Minimum Eigenvalue Based 3-D AR Model Order Selection

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ABSTRACT

This paper deals with the problem of three-dimensional AutoRegressive (3-D AR) model order estimation. Especially, we develop a practical algorithm to estimate the 3-D AR order (p_1, p_2, p_3) corresponding to the Quarter-Space (QS) region of support. The proposed method is derived from the minimum description length (MDL) criterion and uses the minimum eigenvalue of the covariance matrix of the underlying 3-D Gaussian process. Numerical simulations are presented to illustrate the performances of the new proposed algorithm and compare it with a recently developed method.

1. INTRODUCTION

In multidimensional (m-D) system and signal analysis, it is often assumed that the data are modeled by m-D autoregressive (AR) models [1]. It is well known that any model-based approach for data representation and processing involves two important stages, viz., i) determination of the model order, and ii) identification of the model parameters.

Dealing with the model order, we note that in the 1 and 2 dimensional cases, the existing order determination methods can be divided into two classes, namely information criterion methods and linear algebraic methods. The first class covers the AIC criterion [2] developed by Akaike, the MDL (minimum description length) criterion of Rissanen and Schwartz [3], [4], the φ criterion of Hannan [5], the φ_β criterion [6] and their 2-D extensions [7]. The second class covers the Corner method [8], the singular value decomposition (SVD) based on the rank determination [9], [10] and finally the minimum eigenvalue (MEV) method [11], [12]. However, in the 3-D case, a few papers are available and they often deal with the parameters estimation problem assuming that the model order is known [1], [13].

In this paper we focus on the estimation of the 3-D AR model order corresponding to the quarter-space (QS) region of support by using the so-called minimum eigenvalue method asymptotically equivalent to the MDL criterion

2. PROBLEM FORMULATION

The 3-D AR process considered here is described by the following difference equation

$$y(m,n,t) = -\sum_{\substack{k_1=0 \\ (k_1,k_2,k_3)\neq(0,0,0)}}^{p_1} \sum_{k_2=0}^{p_2} \sum_{k_3=0}^{p_3} a_{k_1,k_2,k_3}\, y(m-k_1, n-k_2, t-k_3) + e(m,n,t) \tag{1}$$

where $\{a_{k_1,k_2,k_3}\}$ are the transversal AR parameters coefficients, $\{e(m,n,t)\}$ is a zero mean, white noise, 3-D Gaussian random process, with variance σ_e^2. The problem is to estimate the unknown order (p_1, p_2, p_3).

Recently, we have proposed an algebraic method for Gaussian 3-D QS AR model using a rank test procedure (RTP) [14]. Good results are obtained by this method comparing with the informational criterion ones. However, the remaining problem with the RTP method is the need of a subjective intervention by the user to choose a threshold for determining the rank of a given matrix. So in this paper we present an hybrid between algebraic and informational method which can be automated. Indeed, the proposed model order selection is based on the so-called Minimum EigenValue (MEV) method and uses the information from the covariance matrix as in [15]. Moreover, this MEV method derived from the information MDL criterion [11] is based on the minimum eigenvalue of a covariance matrix computed from the observed data and is shown to have improved accuracy compared with the classical criterion. This work is the sequel to our former work [7], [10], and [14] where we have investigated the problem of AR model order using a RTP based algebraic method.

3. PROPOSED APPROACH

Considering the model equation (1) and assume that we have access to a data of size $M \times N \times T$. Let $Q_1 = M - p_1$, $Q_2 = N - p_2$, and $Q_3 = T - p_3$. Omitting the initial conditions, we arrange the $Q_1 \times Q_2 \times Q_3$ significant data in a compact matrix form as: $Y\theta = e(\mathrm{M,N,T})$

Where Y is the $Q_1Q_2Q_3 \times (p_1+1)(p_2+1)(p_3+1)$ data matrix and $e(M,N,T)$ is the $Q_1Q_2Q_3 \times 1$ vector containing the corresponding values of the input process generator. The $(p_1+1)(p_2+1)(p_3+1) \times 1$ vector parameter θ is given by:

