Reducing Excess Variance in Beamspace Methods for Uniform Circular Array

Fabio Belloni, Andreas Richter and Visa Koivunen
Signal Processing Laboratory, SMARAD CoE
Helsinki University of Technology
P.O. Box 3000, FIN-02015 HUT, Finland
Email: {fbelloni,arichter,visa}@wooster.hut.fi

***Abstract*— In this paper we present a procedure for reducing the excess variance introduced by Beamspace Transform when it is applied to Uniform Circular Array (UCA). Several algorithms for Direction of Arrival (DoA) estimation exploit modal transforms which are based on the phase-mode excitation principle [1]. Here we analyze the inverse Fourier series of the array impulse response, called Effective Aperture Distribution Function (EADF), for defining an efficient criterion for selecting the number of virtual array elements. The proposed criterion is optimum in the sense that maximizes the aperture of the virtual array while satisfying certain constraints on the maximum number of virtual array elements. Simulation results show that we can obtain DoA estimates with a lower variance such that they are closer to the CRB.**

I. Introduction

Circular arrays are of interest in a variety of applications, e.g. in multiantenna transceivers. Moreover, UCA's have almost uniform performance regardless of the angle of azimuth and they can estimate both azimuth and elevation angles simultaneously. Several DoA estimators for UCA, such as MUSIC, root-MUSIC and ESPRIT [2]-[4] employ some modal transforms, e.g. Davies and Beamspace Transform. These transforms essentially map the steering vectors of UCA into the steering vectors of a ULA-like array, called virtual array, with an approximated Vandermonde structure [5].

In [2],[5] it has been shown that the Beamspace Transform (BT) works properly only under certain constraints that may be difficult to satisfy in some applications. As claimed in [5], when the BT does not work under suitable conditions on the array configuration (number of elements, interelement spacing,...), a bias and an excess variance may appear in the DoA estimates which severely degrade the algorithm performance. Here, the excess variance refers to the variance introduced by the modal transform on top of the Cramér-Rao Lower Bound (CRB). An efficient solution for bias cancellation has been proposed in [5]. However, it does not deal with the excess variance that appears after the BT.

In this paper we propose a method to reduce the excess variance term based on the analysis of the EADF [8]. The proposed method selects the appropriate number of virtual array elements $\mathcal{M}$ which leads to reduced variance. The simulation results clearly shown that, by selecting the number of virtual array elements using the proposed method, the variance may get significantly reduced. Consequently, the statistical performance of the DoA estimation algorithm gets closer to the CRB.

This paper is organized as follows. First, the UCA signal model is presented. In Section III, the phase-mode excitation principle is described by focusing on the functions defined on the original and transformed domain. In Section IV, we introduce the modal transforms and in particular the different design criteria for selecting $\mathcal{M}$. In Section V, we introduce the concept of Effective Aperture Distribution Function (EADF). In Section VI we propose a criterion for selecting $\mathcal{M}$ which is based on analyzing the EADF. In Section VII, simulation results demonstrating the reduction of the variance are shown. Finally, Section VIII concludes the paper.

II. Signal model

Let us have a Uniform Circular Array of N sensors. There are P ($P < N$) uncorrelated narrow-band signal sources on the array plane, impinging the array from directions $\phi_1, \phi_2, \ldots, \phi_P$ (ϕ is the azimuth angle). Furthermore we assume that K snapshots are observed by the array. The $N \times K$ element-space array output matrix may be written:

$$\mathbf{X} = \mathbf{AS} + \mathbf{N}, \tag{1}$$

where $\mathbf{X} \in \mathbb{C}^{N \times K}$ is the element-space data matrix, $\mathbf{A} \in \mathbb{C}^{N \times P}$ is the element-space steering matrix, $\mathbf{S} \in \mathbb{C}^{P \times K}$ is the source matrix and $\mathbf{N} \in \mathbb{C}^{N \times K}$ is the noise matrix. The noise is modelled as a stationary, second-order ergodic, zero-mean spatially and temporally white circular complex Gaussian process.

