COMPLEX SINGULAR WISHART MATRICES AND MULTIPLE-ANTENNA SYSTEMS

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ABSTRACT

In this paper, complex singular Wishart matrices and their applications are investigated. In particular, a volume element on the space of positive semidefinite $m \times m$ Hermitian matrices of rank $n < m$ is introduced and some transformation properties are established. The Jacobian for the change of variables in the singular value decomposition of general $m \times n$ complex matrices is derived. Then the density functions are formulated for all rank n complex singular Wishart distributions. From this, the joint eigenvalue densities of low rank complex Wishart matrices are derived. Finally, using these densities spatially correlated multiple-antenna Rayleigh channel capacities are evaluated.

1. INTRODUCTION

Let an $n \times m$ complex Gaussian (or normal) random matrix $\mathbf{A}$ be distributed as $\mathbf{A} \sim \mathcal{C}N(M, I_n \otimes \Sigma)$ with mean $\mathcal{E}\{\mathbf{A}\} = M$ and covariance $\text{cov}\{\mathbf{A}\} = I_n \otimes \Sigma$. Here we read the symbol "$\sim$" as "is distributed as", $\mathcal{C}N$ denotes the complex normal distribution and $\otimes$ denotes the Kronecker product. Then the matrix $\mathbf{W} = \mathbf{A}^H\mathbf{A}$ is called a complex noncentral Wishart matrix. If $M = 0$, then $\mathbf{W}$ is called a complex central Wishart matrix. The complex central and noncentral Wishart distributions are denoted by $\mathcal{C}W_m(n, \Sigma)$ and $\mathcal{C}W_m(n, \Sigma, \Omega)$, respectively, where $\Omega = \Sigma^{-1}M^HM$. The complex Wishart matrices are well studied in the literature only for $n \geq m$, for example, see [4], [5], [9], [10], [12] and references therein.

In this paper, we extend the study of complex central Wishart distributions to the singular case, where $0 < n < m$ and $n, m \in \mathbb{Z}$. Thus the rank of $\mathbf{W} \in \mathbb{C}^{m \times m}$ is n provided the rank of $\mathbf{A} \in \mathbb{C}^{n \times m}$ is n. A volume element on the space of positive semidefinite $m \times m$ Hermitian matrices of rank $n < m$ is introduced (see Theorem 1). The Jacobian of the change of variables in the singular value decomposition of general $m \times n$ complex matrices is derived (see Theorem 2). The density is derived for rank-n complex central Wishart distributions for all integers n, $0 < n < m$ (see Theorem 3). The joint eigenvalue density of low rank complex Wishart matrices is derived (see Theorem 4). It should be noted that singular Wishart and beta distributions are studied in [17] for real random matrices. In [7], the author studied the complex pseudo-Wishart (or singular Wishart) distribution using a linear algebraic technique. However, our derived results and densities are more direct and simple.

The theory of complex singular Wishart matrices is used to evaluate the capacity of multiple-input multiple-output (MIMO) wireless communication systems. Let us denote the number of inputs (or transmitters) and the number of outputs (or receivers) of the MIMO wireless communication system by n_t and n_r, respectively, and assume that the channel coefficients are distributed as complex Gaussian and correlated at both the transmitter and the receiver ends. Then the MIMO Rayleigh flat fading channel can be represented by an $n_r \times n_t$ complex random matrix $\mathbf{H} \sim \mathcal{C}N(0, \Sigma_r \otimes \Sigma_t)$, where Σ_r and Σ_t are positive definite Hermitian matrices which represent the channel correlation at the receiver and transmitter ends, respectively [12]. This means the covariance matrices of the columns and rows of $\mathbf{H}$ are denoted by Σ_r and Σ_t, respectively. Note that if the channel is correlated only at the transmitter end and $n_t > n_r$ then $\mathbf{H}^H\mathbf{H}$ is a complex singular Wishart matrix. Similarly, if the channel is correlated only at the receiver end and $n_r > n_t$ then $\mathbf{H}\mathbf{H}^H$ is a complex singular Wishart matrix. Therefore, the study of complex singular Wishart matrices is important for evaluating the channel capacities in these situations. If $\Sigma_r = \sigma^2 I_{n_r}$ (or I_{n_r}) and $\Sigma_t = I_{n_t}$ (or $\sigma^2 I_{n_t}$) then the channel is said to be an *uncorrelated Rayleigh distributed channel*, and the complex nonsingular Wishart matrix theory can be used to compute the capacities for both cases $n_r \geq n_t$ and $n_t > n_r$, see [16]. However, if $\Sigma_r = I_{n_r}$ and $\Sigma_t = \Sigma_t$ then the complex nonsingular and singular Wishart matrix theories are needed to compute the capacities for the cases $n_r \geq n_t$ and $n_t > n_r$, respectively. See [9] for the case $n_r \geq n_t$. In this paper we assume that