$\theta = \left[\theta_0; \theta_1; \cdots; \theta_{p_1}\right]^T$;

where $\theta_{k_1} = \left[\theta_{k_1,0}; \theta_{k_1,1}; \cdots; \theta_{k_1,p_2}\right]^T$

and $\theta_{k_1,k_2} = \left[a_{k_1,k_2,0}; a_{k_1,k_2,1}; \cdots; a_{k_1,k_2,p_3}\right]^T$

Let $R = Y^T Y$ be the covariance matrix. The MDL criterion [3] is equal to the sum of the log-likelihood function of the maximum likelihood estimator and a function that penalizes the use of large number of model parameters. The MDL criterion is equivalent to:

$$MDL(k_1,k_2,k_3) = \frac{MNT}{2}\log(2\pi\sigma^2) + \frac{\theta^T R\theta}{2\sigma^2} + \frac{(k_1+1)(k_2+1)(k_3+1)}{2}\log(MNT) \tag{2}$$

We will now give the following lemma which will be used to develop our numerical algorithm.

Lemma For a fixed order (k_1,k_2,k_3) and constraining θ to have unit Euclidean norm, the choice of θ that minimizes criterion (2) is found to be the eigenvector associated with the minimum eigenvalue $\lambda_{\min}$ of the covariance matrix R.

Thus, for $\theta = \theta_{\min}$, we have

$$\frac{1}{\mathrm{MNT}}\theta_{\min}^{\mathrm{T}}\mathrm{R}\,\theta_{\min} = \frac{1}{\mathrm{MNT}}\mathrm{e}_1^{\mathrm{T}}\mathrm{e}_1 \approx \sigma^2,$$

thereby
$$\frac{1}{MNT}\lambda_{\min} = \sigma^2 \tag{3}$$

Substituting and dropping all terms that do not depend on k_1, k_2, k_3 or θ we then obtain:

$$MDL(k_1,k_2,k_3) = \frac{MNT}{2}\log(\lambda_{\min}) + \frac{(k_1+1)(k_2+1)(k_3+1)}{2}\log(MNT) \tag{4}$$

The parameter vector θ in the argument of the MDL has been dropped, since the explicit θ dependence is suppressed (it has been incorporated into the term $\lambda_{\min}$).

Multiplying both sides of (4) by $\frac{2}{MNT}$ and combining terms we obtain

$$\frac{2}{MNT} MDL(k_1, k_2, k_3) = \log(\lambda_{\min}) + \log(MNT^{\frac{(k_1+1)(k_2+1)(k_3+1)}{MNT}}) \quad (5)$$

Since $\log(.)$ is a monotonically increasing function we can form a different criterion that contains the same information as the MDL criterion. Consequently, the following MEV criterion has its minimum at the same location as the MDL:

$$MEV(k_1,k_2,k_3) = \lambda_{\min} (MNT)^{\frac{(k_1+1)(k_2+1)(k_3+1)}{MNT}} \quad (6)$$

*Remark***:** From the above expression, it can be seen that when large sample sizes are used (i.e. $MNT \to +\infty$), the term $(MNT)^{\frac{(k_1+1)(k_2+1)(k_3+1)}{MNT}}$ is approximately one, this means that the new criterion asymptotically tends to the minimum eigenvalue.

If we increase the length of θ by increasing k_1 and/or k_2 and/or k_3, the dimension of the covariance matrix R also increases. If k_1, k_2 and k_3 are chosen such $k_1 \geq p_1$, $k_2 \geq p_2$ and $k_3 \geq p_3$ then $\lambda_{\min}$ will be small compared with the cases ($k_1 < p_1$, or $k_2 < p_2$ or $k_3 < p_3$). Indeed, if $k_1 < p_1$, or $k_2 < p_2$ or $k_3 < p_3$, the prediction error is significantly large because the model does not have a sufficient number of parameters to fit the predicted signal to the observed data. Consequently, the determination procedure of the optimal model order consists of computing $MEV(k_1,k_2,k_3)$ for different orders and selecting the triplet which correspond to the corner where $\lambda_{\min}$ drops very quickly. In practice several corners can be found instead of a single corner. In order to select the correct corners in the three-dimensional matrix MEV, we construct the three arrays as follows :

- **row ratio array** by dividing each horizontal plan of MEV array by the previous one; i.e.
$RRA(i, j, m) = MEV(i, j, m) / MEV(i-1, j, m)$;
$i = 2, \cdots k_1$, $j = 1, \cdots k_2$, $m = 1, \cdots k_3$.