The $N \times P$ element-space steering vector matrix may be written as $\mathbf{A} = [\mathbf{a}_1(\zeta, \phi), \mathbf{a}_2(\zeta, \phi), \ldots, \mathbf{a}_P(\zeta, \phi)]$ where each column is of the form

$$\mathbf{a}_p(\boldsymbol{\vartheta}) = [e^{j\zeta \cos(\phi_p - \gamma_0)}, e^{j\zeta \cos(\phi_p - \gamma_1)}, \ldots, e^{j\zeta \cos(\phi_p - \gamma_{(N-1)})}]^T \tag{2}$$

for $p = 1, 2, \ldots, P$. Here $\boldsymbol{\vartheta} = (\zeta, \phi)$ and $\zeta = \kappa r \sin\theta$, r is the radius, $\kappa = \frac{\omega}{c}$ is the wavenumber, c is the speed of light, $\omega = 2\pi f$ is the angular frequency and $\gamma_n = \frac{2\pi n}{N}$ ($n = 0, \ldots N-1$) is the sensor location. The elevation angle θ is measured down from the z-axis (assumed to be $\theta = 90^\circ$) and ϕ is the azimuth angle measured counterclockwise from the x-axis in the xy-plane [5].

III. Phase Mode Excitation

The phase-mode excitation principle was introduced by Davies [1] and it focuses on the study of spatial harmonics [3]. It is essentially a Fourier analysis of the array excitation functions for different array configurations, e.g. for UCA [1]-[2]. The principle forms the background for the Davies, Beamspace and Generalized Beamspace Transform [5].

For circular arrays the study on the spatial harmonics may be described using two different array configurations, the continuous and the discrete circular array (i.e. UCA). A detailed analysis of a continuous circular aperture shows that any excitation function is periodic in γ with a period of 2π. Hence, it can be expressed in terms of Fourier series [1]. A generic excitation function $w(\gamma)$ may be defined using the inverse Fourier series $w(\gamma) = \sum_{m=-\infty}^{\infty} c_m e^{jm\gamma}$, where the m^{th} phase mode $w_m(\gamma) = e^{jm\gamma}$ represents a spatial harmonic of the array excitation and c_m is the corresponding Fourier series coefficient. For continuous circular arrays we can compute the normalized far-field pattern resulting from exciting the aperture with the m^{th} mode as [1]-[2]

$$f_m^c(\boldsymbol{\vartheta}) = \frac{1}{2\pi}\int_0^{2\pi} w_m(\gamma)e^{j\zeta\cos(\phi-\gamma)}\mathrm{d}\gamma = j^m J_m(\zeta)e^{jm\phi} \ , \quad (3)$$

where $J_m(\zeta)$ is the Bessel function of the first kind of order m. As a result we have a continuous periodic function in the transformed spatial domain γ and a discrete aperiodic function in the domain of the excitation mode order m.

In case of discrete circular aperture (UCA) the normalized far-field pattern resulting from exciting the aperture with the m^{th} mode is

$$\begin{aligned} f_m^s(\boldsymbol{\vartheta}) =& j^m J_m(\zeta)e^{jm\phi} + \sum_{q=1}^{\infty}(j^g J_g(\zeta)e^{-jg\phi} + j^h J_h(\zeta)e^{jh\phi}) \\ =& j^m J_m(\zeta)e^{jm\phi} + \varepsilon_m \ , \end{aligned} \quad (4)$$

where ε_m represents a sum on index $q = 1, 2, \ldots, \infty$ for defining the m^{th} excitation mode and the indices g and h are defined as $g = Nq - m$ and $h = Nq + m$, respectively. Eq.(4) is composed of two terms. The first term is known as the *principal term*. The other quantity ε_m, called *residual term*, arises from sampling the continuous aperture by N sensors and may represent either an error (under the transformation point of view [5]) or a set of replicas with index q of the principal term (from a function analysis perspective).

In other words, by sampling the continuous periodic function in the spatial harmonics domain, we get infinite replicas (shifted by $m = N$) of the discrete aperiodic function, eq.(3), in the excitation mode domain. As will be explained in Section V, this leads to aliasing problems which indeed define the choice of the virtual array size.

As explained in [2],[5] the replicas ε_m of the principal term in eq.(4), also called higher-order distortion modes, have to be minimized in order to get closer to the ideal (continuous) case, i.e. for reducing the bias term on the DoA estimates.

IV. Modal Transforms Criteria for Virtual Array Size

In this section two modal transforms are presented. In particular we focus on the criteria used for determining the size of the virtual array $\mathcal{M}$ after the transforms. Some methods originally designed for ULA, e.g. root-MUSIC and spatial smoothing [2]-[4] use these modal transforms in order to build the desired structure of the steering vectors (i.e. the Vandermonde structure) that is exploited in finding the DoA's.

