{t}}\mathbf{R}\mathbf{S}\mathrm{E}\{\mathbf{w}(n-1)\mathbf{w}^{\mathrm{t}}(n-1)\}]\,, \qquad (32)$$

where σ_z^2 denotes the variance of $z(n)$.

Evaluation of (32) requires a recursion for the tap-weight autocorrelation matrix

$$\mathbf{K}(n–1)=\mathrm{E}\{\mathbf{w}(n–1)\mathbf{w}^{\mathrm{t}}(n–1)\}. \qquad (33)$$

Post-multiplying (21) by its transpose and taking the expected value yields

$$\mathbf{K}(n-1)=\mathbf{K}(n-2)+ \\ +\mu\mathrm{E}\{\mathbf{w}(n-2)\}\mathbf{p}_{\hat{L}}^{\mathrm{t}}\hat{\mathbf{S}}-\mu\mathbf{K}(n-2)\mathbf{S}^{\mathrm{t}}\mathbf{R}_{\hat{L}xL}^{\mathrm{t}}\hat{\mathbf{S}}+ \\ +\mu\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{p}_{\hat{L}}\mathrm{E}\{\mathbf{w}^{\mathrm{t}}(n-2)\}-\mu\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{R}_{\hat{L}xL}\mathbf{S}\mathbf{K}(n-2)+ \\ +\mu^2\sigma_z^2\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{R}_{\hat{L}x\hat{L}}\hat{\mathbf{S}}+ \\ +\mu^2\mathrm{E}\{\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{x}_{\hat{L}}(n-1)d(n-1)\mathbf{x}_{\hat{L}}^{\mathrm{t}}(n-1)\hat{\mathbf{S}}d(n-1)\}+ \\ -2\mu^2\mathrm{E}\{\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{x}_{\hat{L}}(n-1)d(n-1)\mathbf{x}_{\hat{L}}^{\mathrm{t}}(n-1)\hat{\mathbf{S}}\mathbf{x}^{\mathrm{t}}(n-1)\mathbf{S}\mathbf{w}(n-2)\}+ \\ +\mu^2\mathrm{E}\{\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{x}_{\hat{L}}(n-1)\mathbf{w}^{\mathrm{t}}(n-2)\mathbf{S}^{\mathrm{t}}\mathbf{x}(n-1) \\ \mathbf{x}_{\hat{L}}^{\mathrm{t}}(n-1)\hat{\mathbf{S}}\mathbf{x}^{\mathrm{t}}(n-1)\mathbf{S}\mathbf{w}(n-2)\}\,, \qquad (34)$$

where the effects of the statistical dependence between weights and signals have been neglected again. Assuming that $\mathbf{x}(n)$ and $d(n)$ are jointly Gaussian of zero mean, the last three expected values in (34) can be evaluated using the moment factoring theorem [7], [8]. Neglecting again the statistical dependence of $d(n)$ and $\mathbf{x}(n)$ with $\mathbf{w}(n–1)$, straightforward calculation leads to:

$$\mathrm{E}\{\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{x}_{\hat{L}}(n-1)d(n-1)\mathbf{x}_{\hat{L}}^{\mathrm{t}}(n-1)\hat{\mathbf{S}}d(n-1)\}= \\ 2\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{p}_{\hat{L}}\mathbf{p}_{\hat{L}}^{\mathrm{t}}\hat{\mathbf{S}}+\sigma_d^2\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{R}_{\hat{L}x\hat{L}}\hat{\mathbf{S}}\,, \qquad (35)$$

$$\mathrm{E}\{\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{x}_{\hat{L}}(n-1)d(n-1)\mathbf{x}_{\hat{L}}^{\mathrm{t}}(n-1)\hat{\mathbf{S}}\mathbf{x}^{\mathrm{t}}(n-1)\mathbf{S}\mathbf{w}(n-2)\}= \\ \hat{\mathbf{S}}^{\mathrm{t}}\mathbf{p}_{\hat{L}}\mathrm{E}\{\mathbf{w}^{\mathrm{t}}(n-2)\}\mathbf{S}^{\mathrm{t}}\mathbf{R}_{\hat{L}xL}^{\mathrm{t}}\hat{\mathbf{S}}+\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{R}_{\hat{L}xL}\mathbf{S}\mathrm{E}\{\mathbf{w}(n-2)\}\mathbf{p}_{\hat{L}}^{\mathrm{t}}\hat{\mathbf{S}} \\ +\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{R}_{\hat{L}x\hat{L}}\hat{\mathbf{S}}\mathbf{p}^{\mathrm{t}}\mathbf{S}\mathrm{E}\{\mathbf{w}(n-2)\} \qquad (36)$$

