# ACTIVITY DETECTION OF A PSK SIGNAL IN UNKNOWN WHITE GAUSSIAN NOISE: OPTIMAL AND SUBOPTIMAL INVARIANT DETECTORS

*A.A. Tadaion* [†‡], *S. Gazor* [‡], *M. Derakhtian* [†] *and M.R. Aref* [†]

[†]Department of Electrical Engineering, Sharif University, Tehran, Iran

[‡]Department of Electrical and Computer Engineering, Queen's University, Kingston, Canada

**Abstract**–We propose three solutions for the detection of the activity of Phase Shift Keying (PSK) signals in additive white Gaussian noise environment. The symbol sequence, the complex amplitude of the signal and the noise variance are assumed to be unknown. We show that the Uniformly Most Powerful Invariant (UMPI) test does exist only if the Signal-to-Noise Ratio (SNR) is known. We use this UMPI test in order to obtain an upper-bound performance for the evaluation of invariant detectors. We also propose two suboptimal tests namely, the Generalized Likelihood Ratio (GLR) test, and the Average Likelihood Ratio (ALR)-GLR test. It turns out that the Computational Complexity (CC) of these detectors increases exponentially with the increase of the sequence length. Therefore, we suggest a suboptimal computationally efficient implementation of the GLR. This implementation requires only 0.02dB higher SNR in order to perform as good as the GLR. Furthermore, we develop a new inexpensive detector for the case of Binary PSK (BPSK) signals, namely Generalized Energy Detector (GED). Simulation results illustrate and compare the performance and the efficiency of these methods.

## 1. INTRODUCTION AND PROBLEM STATEMENT

Signal Activity Detection (SAD) is a critical stage in the implementation of an effective communication system. For instance, detection of the presence of digitally modulated signals is an important problem within the context of wireless packet switching networks. A user may attempts to transmit signals or is inactive. In particular, a reliable activity detection of a signal in the presence of ambient noise plays a major role in the performance of these systems. Some approaches to address such a problem are attempted for a variety of other applications including reconnaissance, surveillance, and some other intelligence gathering activities, e.g., see [1–8]. Detection of a digitally modulated signal is a binary hypothesis testing problem as follows, [8]:

$$\begin{cases} \mathcal{H}_0 : \mathbf{r} = \mathbf{n}, & \text{if PSK signal is absent,} \\ \mathcal{H}_1 : \mathbf{r} = A\mathbf{c} + \mathbf{n}, & \text{if PSK signal is active,} \end{cases} \quad (1)$$

where $\mathbf{r} = [r_0, \cdots, r_{N-1}]^T$ is the base band representation of the received signal vector, $\mathbf{n} = [n_0, \cdots, n_{N-1}]^T$ is the complex zero-mean white Gaussian noise with unknown variance $\sigma^2 = E[|n_i|^2]$, $\mathbf{c} = [c_0, c_1, \cdots, c_{N-1}]^T$ is the data vector and $A = \alpha e^{j\phi_c}$ is the unknown complex amplitude of the received signal, $\alpha$ the magnitude and $\phi_c$ the carrier phase, which are assumed constant unknown during the period of observation [2]. The digitally modulated sequence $c_n$ is selected from the set of Phase Shift Keying (PSK) alphabets $\mathcal{C} = \left\{e^{j\frac{2\pi k}{M}}, k = 0, \cdots, M-1\right\}$, where $M$ is the number of constellation points, e.g., in the Binary PSK (BPSK) scheme $c_n \in \mathcal{C} = \{+1, -1\}$. In this paper, we assume no Inter-Symbol Interference (ISI) and perfect synchronization in the sampling time and in the carrier frequency. In practice, the synchronization may be achieved using several methods, such as transmitting pilot-tones for all users or non-data-aided algorithms [9].

A traditional method (called the energy detector or the radiometer) uses the energy level of the received signal as a criterion for the presence detection of the signal [4]. Such a method is susceptible to the a priori knowledge of the variance of noise and interference [3]. It is apparent that this method does not exploit efficiently the received information. In an Additive White Gaussian Noise (AWGN) environment, a known waveform is often detected by either employing a filter matched to the known waveform or by correlating the received signal with the waveform. Most of these methods not only assume that the signal is known but also they require the noise variance in order to determine the decision threshold [5]. Krasner [1] considered a model for the signal with unknown parameters (such as carrier frequency, carrier phase and symbol sequence) as random variables and developed the coherent and noncoherent detectors by averaging the detection criteria over the distribution of these parameters. If the probability density functions (pdfs) of the unknown parameters are provided, these detectors are able to detect the presence of a weak signal at the expense of higher Computational Complexity (CC). Ramprashad and Parks proposed a locally most powerful invariant test to detect signals [8]. The most common assumption of the existing methods is that the noise variance is considered to be known, which is far from reality in many practical situations.