It can be easily shown that the size of the virtual array $\mathcal{M}$ is always odd [5]. For this reasons we can write that $\mathcal{M} = 2M + 1$ where M is the maximum excitation mode order. Overall it is important to remember that the modal transforms can not make the size of the physical array larger, e.g. UCA [1],[3]. This means that the constraint $\mathcal{M} \leq N$ has to be always verified due to spatial sampling conditions (Nyquist criterion).

A. Davies Transform Criterion

The idea of Davies transform is closely related to the study of phase mode excitation [3]. The key point is to create a transformation matrix (see [1],[6] for details) for mapping from UCA into a virtual array. In literature, some criteria for determine the virtual array size can be found. For instance, in [6] it is suggested to use a square transformation matrix such that the size of UCA and the virtual array are the same $\mathcal{M} = N$. Unfortunately this approach can give some problems for UCA with an even number of elements as shown in Section V [3].

A more advanced criterion can be found in [3],[7] where the maximum excitation mode order M is chosen as

$$\max\left\{ M \middle| M \leq \frac{N-1}{2} \text{ and } \frac{|J_{M-N}(\kappa r)|}{|J_M(\kappa r)|} < \epsilon \right\} . \quad (5)$$

Here the first inequality relates to the spatial sampling condition and the second condition determines the accuracy of the approximation by giving a measure of the effect of the aliasing on the highest mode M. It is important to remember that the aliasing (i.e. the residual term) may introduce an error in the transform that could lead to biased DoA estimates [5]. For example, let us consider Fig.2a with $M = 3$ and $M - N = -4$. For the excitation mode $m = 3$, the $m = -4$ mode of the right replica is interfering with the $m = 3$ mode of the dominant term. The ratio of the two modes gives a measure of aliasing.

The criterion has the drawback of requiring some preprocessing for determining the virtual array size and it shows a trade-off between the aperture of the virtual array and the accuracy of the transform.

B. Beamspace Transform Criterion

The Beamspace Transform (BT) is done by employing a $\mathcal{M} \times N$ beamformer $\mathbf{F}_e^H$ (see [2] for details) for mapping between UCA and the virtual array. Notice that the transformation matrix is in general not square and $\mathcal{M} \leq N$.

A rule of thumb for computing the highest order mode M is to consider the smallest integer that is close or equal to κr, where r is the array radius and $\kappa = \frac{\omega}{c}$ is the wavenumber. In this way the excitation modes are $m \in [-M, M]$. For more details, see [2],[5]. It is interesting to notice that also Davies gave a similar criterion for selecting the maximum excitation mode order [1].

This approach has the advantage of being simple and straight forward. On the other hand it does not optimize the virtual array size (aperture) and, for certain UCA configuration, it does not select a proper number of modes, see Section VI for numerical examples. Consequently, the resolution of the array may reduce and the statistical performance of the DoA estimation algorithm could remain far from the theoretical limit (CRB), see Figures 3 and 4.

V. Effective Aperture Distribution Function

The Effective Aperture Distribution Function (EADF) represents a way of modelling the beampattern of antenna arrays as a function of the azimuth and elevation angles of incoming waves [8].

An antenna array response to a far field source can be model by measuring the directional characteristic of the antenna in an anechoic chamber. For our purposes, we may measure the array response to a far field source by moving it around the array at a fixed elevation angle $\theta = 90^\circ$ along the azimuthal direction in the range $\phi \in [-\pi, \pi)$. This creates a discrete set of measured point along the direction ϕ which represents a discrete periodic function with period 2π in azimuth. Hence, the beam pattern can be expressed by an inverse FFT (Fast Fourier Transform) of the previously measured data. We will refer to this Fourier series as the Effective Aperture Distribution Function.

In Fig.1 we depict three EADF's as a function of the order m for different values of the interelement spacing d. In particular an UCA with $N = 12$ omnidirectional sensors and $d = \{0.3\lambda,\ 0.4\lambda,\ 0.5\lambda\}$ have been considered. From the picture we first observe that the tails of the EADF (for high mode orders) show a significant increase in magnitude as the interelement spacing increases. Hence, see Fig.2, the influence of the replicas of the EADF on the dominant term increases, i.e. the aliasing is larger. Notice that this can also be seen as an amplification of the error after the BT [5]. Moreover, the EADF is a discrete aperiodic function with an infinite number of modes [3] as described in the case of continuous circular array, eq.(3). However, since the magnitude decreases rapidly as the mode order increases, it is reasonable to consider only a finite number of modes [1].

