# Analysis of Direct Position Determination Approach in the Presence of Model Errors

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***Abstract*— The performance of the *Direct Position Determination* (DPD) approach in the presence of model errors is examined. DPD was recently introduced as a promising technique for localization of multiple radio frequency emitters with superior accuracy under low signal to noise ratio conditions. We analyze the performance of DPD in the presence of model errors caused by multipath, calibration errors, mutual coupling, etc. The analysis is general enough to encapsulate various sources of errors. Monte-Carlo simulations are used to validate the analysis. We show that in many cases of interest DPD should be selected as the preferred method of localization.**

***Keywords*— Array-Processing, Emitter-Localization, Angle of Arrival.**

## I. INTRODUCTION

Most localization systems are based on a two step procedure. In the first step the signal parameters (Angle of Arrival, Time of Arrival, Signal Strength) are measured at several stations. In the second step a central station uses the measurements for estimating the emitter location. The two-step procedure is also known as the decentralized approach [1], [2]. To date, not much work have been published that shed light on the performance gap between decentralized and centralized approaches. The DPD is a centralized approach and its performance analysis may help to quantify the accuracy loss caused by using the decentralized approach.

According to the DPD approach each base station transfers the intercepted signals to a central processing unit where the set of positions that best matches all the collected data simultaneously, is determined. Although there are many stray parameters only a two dimensional search is required for a two dimensional geometry. In [1] we compared the performance of DPD with the performance of Angle of Arrival (AOA) for unknown as well as known waveforms. We showed that while for high signal to noise ratio (SNR), both methods converge to the Cramer-Rao Lower Bound (CRLB), for low SNR, DPD achieves better accuracy.

In this correspondence we focus on the performance of DPD in the presence of model errors for signals with known waveforms. Model errors such as calibration errors or multipath can cause errors that are not consistent with the rest of the stations' measurements. In this case the DPD is superior to the traditional methods since it inherently ignores the observations of the bad stations and determines the position based on the rest of the observations. The derivation of the DPD for known signals follows the derivation of the AOA estimator in [3]. We use model errors similar to those proposed in [4]. We assume that the errors can be characterized statistically and we obtain closed form expressions for the covariance matrix of the estimated positions. Monte-Carlo simulations are used to validate the analysis and demonstrate that in the presence of various model errors, DPD provides a better location estimate than the common AOA method.

## II. PROBLEM FORMULATION

Consider $Q$ transmitters and $L$ base stations intercepting the transmitted signals. Each base station is equipped with an antenna array consisting of $M$ elements. Assume that the signal bandwidth is small compared to the inverse of the propagation time over the array aperture. Denote the $q$-th transmitter's position by the vector of Cartezian coordinates $\mathbf{p}_q = [x, y]^T$.
The complex envelopes of the signals observed by the $\ell$-th base station array is given by

$$\mathbf{r}_\ell(t) = \sum_{q=1}^{Q} b_{\ell q}\mathbf{a}_\ell(\mathbf{p}_q)s_q(t - \tau_\ell(\mathbf{p}_q) - t_q^{(0)}) + \mathbf{n}_\ell(t) \quad (1)$$

where $0 \leq t \leq T$, $\mathbf{r}_\ell(t)$ is a complex time-dependent $M \times 1$ observation vector, $b_{\ell q}$ is an *unknown* complex scalar representing the channel attenuation between the $q$-th transmitter and the $\ell$-th base station, $\mathbf{a}_\ell(\mathbf{p}_q)$ is the array response of the $\ell$-th station to a signal transmitted

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from position $\mathbf{p}_q$, $s_q(t)$ is the $q$-th signal transmitted at time $t_q^{(0)}$ and delayed by $\tau_\ell(\mathbf{p}_q)$ and $\mathbf{n}_\ell(t)$ represents noise and interference observed by the array.