It is known that a Uniformly Most Powerful (UMP) test does not exist for all composite hypothesis testing problems in which either one or both of the hypotheses contain unknown parameters (e.g., [10, 11]). For the problem (1), if the parameters $(A, \sigma^2, \mathbf{c})$ are unknown, the UMP test does not exist. We will show in Section 2.1 that the Uniformly Most Powerful Invariant (UMPI) detector does not exist either, except if the Signal-to-Noise Ratio (SNR) is known. We use this UMPI detector as a comparison to evaluate the performance of the Generalized Likelihood Ratio Test (GLRT) introduced in Section 2.2 which is also an invariant test. It is known that in invariant problems, the GLRT is invariant and asymptotically UMPI [12, 13]. In the case of unknown parameters, we propose two alternative detectors namely Generalized Likelihood Ratio (GLR) and Average Likelihood Ratio-GLR (ALR-GLR) detectors in Sections 2.2 and 2.3, respectively. The GLR detector is derived by replacing the unknown parameters by their Maximum Likelihood (ML) estimates and in ALR-GLR detector, the ML estimates of $A$ and $\sigma^2$ are substituted in the Likelihood Ratio (LR) and assuming an independent and identically distributed (i.i.d.) uniform distribution for $c_n$, the resulting LR is averaged over the set of alphabets. The proposed GLR and ALR-GLR detectors require exhaustive search in implementation. To reduce the exponential CC of the GLR detector, we suggest a computationally efficient suboptimal implementation. In addition, we present a suboptimal detector based on matched subspace detectors for the case of BPSK modulated signals with substantially decreased CC at the expense of some small negligible performance loss. Simulation results and performance-CC comparison of these methods are reported in Section 3. Section 4 concludes the paper.

## 2. PROPOSED DETECTORS

### 2.1. UMPI Detector

To evaluate the existence of the UMPI test, our approach is as follows: in [14] we have shown that the problem (1) for PSK signals is invariant under the composition of the following transformation groups

$$G_d = \{g_d : g_d(\mathbf{r}) = d\,\mathbf{r}, \quad \forall d \in \mathbb{C} - \{0\}\}$$

and

$$G_{\mathbf{e}} = \left\{g_{\mathbf{e}} : g_{\mathbf{e}}(\mathbf{r}) = \mathbf{e} \odot \mathbf{r}, \mathbf{e} = [e_0, \cdots, e_{N-1}]^T, \forall e_n \in \mathcal{C}\right\}$$

where $\odot$ denotes the element-wise multiplication. Then in Appendix A, we find the maximal invariant statistic for the composite group $G$ consisting of the above subgroups $G_d$ and $G_{\mathbf{e}}$ [1]:

$$\mathbf{y} = \left[\sqrt[|\mathcal{C}|]{\left(\frac{r_0}{r_{N-1}}\right)^{|\mathcal{C}|}}, \cdots, \sqrt[|\mathcal{C}|]{\left(\frac{r_{N-2}}{r_{N-1}}\right)^{|\mathcal{C}|}}, 1\right]^T. \quad (2)$$

If the LR does not depend on the unknown parameters, the UMPI test is to compare this LR with a threshold. In Appendix B we derive the conditional distribution of the maximal invariant statistic $\mathbf{y}$ under each hypothesis and construct the LR as in (7) where $(.)^H$ is the transpose conjugate, $u_{k,j}$ is the $k^{\text{th}}$ component of the $j^{\text{th}}$ possible sequence, $u_{N-1,j} \triangleq 1, j = 1, \cdots, |\mathcal{C}|^{N-1}$ and the SNR is denoted by $\rho = \frac{|A|^2}{\sigma^2}$. Since $\mathbf{c}$ is an $N$-tuple vector, there are $|\mathcal{C}|^{N-1}$ different possible sequences to be considered. The UMPI test, if it exits, rejects $\mathcal{H}_0$, if $L(\mathbf{r}) > \eta_{\text{UMPI}}$, where the threshold is selected such that the probability of false alarm $P_{\text{fa}}$ requirement satisfies. Since $L(\mathbf{r})$ in (7) depends on the SNR $\rho$, the UMPI test can not be implemented if the SNR is unknown. In addition, provided the SNR value this UMPI test allows us to numerically evaluate an upper bound for the performance of any invariant test, such as the GLR detector derived in the following subsection.