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$\Sigma_r = I_{n_r}$, $\Sigma_t = \Sigma_t$ and $n_t > n_r$, which leads us to represent the channel capacity in the form of a complex singular Wishart matrix. This is the motivation behind this study.

This paper is organized as follows. Section 2 provides the necessary tools for deriving the complex singular Wishart distribution theory. Complex singular Wishart matrices are studied in Section 3. The capacity of a MIMO channel and the computational method are given in Section 4.

2. ESSENTIAL RESULTS

In this section, we derive necessary tools for studying the singular Wishart distribution theory and MIMO channel capacity. If $0 < n < m$, then the density does not exist for $\mathbf{W} \sim \mathcal{C}W_m(n, \Sigma)$ on the space of Hermitian $m \times m$ matrices because $\mathbf{W}$ is singular and of rank n almost surely. It can be shown that the density does exist on the $(2mn - n^2)$-dimensional manifold, $\mathcal{C}S_{m,n}$, of rank n of positive semidefinite $m \times m$ Hermitian matrices $\mathbf{W}$ with n distinct positive eigenvalues. Moreover, the set of all $m \times n$ matrices E_1 with orthonormal columns is called the *Stiefel manifold,* denoted by $\mathcal{C}V_{n,m}$. Thus,

$$\mathcal{C}V_{n,m} = \{E_1 \in \mathbb{C}^{m \times n}; E_1^H E_1 = I_n\}. \tag{1}$$

The elements of E_1 can be regarded as the coordinates of a point on a $(2mn - n^2)$-dimensional surface in the $2mn$-dimensional Euclidean space.

Theorems 1 and 2 below are derived by means of the exterior product approach. See [8], p. 57, for the definition of the exterior product (dX) for real matrices X, such as symmetric, skew-symmetric, upper-triangular and arbitrary matrices. On the other hand, if $X = X_r + iX_c$ is a complex matrix, its exterior product is $(dX) = (dX_r) \wedge (dX_c)$. Jacobian formulas for some important complex matrix factorizations are given in [11].

The volume element (dW) (or Jacobian) in the reduced spectral decomposition $W = E_1 \Lambda E_1^H$ is given by the following theorem. Note that the proofs of the theorems are omitted here for brevity, see [13] for details.

Theorem 1 *Let m and n be two positive integers such that $0 < n < m$ and consider an $m \times m$ positive semidefinite Hermitian matrix $W \in \mathcal{C}S_{m,n}$ of rank n with decomposition $W = E_1 \Lambda E_1^H$, where the diagonal elements of $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ are positive eigenvalues in decreasing order, $\lambda_1 > \cdots > \lambda_n > 0$, and $E_1 \in \mathcal{C}V_{n,m}$. Then the volume element is*

$$\begin{aligned}(dW) &= (2\pi)^{-n} \left(\prod_{k=1}^{n} \lambda_k^{2m-2n} \right) \prod_{k<l}^{n} (\lambda_k - \lambda_l)^2 \, (d\Lambda) \\ &\quad \times \wedge \, (E_1^H \, dE_1),\end{aligned} \tag{2}$$

where

$$(d\Lambda) = \bigwedge_{k=l}^{n} d\lambda_k, \qquad (E_1^H \, dE_1) = \bigwedge_{k=1}^{n} \bigwedge_{l=k}^{m} e_l^H \, de_k,$$

and the matrix E_1 is appended with an $m \times (m-n)$ matrix E_2 such that the compound $m \times m$ matrix, $E = [E_1 : E_2] = [e_1, \dots, e_n : e_{n+1}, \dots, e_m]$ is unitary.