In Fig.2 we show the effect of the spatial sampling on the EADF. In case of UCA, eq.(4) the continuous periodic spatial harmonic function is sampled by the array sensors at a rate $\gamma_n = \frac{2\pi n}{N}$ (for $n = 0, \ldots, N-1$). Replicas of the original EADF, shifted by N, are hence created in the excitation modes order domain. Clearly, these replicas interfere with the original EADF by causing aliasing. Notice that, in order to make the illustration clearer, only the first left and right replicas (for $q = 1$) have been depicted.

The EADF clearly illustrates the impact of the spatial sampling on the excitation modes. Consequently, we can choose the virtual array size optimally such that its aperture is as large as possible, as it will be described in the next section.

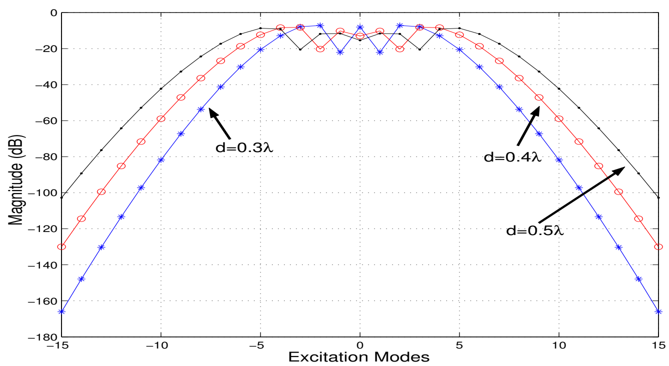

Fig. 1. Comparison between the excitation modes amplitude for an UCA with $N = 12$ sensors at $f = 1.8$ GHz. Clearly the magnitude (in dB) increases significantly as the interelement spacing d increases.

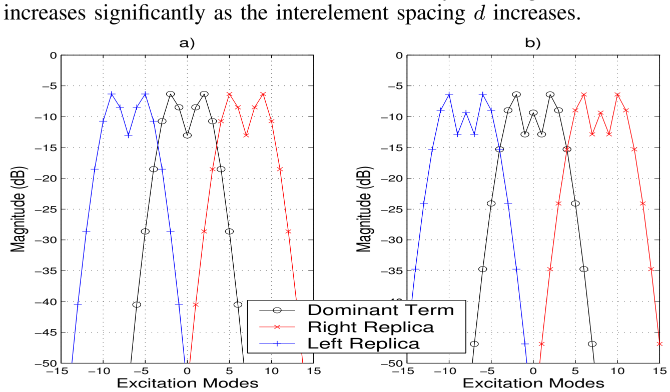

Fig. 2. Representation of the excitation modes belonging to the dominant term and the first left and right replicas of it due to the spatial sampling by the UCA sensors. Here $d = 0.4\lambda$, $f = 1.8$ GHz, $N = 7$ and $N = 8$ in a) and b) respectively.

VI. Virtual Array Size

Here we propose a criterion for selecting the number of virtual array elements $\mathcal{M}$ based on the analysis of the EADF. As it is clearly depicted in Fig.2, for every UCA configuration we want to select all the available excitation modes where the magnitude value of the dominant mode is larger than the magnitude of the replicas (aliasing mode). As a result, the rule of thumb for choosing the total number of excitation modes $\mathcal{M}$ (i.e. the size of the virtual array) is:

$$\begin{cases} \mathcal{M} = N - 1 & \text{for arrays with an EVEN number of sensors} \\ \mathcal{M} = N & \text{for arrays with ODD number of sensors} \end{cases} \quad (6)$$

where N denotes the number of sensors in the UCA.

This approach has the advantage of being simple to use and optimum with respect to the size of the virtual array. In fact, the aperture of the virtual array is relatively large since we choose always as much elements as possible. Notice that the criterion in eq.(6) is similar to the one in eq.(5). However, since in this paper we reduce the bias by preprocessing, we can avoid the second condition in (5) and write a more compact criterion as in (6).

In Table I we compare the total number of excitation modes $\mathcal{M}$ that the BT criterion (Section IV-B) and the criterion in eq.(6) suggest to use in the modal transforms. For several UCA configurations, the criterion in eq.(6) gives a larger $\mathcal{M}$ than the BT criterion, and it always satisfies the constraint $\mathcal{M} \leq N$. Hence, the proposed criterion tends to select a virtual array with as large aperture as possible. Consequently, as will be show in Section VII, the resolution limit of the virtual array is improved and we can significantly reduce the excess variance.