### 2.2. GLR Detector

The binary hypothesis testing problem (1) is equivalent to the following test:

$$\begin{cases} \mathcal{H}_0 : \mathbf{r} \sim \mathcal{N}(\mathbf{0}, \sigma^2 I_N), \\ \mathcal{H}_1 : \mathbf{r} \sim \mathcal{N}(A\mathbf{c}, \sigma^2 I_N), \end{cases} \quad (3)$$

where $\mathcal{N}(., \sigma^2 I_N)$ denotes an $N$-dimensional multivariate complex normal distribution, $\sigma^2$ is the noise variance, $I_N$ is $N \times N$ identity matrix, $A$ is the unknown complex amplitude of the signal and $\mathbf{0}$ is a zero vector of length $N$. In this section we employ the GLR approach to solve the problem. We obtain the ML estimates of the unknown parameters $A$ and $\sigma^2$ under each hypothesis, substitute them in the pdfs and construct the likelihood ratio [15]. In Appendix C, we derived the GLRT, substituting the estimated values of the unknown $A$ and $\sigma^2$ that rejects $\mathcal{H}_0$ if:

$$T = \frac{|\mathbf{c}^H\mathbf{r}|^2}{N\|\mathbf{r}\|^2 - |\mathbf{c}^H\mathbf{r}|^2} > \eta_{\text{T}}, \quad (4)$$

where $\eta_{\text{T}}$ is chosen such that the $P_{\text{fa}}$ requirement satisfies. In order to derive the GLR detector, we should substitute the ML estimate of the unknown sequence $\mathbf{c}$ in (4). This approach results in a joint detection for the data sequence and its activity. Thus, we obtain a GLR detector with the following rejection region:

$$\max_{\mathbf{c}\in\mathcal{C}^N} T = \max_{\mathbf{c}\in\mathcal{C}^N} \frac{|\mathbf{c}^H\mathbf{r}|^2}{N\|\mathbf{r}\|^2 - |\mathbf{c}^H\mathbf{r}|^2} > \eta'_{GLR}, \quad (5)$$

where $\mathbf{c}_{\text{ML}} = \arg\max_{\mathbf{c}\in\mathcal{C}^N} T$ is the ML estimate of $\mathbf{c}$. The above inequality is equivalent to the following inequality[2]:

$$T_{\text{GLR}} = \frac{\max_{\mathbf{c}\in\mathcal{C}^N} |\mathbf{c}^H\mathbf{r}|^2}{N\|\mathbf{r}\|^2} > \eta_{\text{GLR}} = \frac{\eta'_{GLR}}{\eta'_{GLR} + 1}, \quad (6)$$

Regardless of a phase ambiguity[3] the data sequence is also detected in this method. The CC of (6) is of order of $\mathcal{O}(|\mathcal{C}|^N)$, where $|\mathcal{C}|$ denotes the number of constellation points.

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[1] For a complex number $z$, the value $\sqrt[M]{z}$ denotes the unique solution of $x^M = z$ for which $\angle x \in \left[-\frac{\pi}{M}, \frac{\pi}{M}\right)$.

[2] Since, for any three positive number, $a, b$ and $c$ the inequalities $\frac{a}{b} > c$ and $\frac{a}{a+b} > \frac{c}{1+c}$ are equivalent.

[3] For instance in BPSK, this ambiguity is in the sign of vector $\mathbf{c}$. Such an ambiguity is resolvable employing a differential modulation scheme.

$$L(\mathbf{r}) = \frac{e^{-N\rho}}{2^{N-1}} \sum_{j=1}^{|\mathcal{C}|^{N-1}} \left[ \exp\left( \rho \frac{\left| \sum_{k=0}^{N-1} \sqrt[|\mathcal{C}|]{r_k} u_{k,j}^H \right|^2}{\|\mathbf{r}\|^2} \right) \sum_{p=0}^{N-1} \binom{N-1}{p} \left( \rho \frac{\left| \sum_{k=0}^{N-1} \sqrt[|\mathcal{C}|]{r_k} u_{k,j}^H \right|^2}{\|\mathbf{r}\|^2} \right)^p \frac{1}{p!} \right]. \quad (7)$$