and

$$\mathrm{E}\{\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{x}_{\hat{L}}(n-1)\mathbf{w}^{\mathrm{t}}(n-2)\mathbf{S}^{\mathrm{t}}\mathbf{x}(n-1) \\ \mathbf{x}_{\hat{L}}^{\mathrm{t}}(n-1)\hat{\mathbf{S}}\mathbf{x}^{\mathrm{t}}(n-1)\mathbf{S}\mathbf{w}(n-2)\}= \\ 2\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{R}_{\hat{L}xL}\mathbf{S}\mathbf{K}(n-2)\mathbf{S}^{\mathrm{t}}\mathbf{R}_{\hat{L}xL}^{\mathrm{t}}\hat{\mathbf{S}}+ \\ +\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{R}_{\hat{L}x\hat{L}}\hat{\mathbf{S}}\,tr[\mathbf{S}^{\mathrm{t}}\mathbf{R}\mathbf{S}\mathbf{K}(n-2)]\,, \qquad (37)$$

where

$$\mathbf{R}_{\hat{L}x\hat{L}}=\mathrm{E}\{\mathbf{x}_{\hat{L}}(n-1)\mathbf{x}_{\hat{L}}^{\mathrm{t}}(n-1)\} \qquad (38)$$

is the $\hat{L}$ - by - $\hat{L}$ autocorrelation matrix of $\mathbf{x}_{\hat{L}}(n-1)$.

Thus, (34) is a recursion for the weight correlation matrix and (32) for the mean square error behavior.

5. THE ACTIVE NOISE CONTROL CASE

The FXLMS algorithm is typically employed in adaptive systems in which the output of the adaptive filter is linearly filtered (Figure 5). This includes, for example, systems used for active control of sound and vibration.

Now, the estimated gradient at iteration n is given by

$$\hat{\nabla}J_{WS}[\mathbf{w}(n)]=-\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{x}_{\hat{L}}(n)[d(n)+z(n)]+ \\ +\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{x}_{\hat{L}}(n)\sum_{i=0}^{M-1}s_i\mathbf{x}_N^{\mathrm{t}}(n-i)\mathbf{w}(n-1-i)\,, \qquad (39)$$

while in Figure 4

$$\hat{\nabla}J_{SW}[\mathbf{w}(n)]=-\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{x}_{\hat{L}}(n)[d(n)+z(n)]+ \\ +\hat{\mathbf{S}}^{\mathrm{t}}\mathbf{x}_{\hat{L}}(n)\sum_{i=0}^{M-1}s_i\mathbf{x}_N^{\mathrm{t}}(n-i)\mathbf{w}(n-1)\,, \qquad (40)$$

where

$$\mathbf{x}_N(n)=[x(n),x(n-1),\ldots,x(n-N+1)]^{\mathrm{t}}\,. \qquad (41)$$

Comparing (39) with (40), we observe that the only difference between the two equations is in the last term. In (39), all M last weight vector $\mathbf{w}(n–1–i)$, for i=0, 1, …, M–1, have been taken into account in the sum, while in (40) only $\mathbf{w}(n–1)$ is considered. This is a consequence of exchanging the order of the filters **s** and $\mathbf{w}(n)$ in Figure 4.

In deriving the expressions in the last section, we assumed that the order of filters $\mathbf{w}(n)$ and **s** could be exchanged in Figure 5. Thus, **s** is placed before $\mathbf{w}(n)$ in the analysis. Since $\mathbf{w}(n)$ is time-varying, this order exchange cannot be done in general, as it has been observed above. However, it will closely approximate the actual system behavior if the time variations of $\mathbf{w}(n)$ take place with a time constant longer than the memory length of filter **s**. Such approximation has been successfully used in [6] for deriving the FXLMS algorithm.

In the next section, simulations will show the accuracy of the new FXLMS algorithm model for active control of sound and vibration.

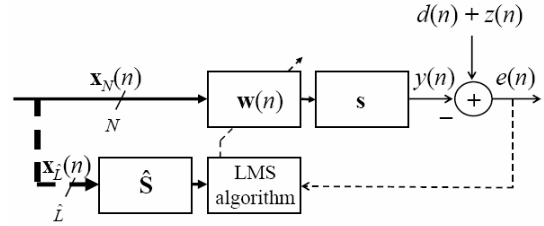
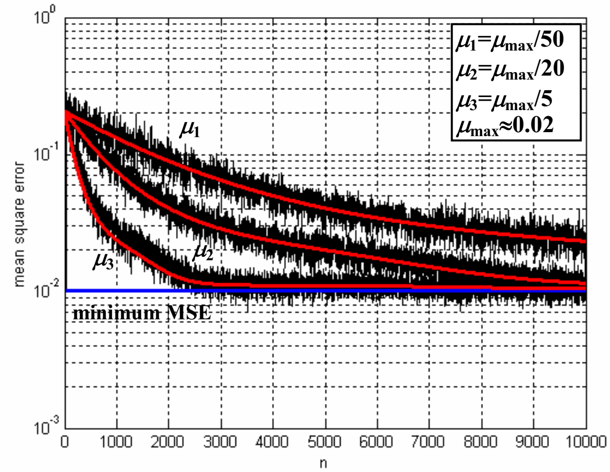


Figure 5: FXLMS algorithm for ANC and AVC systems.