The observed signal is partitioned into $K$ sections each of length of $T/K \gg \max\{\tau_\ell(\mathbf{p}_q)\}$ where the maximum delay is the propagation time between the two most separated stations [2]. Each section can be Fourier transformed and the result of this process is given by

$$\mathbf{r}(j,k) = \mathbf{A}(j)\bar{\mathbf{s}}(j,k) + \mathbf{n}(j,k) \tag{2}$$

where we defined

$$\begin{aligned}
\mathbf{r}(j,k) &\triangleq \left[\mathbf{r}_1^T(j,k), \cdots, \mathbf{r}_L^T(j,k)\right]^T && (3)\\
\mathbf{n}(j,k) &\triangleq \left[\mathbf{n}_1^T(j,k), \cdots, \mathbf{n}_L^T(j,k)\right]^T && (4)\\
\mathbf{A}(j) &\triangleq \left[\mathbf{A}_1^T(j), \cdots, \mathbf{A}_L^T(j)\right]^T && (5)
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{r}_\ell(j,k) &= \mathbf{A}_\ell(j)\bar{\mathbf{s}}(j,k) + \mathbf{n}_\ell(j,k) && (6)\\
\mathbf{A}_\ell(j) &\triangleq [\bar{\mathbf{a}}_\ell(j,\mathbf{p}_1,b_{\ell 1}), \cdots, \bar{\mathbf{a}}_\ell(j,\mathbf{p}_Q,b_{\ell Q})] && (7)\\
\bar{\mathbf{s}}(j,k) &\triangleq [\bar{s}_1(j,k), \cdots, \bar{s}_Q(j,k)]^T && (8)
\end{aligned}$$

with

$$\begin{aligned}
\bar{s}_q(j,k) &\triangleq s_q(j,k)e^{-i\omega_j t_q^{(0)}} && (9)\\
\bar{\mathbf{a}}_\ell(j,\mathbf{p}_q,b_{\ell q}) &\triangleq b_{\ell q}\mathbf{a}_{\ell q}e^{-i\omega_j \tau_{\ell q}} && (10)
\end{aligned}$$

We also assume that the signals and the noise are uncorrelated. Next define the matrices

$$\begin{aligned}
\mathbf{R}_{\mathbf{ss}}(j) &\triangleq \frac{1}{K}\sum_{k=1}^{K}\bar{\mathbf{s}}(j,k)\bar{\mathbf{s}}^H(j,k) && (11)\\
\mathbf{R}_{\mathbf{sr}}(j) &\triangleq \frac{1}{K}\sum_{k=1}^{K}\bar{\mathbf{s}}(j,k)\mathbf{r}^H(j,k) && (12)\\
\mathbf{U}(j) &\triangleq \mathbf{R}_{\mathbf{sr}}^H(j)\mathbf{R}_{\mathbf{ss}}^{-1}(j) && (13)
\end{aligned}$$

The matrix $\mathbf{R}_{\mathbf{ss}}(j)$ becomes diagonal for large $K$ if the signals are uncorrelated. Finally, denote the $q$-th column of $\mathbf{U}(j)$ by $\mathbf{u}_q(j)$ and its $\ell$-th sub vector by $\mathbf{u}_{\ell q}(j)$.

### A. *DPD Estimator*

In [2] we derive the DPD estimator for general noise covariance and the result is summarized below

$$\hat{\mathbf{p}}_q = \underset{\mathbf{p}_q}{\operatorname{argmax}}\left\{\mathbf{u}_q^H\overline{\mathbf{W}}_q^{-\frac{1}{2}}\mathbf{P}_{\bar{\mathbf{A}}_q}\overline{\mathbf{W}}_q^{-\frac{1}{2}}\mathbf{u}_q\right\} \tag{14}$$