The volume element (dW) in the singular value decomposition is given by the following theorem [13].

Theorem 2 *Let Z be an $m \times n$ complex matrix and $Z = E_1 \Upsilon H$ the nonsingular part of the singular value decomposition, where $E_1 \in \mathcal{C}V_{n,m}$, $H \in U(n)$ and the diagonal elements of $\Upsilon = \operatorname{diag}(\upsilon_1, \dots, \upsilon_n)$ are positive real singular values with $\upsilon_1 > \cdots > \upsilon_n > 0$. Then we have*

$$\begin{aligned}(dZ) &= (2\pi)^{-n} \left(\prod_{k=1}^{n} \upsilon_k^{2m-2n+1} \right) \prod_{k<l}^{n} (\upsilon_k^2 - \upsilon_l^2)^2 \\ &\quad \times (d\Upsilon) \wedge (E_1^H \, dE_1) \wedge (H^H dH)\end{aligned} \tag{3}$$

where

$$(d\Upsilon) = \bigwedge_{k=l}^{n} d\upsilon_k, \quad (H^H dH) = \bigwedge_{k=1}^{n} \bigwedge_{l=k}^{n} h_l \, dh_k,$$

$$(E_1^H \, dE_1) = \bigwedge_{k=1}^{n} \bigwedge_{l=k}^{m} e_l^H \, de_k,$$

and the matrix E_1 is appended with an $m \times (m-n)$ matrix E_2 such that the compound $m \times m$ matrix, $E = [E_1 : E_2] = [e_1, \dots, e_n : e_{n+1}, \dots, e_m]$ is unitary.

The volume of the Stiefel manifold $\mathcal{C}V_{n,m}$ is given by

$$\operatorname{Vol}(\mathcal{C}V_{n,m}) = \int_{\mathcal{C}V_{n,m}} (E_1^H \, dE_1) = \frac{2^n \pi^{mn}}{\mathcal{C}\Gamma_n(m)}, \tag{4}$$

where the complex multivariate gamma function is

$$\mathcal{C}\Gamma_n(a) = \pi^{n(n-1)/2} \prod_{k=1}^{n} \Gamma(a - k + 1), \quad \operatorname{Re}(a) > n - 1,$$

and the gamma function is define by $\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} \, dt$. The differential form

$$(dE_1) \triangleq \frac{1}{\operatorname{Vol}[\mathcal{C}V_{n,m}]} \, (E_1^H dE_1) = \frac{\mathcal{C}\Gamma_n(m)}{2^n \pi^{mn}} \, (E_1^H \, dE_1) \tag{5}$$

has the property that $\int_{\mathcal{C}V_{n,m}} (dE_1) = 1$, and it represents the Haar invariant probability measure on $\mathcal{C}V_{n,m}$. If $m = n$, then we get a special case of Stiefel manifold, the so-called unitary manifold, defined by

$$\mathcal{C}V_{n,n} \equiv U(n) = \{E \in \mathbb{C}^{n \times n}; E^H E = I_n\},$$

that is, the set of unitary $n \times n$ matrices. The volume of $U(n)$ is given by

$$\mathrm{Vol}\left[U(n)\right] = \int_{U(n)} (E^H\, dE) = \frac{2^n \pi^{n^2}}{\mathcal{C}\Gamma_n(n)}. \qquad (6)$$