It is also interesting to notice that the BT criterion gives an improper value for $\mathcal{M}$ in case of even sized UCA and $d = 0.5\lambda$ because it does not satisfy the constraint $\mathcal{M} \leq N$. On the other hand the proposed criterion, eq.(6), still gives a reasonable/optimal value.

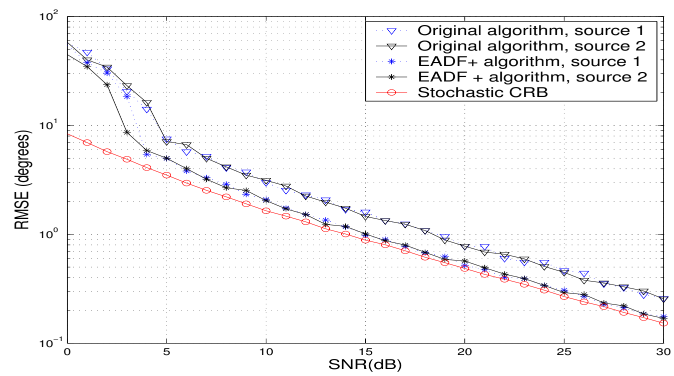

Fig. 3. Performances of UCA Unitary root-MUSIC (with bias correction) when the total number of modes is selected either with the conventional rule of thumb ($\mathcal{M} = 5$) or with our approach based on EADF ($\mathcal{M} = 7$). In the latter case we get closer to the CRB. Settings: $N = 7$, $f = 1.8$ GHz, $d = 0.3\lambda$ and $(\phi_1, \phi_2) = (10°, 20°)$.

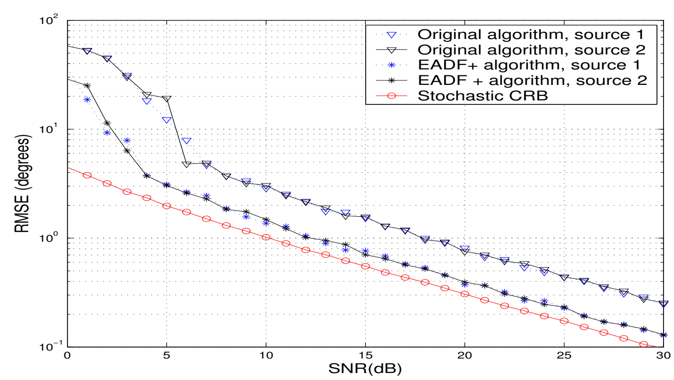

Fig. 4. Performances of UCA Unitary root-MUSIC (with bias correction) when the total number of modes is selected either with the conventional rule of thumb ($\mathcal{M} = 5$) or with our approach based on EADF ($\mathcal{M} = 7$). In the latter case the excess variance is reduced. Settings: $N = 7$, $d = 0.4\lambda$ and $(\phi_1, \phi_2) = (10°, 20°)$.

VII. Simulation Results

In this section some simulation results are presented. In order to provide DoA estimates the UCA Unitary root-MUSIC algorithm [4], in conjunction with the bias correction procedure [5], has been used.

TABLE I

TOTAL NUMBER OF EXCITATION MODES $\mathcal{M}$ FOR UCA'S WITH SEVERAL NUMBER OF SENSORS N AND DIFFERENT INTERELEMENT SPACINGS d. HERE WE COMPARE THE TOTAL NUMBER OF MODES COMPUTED: FIRST BY THE RULE OF THUMB OF THE ORIGINAL BEAMSPACE TRANSFORM (BT) ($M \approx \kappa r$) AND SECOND BY ANALYZING THE NON-ALIASED TERMS OF THE EFFECTIVE APERTURE DISTRIBUTION FUNCTION (EADF).

	$d = 0.3\lambda$			$d = 0.4\lambda$			$d = 0.5\lambda$		
N	Radius	$\mathcal{M}$ (BT)	$\mathcal{M}$ (EADF)	Radius	$\mathcal{M}$ (BT)	$\mathcal{M}$ (EADF)	Radius	$\mathcal{M}$ (BT)	$\mathcal{M}$ (EADF)
3	0.1732λ	3	3	0.2309λ	3	3	0.2887λ	3	3
4	0.2121λ	3	3	0.2828λ	3	3	0.3536λ	5	3
5	0.2552λ	3	5	0.3403λ	5	5	0.4253λ	5	5
6	0.3λ	3	5	0.4λ	5	5	0.5λ	7	5
7	0.3457λ	5	7	0.4610λ	5	7	0.5762λ	7	7
8	0.3920λ	5	7	0.5226λ	7	7	0.6533λ	9	7
9	0.4386λ	5	9	0.5848λ	7	9	0.7310λ	9	9
10	0.4854λ	7	9	0.6472λ	9	9	0.8090λ	11	9
11	0.5324λ	7	11	0.7099λ	9	11	0.8874λ	11	11
12	0.5796λ	7	11	0.7727λ	9	11	0.9659λ	13	11
24	1.1492λ	15	23	1.5323λ	19	23	1.9153λ	25	23
32	1.5303λ	19	31	2.0405λ	25	31	2.5506λ	33	31