where

$$\begin{aligned}
\mathbf{u}_q &\triangleq \left[\mathbf{u}_q^T(1), \cdots, \mathbf{u}_q^T(J)\right]^T && (15)\\
\bar{\mathbf{A}}_q &\triangleq \overline{\mathbf{W}}_q^{-\frac{1}{2}}\mathbf{\Gamma}_q\mathbf{H} && (16)\\
\mathbf{\Gamma}_q &\triangleq \left[\mathbf{\Gamma}_q^T(1), \cdots, \mathbf{\Gamma}_q^T(J)\right]^T && (17)\\
\overline{\mathbf{W}}_q &\triangleq \mathbf{G}_{q,q}^{-1} \otimes \mathbf{W} && (18)\\
\mathbf{G}_{q,q} &\triangleq \operatorname{diag}\left\{[\mathbf{R}_{\mathbf{ss}}(1)]_{q,q}, \cdots, [\mathbf{R}_{\mathbf{ss}}(J)]_{q,q}\right\} && (19)\\
\mathbf{P}_{\bar{\mathbf{A}}_q} &\triangleq \bar{\mathbf{A}}_q\bar{\mathbf{A}}_q^{+} && (20)
\end{aligned}$$

with $\mathbf{\Gamma}_q(j) \triangleq \operatorname{diag}\{\mathbf{a}_{1q}^T e^{-i\omega_j\tau_{1q}}, \cdots, \mathbf{a}_{Lq}^T e^{-i\omega_j\tau_{Lq}}\}$, $\mathbf{b}_q \triangleq [b_{1q}, \cdots, b_{Lq}]^T$, $\mathbf{H} \triangleq \mathbf{I}_L \otimes \mathbf{1}_M$ and $\bar{\mathbf{A}}_q^+$ is the pseudo-inverse (moore-penrose inverse) of $\bar{\mathbf{A}}_q$. Note that each emitter is estimated separately and the similarity to [3].

## III. Model Errors

Errors in the position estimates can arise from different number of sources including non-line-of-sight propagation, array calibration errors, sensor positioning errors, *etc.* We first discuss errors in the response matrix. In this case the observation vector in (2) is given by

$$\hat{\mathbf{r}}(j,k) = \left(\mathbf{A}(j) + \tilde{\mathbf{A}}(j)\right)\bar{\mathbf{s}}(j,k) + \mathbf{n}(j,k) \tag{21}$$

We reserve the $\tilde{\cdot}$ sign for the errors and the $\hat{\cdot}$ sign for the model with the errors.
We examine the statistical effects of the errors. Similarly to the analysis in [4], we assume that the first and second order statistics of the errors are given and that $\tilde{\mathbf{A}}(j)$ and $\mathbf{n}(j,k)$ are statistically independent. Define the operator vec$(\cdot)$ as the concatenation of a matrix columns. We assume that $\mathbf{m}_{\tilde{\mathbf{A}}} = \mathrm{E}[\tilde{\mathbf{A}}(j)]$ is zero and the covariance matrix of vec$(\tilde{\mathbf{A}})$ is

$$\mathbf{\Sigma}_{\tilde{\mathbf{A}}} = \mathrm{E}[\mathrm{vec}(\tilde{\mathbf{A}}(j))\mathrm{vec}^H(\tilde{\mathbf{A}}(j))]; \forall j \tag{22}$$

We also assume that the errors associated with different sources are statistically independent and the errors associated with different base stations are also statistically independent. Under these assumptions, $\mathbf{\Sigma}_{\tilde{\mathbf{A}}}$ is a block diagonal matrix.

### A. *Small Error Analysis*

The error covariance matrix of the DPD estimated positions, $\widehat{\mathbf{C}}_{q,q} \triangleq \mathrm{E}\left[\Delta\hat{\mathbf{p}}_q\Delta\hat{\mathbf{p}}_q^T\right]$, is analyzed in [2] ,using small error assumptions and is summarized below