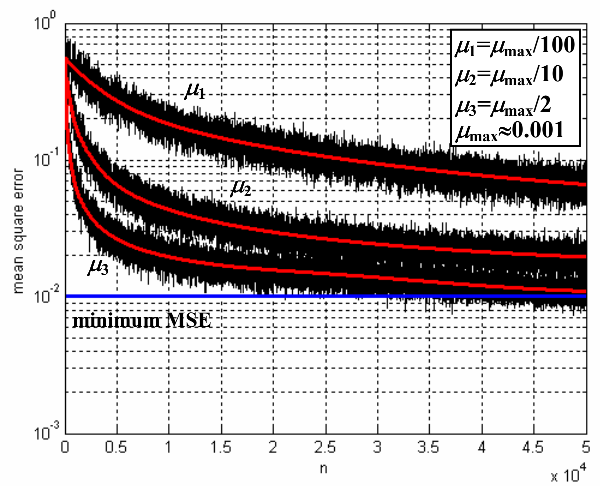
6. SIMULATION RESULTS

To verify the accuracy of the proposed analytical model, consider an ANC system (Figure 5) with the following characteristics. The primary, secondary (**s**) and estimated secondary ($\hat{\mathbf{s}}$) path are bandpass (300 Hz – 5 kHz) FIR filters of order J=68, M=19 and $\hat{M} = 15$, respectively. The controller $\mathbf{w}(n)$ is an adaptive filter of order N=49. The input signal $x(n)$ are samples of white and colored Gaussian processes with σ_x^2=1, and σ_z^2=10^{-2}. Different step size values were used, which are large, moderate and small when compared to the convergence limit $\mu_{\max}$ (determined by simulation). Notice that the orders of the primary path and of the adaptive filter do not match. The same thing happens with the orders of $\mathbf{s}$ and $\hat{\mathbf{s}}$. These order mismatches were deliberately chosen to illustrate the generality of the analytical model.

Figure 6 compares the simulated (averaged over 100 independent runs) and the theoretical MSE behaviors for white and colored Gaussian noises. It can be verified that the analytical model accurately predicts the convergence behavior of the FXLMS algorithm even for relatively moderate and large step-sizes.



(a): white Gaussian noise



(b): colored Gaussian noise

Figure 6: Mean square error behavior.

7. SUMMARY

This paper has presented a simple and efficient analytical model for the convergence behavior of the filtered-x LMS algorithm with Gaussian input data. Deterministic recursions have been derived for the mean weight vector and the mean square error.

Although the statistical analysis departs from a scheme in which the controller is placed after the secondary path filter, it accurately predicts the FXLMS algorithm behavior for active control of sound and vibration. Simulations have confirmed this and it is theoretically based on the fact that the time variations of the adaptive filter tap-weights take place with a time constant longer than the memory length of the secondary path filter.

As a final remark, the model can also be easily extended to the case of complex parameters.

8. REFERENCES

[1] S.M. Kuo and D.R. Morgan, *Active Noise Control System: Algorithms and DSP Implementations*, Wiley, New York, 1996.

[2] S.J. Elliot and P.A. Nelson, "Active Noise Control," IEEE Signal Processing Magazine, pp. 12-35, October 1993.

[3] O.J. Tobias, J.C.M. Bermudez, and N.J. Bershard, "Mean Weight Behavior of the Filtered-x LMS Algorithm," IEEE Transactions on Signal Processing, vol. 48, no. 4, pp. 1061-1075, April 2000.

[4] O.J. Tobias, J.C.M. Bermudez, R. Seara, and N.J. Bershard, "An Improved Model for the Second Moment of the Filtered-x LMS Algorithm," Proceedings of the IEEE 2000 Adaptive Systems for Signal Processing, Communications, and Control Symposium, pp. 337-341, Lake Louise, Alberta-Canada, October 2000.

[5] M.H. Costa, J.C.M. Bermudez, and N.J. Bershard, "Stochastic Analysis of the Filtered-x LMS Algorithm in Systems with Nonlinear Secondary Paths," IEEE Transactions on Signal Processing, vol. 50, no. 6, pp. 1327-1342, June 2002.

[6] B. Widrow and S.D. Stearns, *Adaptive Signal Processing*, Prentice-Hall, New Jersey, 1985.

[7] S. Haykin, *Adaptive Filter Theory*, 4th edition, Prentice-Hall, New Jersey, 2002.

[8] A. Papoulis, *Probability, Randow Variables and Stochastic Processes*, 3rd edition, McGraw-Hill, New York, 1991.