If $|\mathcal{C}|$ is a large number (or if $|\mathcal{C}|$ is unknown), the CC of this detector could be substantially reduced by approximating $c_i \simeq e^{j\angle r_i}$ or by selecting the nearest neighbor of $e^{j\angle r_i}$ in the constellation set. In such a case, the detection variable becomes the ratio of $L_1$-norm to $L_2$-norm of the observed data with the threshold, i.e., $\frac{\left(\sum_{k=0}^{N-1} |r_i|\right)^2}{N \sum_{k=0}^{N-1} |r_i|^2} \gtrless \eta_{\mathrm{GLR}}$. If the number of constellation points is small, we propose a better suboptimal implementation as follows. Instead of this exhaustive search in (6), the following procedure iteratively chooses the data symbols:

**Step 1:** Choose $\widehat{c}_0 \in \mathcal{C}$ arbitrarily and set $G_0 = \widehat{c}_0^H r_0$.

**Step 2:** For $k=1, \cdots, N-1$, choose $\widehat{c}_k$ by

$$\widehat{c}_k = \arg\max_{c_k \in \mathcal{C}} \left| G_{k-1} + c_k^H r_k \right|; \ \ G_k = G_{k-1} + \widehat{c}_k^H r_k. \quad (8)$$

**Step 3:** Compare $\frac{|G_{N-1}|^2}{N\|\mathbf{r}\|^2} \triangleq \frac{\widehat{\max_{\mathbf{c}\in\mathcal{C}^N} |\mathbf{c}^H \mathbf{r}|^2}}{N\|\mathbf{r}\|^2}$ with $\eta_{\mathrm{GLR}}$.

This detector requires only a CC of the order of $\mathcal{O}(N\ |\mathcal{C}|)$, while simulation results illustrate that this Suboptimal implementation of GLR (we call it S-GLR) performs almost similar to the the GLR detector using exhaustive search in (6), i.e., the S-GLR requires only less than 0.02dB more SNR in order to perform as good as the GLR in (6).

### 2.3. ALR-GLR Detector

In ALR-GLR approach, $\{c_k\}_{k=0}^{N-1}$ is assumed as an i.i.d. random sequence uniformly distributed over $\mathcal{C}^N$. Since in practice, the data sequence is encoded such an assumption is reasonable in most applications. In this case, the likelihood ratio (4) is averaged over $\mathcal{C}^N$ and the rejection region for ALR-GLR detector is given by:

$$\frac{1}{|\mathcal{C}|^N} \sum_{j=1}^{|\mathcal{C}|^N} \frac{|\mathbf{c}_j^H \mathbf{r}|^2}{N\|\mathbf{r}\|^2 - |\mathbf{c}_j^H \mathbf{r}|^2} > \eta_{\mathrm{ALR-GLR}}, \quad (9)$$

where $\mathbf{c}_j \in \mathcal{C}^N$ is any possible sequence. Since $\mathbf{c}_j$ is an N-tuple vector and its components are different elements of $\mathcal{C}$, it is evident that the CC of (9) is of order of $\mathcal{O}(N|\mathcal{C}|^N)$.

### 2.4. Generalized Energy Detector (GED)

In this section, we propose an inexpensive suboptimal detector with very low CC for the case of BPSK signals. Let $t_k \triangleq r_k^2$ in (1), under $\mathcal{H}_1$ we get, $t_k = A^2 + n_k^2 + 2Ac_k n_k \triangleq A^2 + w_k$, since for a BPSK signal we have $c_k^2 = 1$. Under $\mathcal{H}_1$ assuming that the noise variance is small compared with the signal power, we have $w_k \approx 2Ac_k n_k$; obviously $2Ac_k n_k$ is a zero-mean normal random variable with variance $4|A|^2\sigma^2$ where $A$ is an unknown parameter. Thus, under $\mathcal{H}_1$, we have $E[t_k] \approx A^2$ while under $\mathcal{H}_0$, we have $E[t_k] \approx 0$. The recent problem is a particular case of the matched subspace detector (see e.g. [13, 16, 17]) and the resulting Generalized Energy Detector (GED) rejects $\mathcal{H}_0$ if:

$$T_{\mathrm{GED}} = \frac{\left| \sum_{k=0}^{N-1} t_k \right|^2}{\sum_{k=1}^{N} |t_k|^2} = \frac{\left| \sum_{k=0}^{N-1} r_k^2 \right|^2}{\sum_{k=1}^{N} |r_k^2|^2} > \eta_{\mathrm{GED}}, \quad (10)$$

where $\eta_{\mathrm{GED}}$ is chosen such that the $P_{\mathrm{fa}}$ requirement satisfies.