$$\begin{aligned}
\widehat{\mathbf{C}}_{q,q} &= \mathbf{C}_{q,q} + \widetilde{\mathbf{C}}_{q,q} && (23)\\
\mathbf{C}_{q,q} &= \frac{1}{2K}\mathrm{Re}\left\{(\mathbf{\Omega}_q^H\mathbf{\Omega}_q)^{-1}\right\} && (24)\\
\widetilde{\mathbf{C}}_{q,q} &= \frac{1}{2}\mathrm{Re}\left\{\mathbf{\Omega}_q^+\overline{\mathbf{W}}_q^{-\frac{1}{2}}\left[\mathbf{\Sigma}_{\tilde{\mathbf{A}}}\right]_{q,q}^{ML}\overline{\mathbf{W}}_q^{-\frac{1}{2}}\mathbf{\Omega}_q^{+H}\right\} && (25)
\end{aligned}$$

where $\mathbf{C}_{q,q}$ is associated with measurement noise and $\widetilde{\mathbf{C}}_{q,q}$ is associated with model errors and

$$\mathbf{\Omega}_q \triangleq \mathbf{P}^{\perp}_{\bar{\mathbf{A}}_q} \widetilde{\mathbf{D}}_q \left[\mathbf{I}_2 \otimes \mathbf{b}_q\right] \quad (26)$$

$$\widetilde{\mathbf{D}}_q \triangleq \left[\begin{array}{cc} \overline{\mathbf{W}}_q^{-\frac{1}{2}} \frac{\partial \mathbf{\Gamma}_q}{\partial x_q}\mathbf{H} & \overline{\mathbf{W}}_q^{-\frac{1}{2}} \frac{\partial \mathbf{\Gamma}_q}{\partial y_q}\mathbf{H} \end{array}\right] \quad (27)$$

where

$$\mathbf{P}^{\perp}_{\bar{\mathbf{A}}_q} \triangleq \mathbf{I} - \mathbf{P}_{\bar{\mathbf{A}}_q} \quad (28)$$

and $[\cdot]^{ML}_{q,q}$ stands for the $q,q$-th $ML \times ML$ submatrix, $\mathbf{P}_{\bar{\mathbf{A}}_q}$ is defined in (20) and $\mathbf{\Omega}_q^{+}$ is the pseudo-inverse (Moore-Penrose inverse) matrix of $\mathbf{\Omega}_q$.

## IV. CASES OF MODEL ERRORS

In this section we analyze two special cases of model errors: imprecise knowledge of the sensor positions and scattering environment (multipath).

### *A. Imprecise Knowledge of Sensor Locations*

In practice the assumption that the sensor locations are known precisely does not always hold. Sensors mounted on moving platforms such as cars, ships, airplanes are prone to location errors. Assume that $L_e$ out of the $L$ arrays suffer from position errors and in each array only $M_e$ of the $M$ elements is displaced. The response of the $m$-th element in the $\ell$-th array is

$$\hat{\mathbf{a}}_{\ell,q}(m) = e^{-i\xi[(x_{\ell m}+\tilde{x}_{\ell m})\cos\theta_{\ell q}+(y_{\ell m}+\tilde{y}_{\ell m})\sin\theta_{\ell q}]} \quad (29)$$

where $\xi \triangleq 2\pi/\lambda$ and $(x_{\ell m}, y_{\ell m})$ are the assumed sensor coordinates while $(\tilde{x}_{\ell m}, \tilde{y}_{\ell m})$ are the errors. We assume that the errors are Gaussian random variables with zero mean and variance of $\epsilon^2$.

In [2] we show that the effect of the sensor position errors is approximately the same order as the effect of the noise if

$$\epsilon_t^2 \cong \left(\frac{2}{\xi}\right)^2 \frac{M^2 L}{2 M_e L_e}\gamma^2 \quad (30)$$

where $\gamma^2 \triangleq \sigma^2/KE_q$ where $E_q \triangleq J^{-1}\sum_j [\mathbf{R}_{\mathbf{ss}}(j)]_{q,q}$. A comparison with AOA under sensor location errors is provided in section V.