The correction procedure indeed mitigates the effect of the replicas of the EADF by cancelling the bias term. Moreover it allows us to focus on the excess variance introduced by the modal transform since it is the only remaining error term. In fact, in both pictures, the constant gap in between the CRB and the estimated Root Mean Square Error (RMSE) is due to the excess variance term.

In Fig.3 we can see that the extra variance term can be significantly reduced by properly selecting the number of virtual array elements $\mathcal{M}$. The conventional beamspace transform suggest to select $\mathcal{M} = 5$ elements for this array configuration. Instead, the proposed criterion based on the EADF suggests to increase the aperture of the virtual array by selecting $\mathcal{M} = 7$ elements. Notice that in this way we optimize the virtual array size subjected to $\mathcal{M} \leq N$. As a result the variance of the estimates is closer to the CRB.

In Fig.4 we also demonstrate that the excess variance can be reduced by a proper selection of $\mathcal{M}$.

In [9] it has been proven that working in the beamspace domain leads to a loss in statistical performance. Consequently, the CRB can not be achieved. The gap left between the RMSE and CRB curves depends on the beamformer [9]. However, since the beamformer used in the BT is a function of d [5], the loss in performance is proportional to d as well. Consequently, the gap increases as d increases.

Comparing Fig.3 and 4, we observe that even though we are more far from the CRB for large d, the RMSE (and of course the CRB) are smaller in case of $d = 0.4\lambda$ than $d = 0.3\lambda$. As a result, it is better to use an array with large aperture.

VIII. CONCLUSIONS

In this paper we have presented a simple and efficient approach for selecting the number of elements of the virtual array. As a result we optimize the size of the virtual array by generally making its size larger. Notice that the proposed criterion always satisfies the modal transformation constraints. Consequently, the excess variance in the DoA estimations is reduced and we get closer to the CRB.

REFERENCES

[1] Davies, D.E.N.; *The handbook of antenna design*. A. W. Rudge et al., Edit. London: Peregrinus, 1983, vol. 2, Ch.12.

[2] Mathews, C.P., Zoltowski, M.D., *Eigenstructure techniques for 2-D angle estimation with uniform circular arrays*. IEEE Tr. Signal Processing, Vol. 42 No. 9, 1994, pp. 2395-2407.

[3] Lau, B.K.; *Applications of Adaptive Antennas in Third-Generation Mobile Communications Systems*. PhD Thesis Dissertation, Australian Telecommunications Research Institute, Curtin University of Technology, Australia, 2002.

[4] Belloni, F., Koivunen, V.; *Unitary root-MUSIC technique for Uniform Circular Array*. Proc. of ISSPIT 2003, Darmstadt, Germany, December 14-17, 2003.

[5] Belloni, F., Koivunen, V.; *Beamspace Transform for UCA: Error Analysis and Bias Reduction*. Technical Report No.48, ISBN: 951-22-7292-X, Signal Processing Laboratory, Helsinki University of Technology, 2004.

[6] Tewfik, A.H.; Hong, W.; *On the application of Uniform Linear Array bearing estimation techniques to Uniform Circular Array*. IEEE Tr. Signal Processing, Vol. 40 No. 4, April 1992, pp. 1008-1011.

[7] Wax, M.; Sheinvald, J.; *Direction finding of coherent signals via spatial smoothing for uniform circular arrays*. IEEE Tr. Antennas and Propagation, Vol. 42 No. 5, May 1994, pp. 613-620.

[8] Richter, A.; *Estimation of Radio Channel Parameters: Models and Algorithms*. PhD Thesis Dissertation, Technische Universität Ilmenau, Ilmenau, Germany, 2005.

[9] Stoica, P.; Nehorai, A.; *Comparative performance of element-space and beam-space MUSIC estimators*. Circuits, Syst. Signal Processing, Vol.10, 1991, pp. 285-291.