**Table 1**. **Trade-off between CC and Performance**: The number of multiplications (a measure of CC) and performance in each of the proposed detectors for $N = 10$.

| Detector | No. of Mul. | Performance Loss (dB) |
|---|---|---|
| UMPI | $> N^2|\mathcal{C}|^N$ | 0 |
| GLR | $N + 2N|\mathcal{C}|^N$ | 0.05 |
| S-GLR | $3N + N|\mathcal{C}|$ | 0.075 |
| ALR-GLR | $4N|\mathcal{C}|^N$ | 0.31 |
| GED | $3N$ | 0.37 |

## 3. SIMULATION RESULTS

We evaluate and compare the performance of the proposed detectors by simulations. In all simulations, the complex amplitude of the signal and the variance of the noise are unknown to the detectors, except in Figure 1 where the SNR value is provided to the the UMPI detector. The SNR is defined as the ratio between the signal power and the noise variance. Figure 1 depicts the probability of detection $P_{\mathrm{d}}$ versus SNR for the UMPI test and GLRT for SAD where $N = 5$ or $N = 10$, and $P_{\mathrm{fa}} = 0.1$ or $P_{\mathrm{fa}} = 0.01$ all for BPSK signals. Simulations illustrate how close the GLR performance to UMPI performance is even at small $N$, the length of the sequence; e.g. UMPI detector outperforms the GLR detector for only 0.05dB, while UMPI uses the SNR information for the detection with high CC and GLR detects the signal without this information with low CC.

Figure 2 illustrates the performances of the GLRT, both the search method implementation and the suboptimal implementation (Section 2.2). We also evaluate the performance of the GED in this figure, comparing to the GLR. Different number of data symbols is considered for simulations. We observe that the performance improves as SNR or the number of samples increases. The performances of the

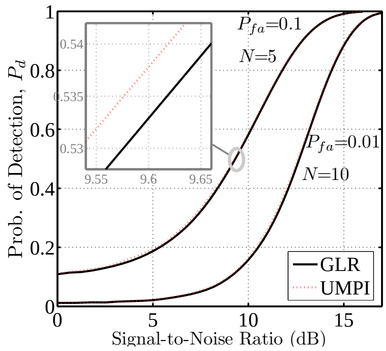


**Fig. 1**. **Performance Comparison of GLR and UMPI tests** in terms of the Probability of Detection versus SNR for a BPSK signal with different number of samples $N = 5, 10$ and different $P_{\mathrm{fa}} = 0.1, 0.01$.

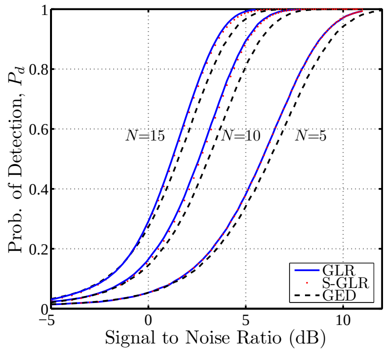


**Fig. 2**. **Performance Comparison of GLR, S-GLR and GED** for the SAD of a PSK signal. We considered a BPSK signal with different number of samples $N = 5, 10, 15$ and the probability of false alarm is $P_{\mathrm{fa}} = 0.01$.

search method implementation and the suboptimal implementation are indeed comparable (only about 0.02dB apart) while the CC of the suboptimal implementation is substantially less than that of the search method. If the number of samples is large, e.g., for $N = 10$ at a given $P_{\mathrm{d}}$, the GLR results in 0.32dB improvement gain in SNR compared with the GED. The price paid for these gains is their higher CC. The trade-off between the number of multiplications performed in each detection algorithm (as the order of CC) and the performance loss (dB) is depicted in table 1 where the performance results are derived for $N = 10$. It is notable that GED is only applicable for BPSK signals. The low CC of S-GLR and GED in this comparison is observable. The performance of the S-GLR is compared for BPSK and QPSK at different sample numbers in Figure 3.

We have compared the performances of the proposed detectors, GLR, ALR-GLR and GED in Figure 4. The GLR and ALR-GLR detectors outperform the GED. If the number of samples is large, e.g., for $N = 10$ at a given $P_{\mathrm{d}}$, the ALR-GLR gives 0.06dB improvement gain in SNR compared with GED. The price paid for these gains is its higher CC. It is seen that the performance of the GLR and ALR-GLR detectors are somehow comparable for different number of symbols. Note that the order of complexity for both algorithms are roughly the same; in the GLR method we should search among all possible symbol sequences to find the maximum value of $|\mathbf{c}^H\mathbf{r}|^2$ in (6). Similarly, in ALR-GLR all possible values of the symbol sequence are used in calculating the likelihood ratio.