The probability distributions of random matrices are often derived in terms of hypergeometric functions of matrix arguments. The following complex hypergeometric function of two Hermitian matrix arguments is required in the sequel, i.e.,

$${}_0F_0^{(n)}(X,Y) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(X) C_\kappa(Y)}{k! C_\kappa(I_m)}, \qquad (7)$$

where $X \in \mathbb{C}^{m\times m}$, $Y \in \mathbb{C}^{n\times n}$ and $0 < n < m$. Moreover, $\kappa = (k_1, \ldots, k_n)$ denotes a partition of the integer k with $k_1 \geq \cdots \geq k_n \geq 0$ and $k = k_1 + \cdots + k_n$ and $\sum_\kappa$ denotes summation over all partitions κ of k. The complex zonal polynomial of a Hermitian matrix Y defined in [5] is

$$C_\kappa(Y) = \chi_{[\kappa]}(1) \chi_{[\kappa]}(Y), \qquad (8)$$

where $\chi_{[\kappa]}(1)$ is the dimension of the representation $[\kappa]$ of the symmetric group,

$$\chi_{[\kappa]}(1) = k! \frac{\prod_{i<j}^n (k_i - k_j - i + j)}{\prod_{i=1}^n (k_i + n - i)!}, \qquad (9)$$

and $\chi_{[\kappa]}(Y)$ is the character of the representation $[\kappa]$ of the linear group given as a symmetric function of the eigenvalues, $\lambda_1, \ldots, \lambda_n$, of Y by

$$\chi_{[\kappa]}(Y) = \frac{\det\left[\left(\lambda_i^{k_j+n-j}\right)\right]}{\det\left[\left(\lambda_i^{n-j}\right)\right]}. \qquad (10)$$

Note that both the real and complex zonal polynomials are particular cases of the (general α) Jack polynomials $C_\kappa^{(\alpha)}(Y)$, where $\alpha = 1$ for complex and $\alpha = 2$ for real zonal polynomials, respectively. See [1] and [6] for details. In this paper we only consider the complex case; therefore, for notational simplicity we drop the superscript of Jack polynomials, as was done in equation (8), i.e., $C_\kappa(Y) := C_\kappa^{(1)}(Y)$. Finally, we have

$$C_\kappa(I_n) = k! \frac{\left[\prod_{i<j}^n (k_i - k_j - i + j)\right]^2}{\prod_{i=1}^n \Gamma(k_i + n - i + 1) \prod_{i=1}^n \Gamma(n - i + 1)}.$$

3. COMPLEX SINGULAR WISHART MATRICES

In this section, we derive the complex singular Wishart density and the joint eigenvalue density of the complex singular Wishart matrix.

Theorem 3 *Let m and n be two positive integers such that $0 < n < m$. The density of $\mathbf{W} \sim \mathcal{C}W_m(n, \Sigma)$ on the space $\mathcal{C}S_{m,n}$ of $m \times m$ positive semidefinite Hermitian matrices of rank n is given by*

$$f(W) = \frac{\pi^{n(n-m)}}{\mathcal{C}\Gamma_n(n)(\det \Sigma)^n} \,\mathrm{etr}\left(-\Sigma^{-1} W\right) (\det \Lambda)^{n-m}, \qquad (11)$$

where $W = E_1 \Lambda E_1^H$, $E_1 \in \mathcal{C}V_{n,m}$, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, and etr denotes the exponential of the trace, $\mathrm{etr}(\cdot) = \exp(\mathrm{tr}(\cdot))$.

The following theorem gives the joint density of the eigenvalues of a complex singular Wishart matrix.

Theorem 4 *Let m and n be two positive integers such that $0 < n < m$ and consider the $m \times m$ positive semidefinite Hermitian matrix $\mathbf{W} \sim \mathcal{C}W_m(n, \Sigma)$. The joint density of the positive eigenvalues, $\lambda_1, \ldots, \lambda_n$, of $\mathbf{W}$ is*

$$f(\Lambda) = \frac{\pi^{n(n-1)}(\det \Sigma)^{-n}}{\mathcal{C}\Gamma_n(n)\mathcal{C}\Gamma_n(m)} \left(\prod_{k=1}^n \lambda_k^{m-n}\right) \prod_{k<l}^n (\lambda_k - \lambda_l)^2 \times {}_0F_0^{(n)}(-\Sigma^{-1}, \Lambda), \qquad (12)$$

where $W = E_1 \Lambda E_1^H$, $E_1 \in \mathcal{C}V_{n,m}$, and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