### *B. Scattering Environment*

We discuss *small local scattering* near the sources that was introduced in [6]. We assume that the $\ell$-th array response to a signal transmitted from the *q*-th source is

$$\hat{\mathbf{a}}_{\ell q} = \sum_{n=0}^{N} \gamma_{\ell,q}(n)\mathbf{a}_\ell\left(\mathbf{p}_q + \Delta\mathbf{p}_{\ell,q}(n)\right) \quad (31)$$

where $n$ is the index of the scatterers, $\gamma_{\ell,q}(n)$ is the attenuation of the *n*-th scatterer and $\Delta\mathbf{p}_{\ell,q}(n)$ is the position vector of the *n*-th scatterer w.r.t. the *q*-th source position. Assume that $L_{eq}$ of the $L$ arrays are affected by scattering while intercepting the $q$-th source. Since we assumed small local scattering near each source, we can approximate the array response using a first order *Taylor* expansion

$$\mathbf{a}_\ell\left(\mathbf{p}_q + \Delta\mathbf{p}_{\ell,q}(n)\right) \cong \mathbf{a}_{\ell q} + \frac{\partial \mathbf{a}_{\ell q}}{\partial \mathbf{p}_q}\Delta\mathbf{p}_{\ell,q}(n) \quad (32)$$

We also adopt some assumptions from [6]: each of the scatterers attenuation is a complex Gaussian random variable with zero mean and variance $1/N$ and the coordinates of the scatterer location are i.i.d. with zero mean and standard deviation $\Delta_q$ (also referred to as *scattering radius*). In [2] we show that for a uniform linear array (ULA), the effect errors due to scattering is approximately the same order as the effect of the noise if

$$\Delta_q^2 \cong \frac{ML}{\sum_{\ell=1}^{L_{eq}}\left(\sum_m \mathbf{a}_{\ell q}(m)\right)^2 \left(\frac{\xi\delta\sin\theta_{\ell q}}{r_{\ell q}}\right)^2}\gamma^2 \quad (33)$$

where $\xi = 2\pi/\lambda$, the separation between array elements is denoted by $\delta$ and $r_{\ell q}, \theta_{\ell q}$ are the distance and AOA associated with the *q*-th source and the $\ell$-th station.

## V. NUMERICAL RESULTS

In order to compare the DPD with AOA based localization we performed extensive Monte-Carlo simulations. We considered the following setting: Four stations located at the corners of a 4[Km]× 4[Km], single emitter positioned at the square center with LoS attenuation to each station equal to one square, each station is equipped with a uniform linear array with broadside towards the square center. Each point on each plot is the result of 100 Monte-Carlo experiments. Each location determination is based on a specified number, $K$, of 4.5 [msec] snapshots using a single frequency bin (*i.e.*, $J = 1$). The snapshot length ensures that the errors introduced by the finite length FFT are 30 [dB] below the signal level. The transmitted waveforms are realizations of a normal Gaussian random process. To measure accuracy we used the Root-Mean-Square (RMS) error of the miss distance.

**Displaced Sensors** Each of the stations is equipped with a uniform linear array of 11 elements. Only the array elements of the first station are displaced. The standard deviation of the location error is $\epsilon$. We discuss the performance of DPD and of AOA as a function of $\epsilon/\lambda$ and the number of displaced elements, $M_e$. The number of displaced elements was changed between 1 and 11. The RMS errors of both DPD and AOA are shown in