## 4. DISCUSSION AND CONCLUSION

In this paper we proposed three detectors for the activity detection of PSK signals with unknown symbol sequence and unknown complex amplitude in a white Gaussian noise environment with unknown variance. We showed that the UMPI test does not exist. In order to obtain an upper-bound performance for any invariant test, we derived a UMPI test that requires the information of the SNR. Obviously, such a test outperforms any other invariant test without the SNR information. We also proposed a GLR detector by substituting the ML estimates of the unknown parameters in likelihood ratio. In addition, we suggested an ALR-GLR detector by substituting the ML estimates of the complex amplitude of the signal and the noise variance and averaging the resulting LR over all set of possible symbol sequences. The GLR detector requires to search over $|\mathcal{C}|^N$ sequences, thus its CC increases exponentially when the number of samples $N$ increases. Similarly, the calculation of the ALR-GLR criterion involves a summation over all possible $|\mathcal{C}|^N$ sequences. Thus, the ALR-GLR detector has the same order of complexity. We proposed a suboptimal implementation of the GLR method and illustrated by simulation that the performances of the resulting detectors are comparable to the optimal implementations via search. Furthermore, for BPSK signals, we derived a computationally inexpensive detector called GED with a CC of order of $N$. Our simulation results show that the GED performance is comparable to GLR and ALR-GLR, i.e., the performance of the GED compromises 0.3dB loss in SNR compared with the upper bound performance.

## Appendix A: Maximal Invariant Statistic for the Composite Group $G$

In this appendix, using the following theorem [13], we obtain the maximal invariant statistic for the composite group $G$ in two steps, each related to finding a maximal invariant statistic for one of the subgroups.

**Theorem 1** *Let $G$ be a group of transformations and let $G_d$ and $G_{\mathbf{e}}$ be two subgroups generating $G$. Suppose that $\mathbf{x} = M_d(\mathbf{r})$ is a maximal invariant with respect to $G_d$ and for any $g_{\mathbf{e}} \in G_{\mathbf{e}}$, any $\mathbf{r}^{(1)}$ and any $\mathbf{r}^{(2)}$, we have*

$$M_d(\mathbf{r}^{(1)}) = M_d(\mathbf{r}^{(2)}) \Rightarrow M_d(g_{\mathbf{e}}(\mathbf{r}^{(1)})) = M_d(g_{\mathbf{e}}(\mathbf{r}^{(2)})),$$

*and there exists a maximal invariant statistic $\mathbf{y} \triangleq M_{\mathbf{h}}(\mathbf{x})$*

*under the group $H$ of transformations $g_{\mathbf{h}}(\mathbf{x}) \triangleq M_d(g_{\mathbf{e}}(\mathbf{r}))$, then the statistic $\mathbf{y} = M_{\mathbf{h}}(M_d(\mathbf{r}))$ is maximal invariant with respect to $G$.*

*Proof:* See [13, Chapter 6, Theorem 2].

A maximal invariant for the scale group $G_d$ is given by [14]

$$\mathbf{x} = M_d(\mathbf{r}) = \left[\frac{r_0}{r_{N-1}}, \cdots, \frac{r_{N-1}}{r_{N-1}}\right]^T. \tag{11}$$

Following Theorem 1, in order to derive a maximal invariant for the second group, assuming $M_d(\mathbf{r}^{(1)}) = M_d(\mathbf{r}^{(2)})$, we can see that

$$M_d\left(g_{\mathbf{e}}(\mathbf{r}^{(1)})\right) = M_d\left(g_{\mathbf{e}}(\mathbf{r}^{(2)})\right). \tag{12}$$

Therefore, the theorem condition is satisfied. Now, we should find a group $G_{\mathbf{h}}$ which acts on $\mathbf{x}$ and then a maximal invariant under that group. According to Theorem 1, $g_{\mathbf{h}}(\mathbf{x}) = M_d(g_{\mathbf{e}}(\mathbf{r}))$, therefore $g_{\mathbf{h}}(\mathbf{x}) = [h_0 x_0, \cdots, h_{N-2}x_{N-2}]^T$ where $h_n \triangleq \frac{e_n}{e_{N-1}} \in \mathcal{C}$ and a maximal invariant under $G_{\mathbf{h}}$ is [14]

$$\mathbf{y} \triangleq M_{\mathbf{h}}(\mathbf{x}) = \left[\sqrt[|\mathcal{C}|]{x_0^{|\mathcal{C}|}}, \cdots, \sqrt[|\mathcal{C}|]{x_{N-2}^{|\mathcal{C}|}}\right]^T. \tag{13}$$