4. MULTIPLE-ANTENNA SYSTEMS

Recently, in response to the demand for higher bit rates in wireless communications, researchers have exploited the use of MIMO systems. These studies show that MIMO systems increase capacity significantly over single-input single-output (SISO) systems. For example, if $n = \min\{n_t, n_r\}$, a MIMO uncorrelated Rayleigh distributed channel achieves almost n more bits per hertz for every 3 dB increase in signal-to-noise ratio (SNR) compared to a SISO system, which achieves only one additional bit per hertz for every 3 dB increase in SNR [16]. But the channel coefficients from different transmitter antennas to a single receiver antenna can be correlated. This channel correlation, which degrades the channel capacity (see [3], [14], [9] and references therein), depends on the physical parameters of a MIMO system and the scatterer characteristics. The physical parameters include the antenna arrangement and spacing, the angle spread, the angle of arrival, etc. One of the objectives of this paper is to evaluate this capacity degradation for the channel matrix $\mathbf{H} \sim \mathcal{C}N(0, I_{n_r} \otimes \Sigma_t)$ with $n_t > n_r$. This will be done by deriving closed form ergodic capacity formulas for correlated channels and their numerical evaluation.

The complex signal received at the jth output can be written as

$$y_j = \sum_{i=1}^{n_t} h_{ij} x_i + v_j, \qquad (13)$$

where h_{ij} is the complex channel coefficient between input i and output j, x_i is the complex signal at the ith input and v_j is complex Gaussian noise. The signal vector received at the output can be written as

$$\begin{bmatrix} y_1 \\ \vdots \\ y_{n_r} \end{bmatrix} = \begin{bmatrix} h_{11} & \cdots & h_{n_t 1} \\ \vdots & \vdots & \vdots \\ h_{1 n_r} & \cdots & h_{n_t n_r} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n_t} \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_{n_r} \end{bmatrix},$$

i.e., in vector notation,

$$y = Hx + v, \tag{14}$$

where $y, v \in \mathbb{C}^{n_r}$, $H \in \mathbb{C}^{n_r \times n_t}$, $x \in \mathbb{C}^{n_t}$ and $v \sim \mathcal{C}N(0, I_{n_r})$. It should be noted that the noise v is independent of the input signal x and channel matrix H. The total power of the input is constrained to ρ,

$$\mathcal{E}\{x^H x\} \le \rho \quad \text{or} \quad \operatorname{tr} \mathcal{E}\{x x^H\} \le \rho.$$

We assume that $\mathbf{H}$ is a complex Gaussian random matrix whose realization is known to the receiver, or equivalently, the channel output consists of the pair $(y, \mathbf{H})$. Note that the transmitter does not know the channel and its statistics (i.e., Σ_t) and the input power is distributed equally over all transmitting antennas, which is a natural thing to do in this case. Therefore, the ergodic capacity achieved by the input or transmitted signal x is distributed as $x \sim \mathcal{C}N(0, R_{xx})$, where $R_{xx} = (\rho/n_t) I_{n_t}$ and ρ is the total transmitted power, see [16]. Moreover, if we assume a block-fading model and coding over many independent fading intervals, then the Shannon or ergodic capacity of the random MIMO channel [16] is given by

$$C = \mathcal{E}_{\mathbf{H}} \left\{ \log\det\left(I_{n_t} + \frac{\rho}{n_t} \mathbf{H}^H \mathbf{H} \right) \right\}, \tag{15}$$

where the expectation is evaluated using a complex Gaussian density. If $\mathbf{H} \sim \mathcal{C}N(0, I_{n_r} \otimes \Sigma_t)$ then the channel is Rayleigh distributed and correlated at the transmitter end. A typical example of this situation is an uplink communication from a mobile unit to a basestation. Here, the antennas at the basestation can be spaced sufficiently far apart to achieve uncorrelation at the receiver end but due to physical size constrains it is more difficult to space the antennas far apart at the mobile unit, which leads to correlation at the transmitter end. Here the covariance matrix of the rows of $\mathbf{H}$ is denoted by Σ_t, which is an $n_t \times n_t$ positive definite Hermitian matrix. Let $\mathbf{W} = \mathbf{H}^H \mathbf{H} \sim \mathcal{C}W_{n_t}(n_r, \Sigma_t)$ and $n_t > n_r$. Then the channel capacity can be written as