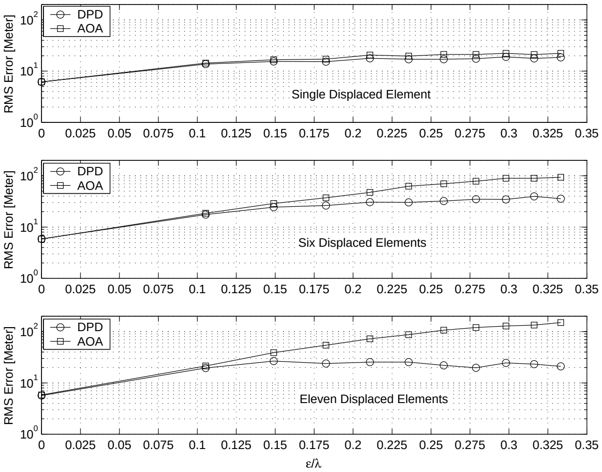

Fig. 1. RMS Error of AOA and DPD for One, Six and Eleven Displaced Elements

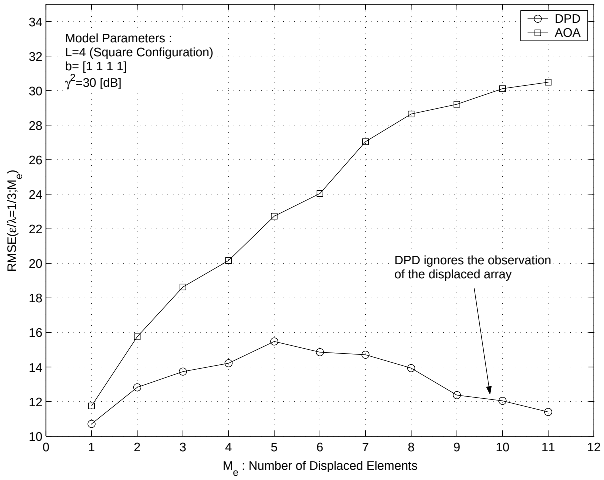

Fig. 2. RMS Error of AOA and DPD as a Function of the Number of Displaced Elements

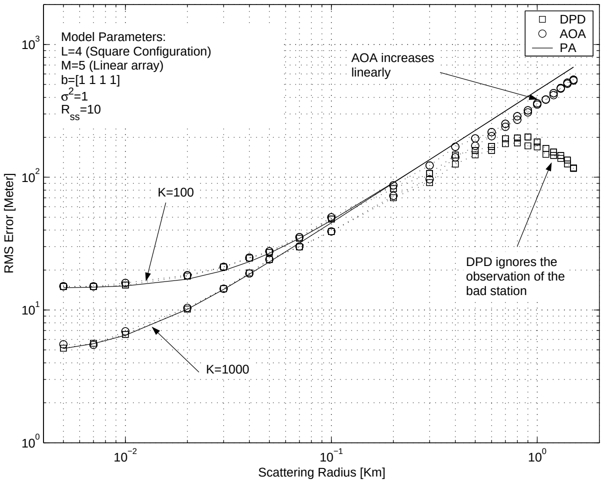

Fig. 3. RMS error of DPD, AOA and the Performance Analysis (PA) in the Presence of Multipath

figure 1 where we plotted the error of DPD and AOA as a function of the standard deviation for three cases :1,6 and 11 displaced elements. In figure 2 we plotted the RMS vs number of displaced elements for $\epsilon/\lambda = 1/3$. For small $M_e$, the difference between DPD and AOA is small. However, as $M_e$ increases, the AOA error increases, while the DPD error is not monotonous. The explanation of this phenomena is that DPD inherently ignores large errors that are not consistent with the rest of the measurements.

**Scattering Environment:** Each of the three base stations is affected by 50 scatterers positioned at random in close proximity to the emitter. The scatterers affecting the forth station were randomly positioned with radius increasing from 5 [meters] to 1500 [meters]. We examined the performance for $\sigma^2 = 1, \mathbf{R_{ss}} = 10$ and the two cases $K$=100,1000. The comparison between the two methods are shown in figure 3 where we also plotted the performance analysis (PA) introduced in section IV-B. When the scattering radius is small, the errors of both methods are comparable. However, as the radius increases, DPD outperforms AOA. The RMS of the AOA increases monotonically with the scattering radius, while that of DPD starts to decrease and ignores the contribution of the bad station. Observe that above certain radius the first order approximation no longer predicts the performance.

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