- **column ratio array** by dividing each vertical plan of MEV array by the previous one;
$CRA(i, j, m) = MEV(i, j, m) / MEV(i, j-1, m)$;
$i = 1, \cdots k_1$, $j = 2, \cdots k_2$, $m = 1, \cdots k_3$.

- **layer ratio array** by dividing each layer of MEV array by the previous one;
$LRA(i, j, m) = MEV(i, j, m) / MEV(i, j, m-1)$;
$i = 1, \cdots k_1$, $j = 1, \cdots k_2$, $m = 2, \cdots k_3$.

An estimate of p_1 is set equal to the row number that contains the minimum value in the row ratio array.

The number of column which haves the minimum value in column ratio array will be the estimate of p_2.

Finally, the number of layers which have the minimum value in layer ratio array will be the estimate of p_3.

4. SIMULATION RESULTS

Here, we give a numerical example to provide verification of the theoretical results. For this purpose, we generate a $64 \times 64 \times 64$ 3-D AR process of fixed order and we compare the MEV based method with the four AIC, MDL, φ_β, and RTP criteria. The model in the example is a 3-D AR $(2,2,2)$ with parameters described in the top of the following page

For 100 runs Monte-Carlo the empirical probability of the estimated order using the five methods are depicted in the table 1. The value of $\beta = 0.2$ and the threshold used to evaluate the rank of matrices in RTP method is $\delta = 0.995$.

The order $k_i, i = 1, 2, 3$ are assumed to be in $1 \le k_i \le 5$.

$$\begin{bmatrix} a(0,0,0) = 1 & a(0,1,0) = -1.2700 & a(0,2,0) = 0.3900 \\ a(1,0,0) = -1.300 & a(1,1,0) = 1.6510 & a(1,2,0) = -0.5070 \\ a(2,0,0) = 0.3825 & a(2,1,0) = -0.4858 & a(2,2,0) = 0.1492 \end{bmatrix}$$

$$\begin{bmatrix} a(0,0,1) = -1.1300 & a(0,1,1) = 1.4251 & a(0,2,1) = -0.4407 \\ a(1,0,1) = 1.4696 & a(1,1,1) = -1.8656 & a(1,2,1) = 0.5729 \\ a(2,0,1) = -0.4322 & a(2,1,1) = 0.5489 & a(2,2,1) = -0.1686 \end{bmatrix}$$

$$\begin{bmatrix} a(0,0,2) = 0.3120 & a(0,1,2) = -0.3962 & a(0,2,2) = 0.1217 \\ a(1,0,2) = -0.4056 & a(1,1,2) = 0.5151 & a(1,2,2) = -0.1582 \\ a(2,0,2) = 0.1193 & a(2,1,2) = 0.1516 & a(2,2,2) = -0.0465 \end{bmatrix}$$

TABLE 1: EMPIRICAL PROBABILITY OF ESTIMATED ORDERS BY THE AIC, MDL, φ_β, RTP, AND MEV FIVE METHODS FOR 3-D AR(2,2,2)

Order	(2,2,2)	(2,1,2)	(2,2,3)	(2,3,2)	(2,3,3)	(3,2,3)	(3,3,2)
AIC %	24	9	20	22	25	0	0
MDL %	60	15	15	0	10	0	0
φ_β %	90	6	4	0	0	0	0
RTP %	84	16	0	0	0	0	0
MEV %	98	2	0	0	0	0	0

5. COMMENTS

The MEV approach developed in this paper shows a serious improvement against the AIC, MDL, $\varphi\beta$, RTP four other methods. On the other hand, the recently developed RTP method depends on a threshold under which we can suppose that singular values are or not null, this makes it some what subjective. Moreover the three objective informational criteria AIC, MDL, and φ_β are based on the estimation of model parameters and the prediction error variance. Before selecting the model order, the criteria have to be computed for several order and select one which minimizes the criteria hence this makes them time consuming. The use of the three row ratio, column ratio, and layers ratio arrays needs an over extra time and places memory but the cost is the best result quality.