## Appendix B: Derivation of pdf of the Maximal Invariant

In this appendix, we derive the pdf of the maximal invariant $\mathbf{y}$ under each hypothesis. It will be done in steps. The density of the observation $\mathbf{r}$ under $\mathcal{H}_1$ is:

$$f(\mathbf{r};\mathcal{H}_1) = \frac{1}{(\pi\sigma^2)^N}\exp\left\{-\frac{1}{\sigma^2}\|\mathbf{r} - A\mathbf{c}\|^2\right\}. \tag{14}$$

The distribution under $\mathcal{H}_0$ is obtained by substituting $A = 0$ in the above. The pdf of $\mathbf{x}$ under $\mathcal{H}_1$ and $\mathcal{H}_0$ are derived in [14] and are presented in (25) and (15), respectively where $\rho = \frac{|A|^2}{\sigma^2}$.

$$f(\mathbf{x};\mathcal{H}_0) = \frac{(N-1)!}{\pi^{N-1}}\frac{1}{(\|\mathbf{x}\|^2+1)^N}. \tag{15}$$

The pdf of $\mathbf{y}$ in (13) is as in (26) where $u_{k,j} = t_{k,j}c_k \in \mathcal{C}$ is the $k^{\text{th}}$ component of the $j^{\text{th}}$ possible sequence and since $\mathbf{c}$ is an $N$-tuple vector, there are $|\mathcal{C}|^{N-1}$ different possible sequences to be considered ($t_{N-1,j} \triangleq 1, j = 1, \cdots, |\mathcal{C}|^{N-1}$) similarly there are $|\mathcal{C}|^{N-1}$ different possible such sequences to be considered ($u_{N-1,j} \triangleq 1, j = 1, \cdots, |\mathcal{C}|^{N-1}$). By replacing $\rho = 0$ in (26), the pdf of $\mathbf{y}$ under $\mathcal{H}_0$ is

$$f(\mathbf{y};\mathcal{H}_0) = \frac{(N-1)!2^{N-1}}{\pi^{N-1}}\sum_{j=1}^{|\mathcal{C}|^{N-1}}\frac{1}{\|\mathbf{y}\|^2}. \tag{16}$$

We obtain (7) by constructing $L(\mathbf{r}) = \frac{f(\mathbf{y};\mathcal{H}_1)}{f(\mathbf{y};\mathcal{H}_0)}$ and some manipulation from (26), (16) and (2). Since, the LR in (7) depends on SNR of the probable signal, the UMPI test exists only if the SNR is known.

## Appendix C: Generalized Likelihood Ratio Test

We calculate the ML estimates of the unknown parameters $A$ and $\sigma^2$ under each hypothesis and construct the likelihood ratio substituting the unknown parameters by the ML estimates. The pdf of the transformed observation $\mathbf{r}$ under the hypothesis $\mathcal{H}_1$ and $\mathcal{H}_0$ are the followings:

$$f(\mathbf{r};\mathcal{H}_1) = \frac{1}{\pi^N\sigma^{2N}}\exp\left\{-\frac{1}{\sigma^2}\sum_{k=0}^{N-1}|r_k - Ac_k|^2\right\} \tag{17}$$

$$f(\mathbf{r};\mathcal{H}_0) = \frac{1}{\pi^N\sigma^{2N}}\exp\left\{-\frac{1}{\sigma^2}\sum_{k=0}^{N-1}|r_k|^2\right\}. \tag{18}$$

Maximizing the above density functions, the ML estimates of $A$ and $\sigma^2$ are easily obtained as follows:

$$\mathcal{H}_1 \;:\; \begin{cases} \hat{A} = \frac{1}{N}\mathbf{c}^H\mathbf{r}, \\ \widehat{\sigma_1^2} = \frac{1}{N}\sum_{k=0}^{N-1}|r_k - \hat{A}c_k|^2, \end{cases} \tag{19}$$

$$\mathcal{H}_0 \;:\; \widehat{\sigma_0^2} = \frac{1}{N}\sum_{k=0}^{N-1}|r_k|^2. \tag{20}$$

Since $|c_k| = 1, k = 0, \cdots, N-1$, we have

$$\widehat{\sigma_1^2} = \frac{1}{N}\sum_{k=0}^{N-1}|c_k^H r_k - \hat{A}|^2 = \sum_{k=0}^{N-1}|r_k|^2 - \frac{1}{N}\left|\sum_{k=0}^{N-1}c_k^H r_k\right|^2.$$

Substituting (19) and (20) in (17) and (18) respectively, results in:

$$f(\mathbf{r};\mathcal{H}_1)|_{(19)} = \left(\frac{1}{(\pi e)\widehat{\sigma_1^2}}\right)^N, \tag{21}$$

$$f(\mathbf{r};\mathcal{H}_0)|_{(20)} = \left(\frac{1}{(\pi e)\widehat{\sigma_0^2}}\right)^N, \tag{22}$$

where $a|_{(b)}$ means substituting the results of equation numbered by (b) in the expression of $a$. The ratio of likelihood functions in (21) and (22) using the ML estimates under each hypothesis is given by,

$$\frac{f(\mathbf{r};\mathcal{H}_1)|_{(19)}}{f(\mathbf{r};\mathcal{H}_0)|_{(20)}} = \left(\frac{\frac{1}{N}\sum_{k=0}^{N-1}|r_k|^2}{\sum_{k=0}^{N-1}|r_k|^2 - \frac{1}{N}\left|\sum_{k=0}^{N-1}c_k^H r_k\right|^2}\right)^N. \tag{23}$$

Since $g(\alpha) = (1+\alpha)^N$, is an increasing function of $\alpha$, the above GLRT rejects $\mathcal{H}_0$ if:

$$T = \frac{|\mathbf{c}^H\mathbf{r}|^2}{N\|\mathbf{r}\|^2 - |\mathbf{c}^H\mathbf{r}|^2} > \eta_{\text{T}}, \tag{24}$$

where $\eta_{\text{T}}$ is such that $P_{\text{fa}}$ satisfies the required condition.

$$f(\mathbf{x};\mathcal{H}_1) = \frac{(N-1)!e^{-N\rho}}{\pi^{N-1}} \frac{e^{\frac{\left|A\left(\mathbf{x}^H\mathbf{c}+c_{N-1}\right)\right|^2}{\sigma^2\left(\|\mathbf{x}\|^2+1\right)}}}{\left(\|\mathbf{x}\|^2+1\right)^N} \sum_{p=0}^{N-1} \binom{N-1}{p} \left(\frac{\left|A\left(\mathbf{x}^H\mathbf{c}+c_{N-1}\right)\right|^2}{\sigma^2\left(\|\mathbf{x}\|^2+1\right)}\right)^p \frac{1}{p!}. \tag{25}$$

$$f(\mathbf{y};\mathcal{H}_1) = \frac{(N-1)!e^{-N\rho}}{\pi^{N-1}\,\|\mathbf{y}\|^{2N}} \sum_{j=1}^{|\mathcal{C}|^{N-1}} e^{\rho\frac{\left|\sum_{k=0}^{N-1} y_k^H u_{k,j}\right|^2}{\|\mathbf{y}\|^2}} \sum_{p=0}^{N-1} \binom{N-1}{p} \left(\rho\frac{\left|\sum_{k=0}^{N-1} y_k^H u_{k,j}\right|^2}{\|\mathbf{y}\|^2}\right)^p \frac{1}{p!}. \tag{26}$$

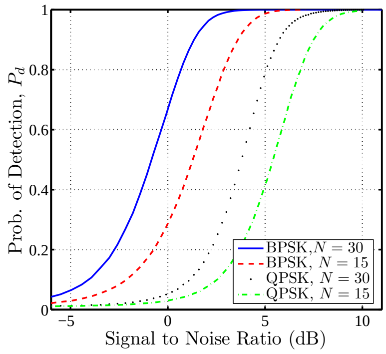


**Fig. 3**. **Performance evaluation of S-GLR** for the SAD of a PSK signal. We considered BPSK and QPSK signals with different number of samples $N = 15, 30$, and the probability of false alarm is $P_{\text{fa}} = 0.01$.

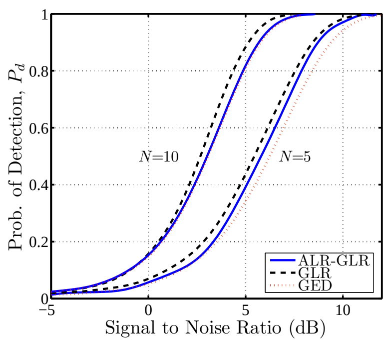


**Fig. 4**. **Performance Comparison of GLR, ALR-GLR and GED** for the SAD of a PSK signal. We considered a BPSK signal with different number of samples $N = 5, 10$ and the probability of false alarm is $P_{\text{fa}} = 0.01$.

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