6. CONCLUSION

In this paper, we have addressed the model order selection problem of 3-D causal and shift invariant AR model with quarter-space region of support. A new minimum eigenvalue method is derived from the minimum description length criterion. Comparing minimum eigenvalue method with the rank test recently proposed method and with the standard informational AIC, MDL, and φ_β criteria, the result shows well improvement

7. REFERENCES

[1] V. V. Digalakis, V. K. Ingle, and D. G. Manolakis, "Three-dimensional linear prediction and its application to digital angiography," *Multidimensional Systems and Signal Processing*, vol. 4, pp. 307-329, 1993.

[2] H. Akaike, "A new look at the statistical model identification," *IEEE Trans. Automatic Control*, vol. AC-19, pp. 716-723, Dec. 1974.

[3] J. Rissanen, "Modeling by shortest data description," *Automatica*, vol. 14, pp. 465-471, 1978.

[4] G. Schwartz, "Estimation of the dimension of a model," *Ann. Statist*., vol. 6, pp. 461-464, 1978.

[5] E. J. Hannan and B. G. Quinn, "The determination of the order of an autoregression," *Journal of the Roy. Stat. Soc*., B 41, pp.190-195, 1979.

[6] A. El Matouat and M. Hallin, "Order selection, stochastic complexity and Kullback-Leibler information," in *Time Series Analysis*, vol. 2, pp. 291-299, in memory of E. J. Hannan, Spriger Verlag: New York, 1996.

[7] B. Aksasse and L. Radouane, "Two-dimensional autoregressive (2-D AR) model order estimation," *IEEE Trans. Signal Processing*, vol. 47, pp. 2072-2077, July 1999.

[8] J. M. Beguin, C. C. Gourieroux, and A. Monfort, "Identification of a mixed autoregressive moving average process: the corner method," in *Time Series,* O. D. Anderson, Ed. Amsterdam, The Netherlands: North-Holand, 1980, pp. 423-436.

[9] J. A. Cadzow, "Spectral estimation an overdetermined rational model equation approach," *Proc. IEEE* vol. 70, pp. 907-938, 1982

[10] B. Aksasse, L. Badidi, and L. Radouane, "A rank test based approach to order estimation-part I: 2-D AR models application," *IEEE Trans. Signal Processing*, vol. 47, pp. 2069-2072, July 1999.

[11] G. Liang, D. M. Wilkes, and J. A. Cadzow, "ARMA model order estimation based on the eigenvalues of the covariance matrix," *IEEE Trans. Signal Processing*, vol. 41, no. 10, pp. 3003-3009, October 1993.

[12] A. Al-Smadi, and D. M. Wilkes, "Robust and accurate ARX and ARMA model order estimation of non-Gaussian processes," *IEEE Trans. Signal Processing*, vol. 50, no. 3, pp. 759-763, March 2002.

[13] B. S. Choi, "An order recursive algorithm to solve the 3-D Yule-Walker equations of causal 3-D AR models*," IEEE Trans. Signal Processing*, vol. 47, pp. 2491-2502, 1999.

[14] B. Aksasse, Y. Stitou, Y. Berthoumieu and M. Najim, "3D AR model order selectiuon via rank test procedure," in revision *IEEE Trans. Signal Processing*, 2004 (see *also Proc. 19e Colloque GRETSI sur le traitement du signal et des images*, vol. II, pp. 119-122, Paris, France 2003).

[15] M. Wax and T. Kailath, "Detection of signals by information theoretic criteria," *IEEE Trans. Acoustic, Speech, Signal Processing*, vol. ASSP-33, no. 2, pp. 387-392, April 1985.