$$C = \mathcal{E}_{\mathbf{W}} \left\{ \log\det\left(I_{n_t} + \frac{\rho}{n_t} \mathbf{W} \right) \right\}, \tag{16}$$

where the expectation is evaluated using a complex singular Wishart density given in Theorem 3. Let $\lambda_1 > \cdots > \lambda_{n_r}$ be the eigenvalues of $\mathbf{W}$ and $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_{n_r})$. Then the capacity can also be computed using a joint eigenvalue density, $f(\Lambda)$, i.e.,

$$C = \mathcal{E}_{\Lambda} \left\{ \log\left(\prod_{k=1}^{n_r} \left[1 + \frac{\rho}{n_t} \lambda_k \right] \right) \right\}. \tag{17}$$

The joint eigenvalue density of a complex singular Wishart matrix is given in Theorem 4.

As a numerical example, we compute the channel capacity of a correlated 2×4 channel matrix ($n_r = 2$ and $n_t = 4$), i.e., $\mathbf{H} \sim \mathcal{C}N(0, I_2 \otimes \Sigma_t)$, where we assume the eigenvalues of the positive definite Hermitian matrix Σ_t are

$$1.8090, \quad 1.3090, \quad 0.6910, \quad 0.1910.$$

In this case $\mathbf{W}$ is 4×4 complex singular Wishart matrix with two nonzero eigenvalues λ_1 and λ_2. The joint eigenvalue distribution is given by

$$f(\lambda_1, \lambda_2) = \frac{(\lambda_1 \lambda_2)^2}{12 (\det \Sigma_t)^2} (\lambda_1 - \lambda_2)^2 {}_0F_0^{(2)}(-\Sigma_t^{-1}, \Lambda), \tag{18}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2)$. The capacity of the correlated channel, $\mathbf{H} \sim \mathcal{C}N(0, I_2 \otimes \Sigma_t)$, is given by

$$\begin{aligned} C_c \;=\; & \int_0^\infty \int_0^{\lambda_1} \left[\log\left(1 + \frac{\rho}{4}\lambda_1\right) + \log\left(1 + \frac{\rho}{4}\lambda_2\right) \right] \\ & \times f(\lambda_1, \lambda_2)\, d\lambda_2\, d\lambda_1. \end{aligned} \tag{19}$$

For comparison purpose we also compute the capacity of the uncorrelated 2×4 channel matrix, i.e., $\mathbf{H} \sim \mathcal{C}N(0, I_2 \otimes \sigma^2 I_4)$. In this case, the joint eigenvalue density of the complex singular Wishart matrix is given by

$$g(\lambda_1, \lambda_2) = \frac{1}{12 \sigma^{16}} (\lambda_1 \lambda_2)^2 (\lambda_1 - \lambda_2)^2 e^{-(\lambda_1 + \lambda_2)/\sigma^2}. \tag{20}$$

From (20), we can evaluate the single unordered eigenvalue density, $f(\lambda)$, as

$$g(\lambda) = \frac{\lambda^2 e^{-\lambda/\sigma^2}}{12 \sigma^6} \left(\frac{\lambda^2}{\sigma^4} - 6\frac{\lambda}{\sigma^2} + 12 \right). \tag{21}$$

The capacity of this uncorrelated channel $\mathbf{H} \sim \mathcal{C}N(0, I_2 \otimes \sigma^2 I_4)$ is given by

$$C_u = 2 \int_0^\infty \log\left(1 + \frac{\rho}{4}\lambda\right) g(\lambda)\, d\lambda. \tag{22}$$

Figure 1 shows the capacity in nats[1] vs signal-to-noise ratio for a 2×4 correlated/uncorrelated Rayleigh fading channel matrix. From this figure we note the following: (i) the capacity is decreasing due to channel correlation, (ii) the capacity is increasing with increasing SNR.

[1] In equation (17), if we use $\log_e$ then the capacity is measured in nats. If we use $\log_2$ then the capacity is measured in bits. Thus, one nat is equal to $1/\log_e(2)$ bits/sec/Hz ($e = 2.718\ldots$).

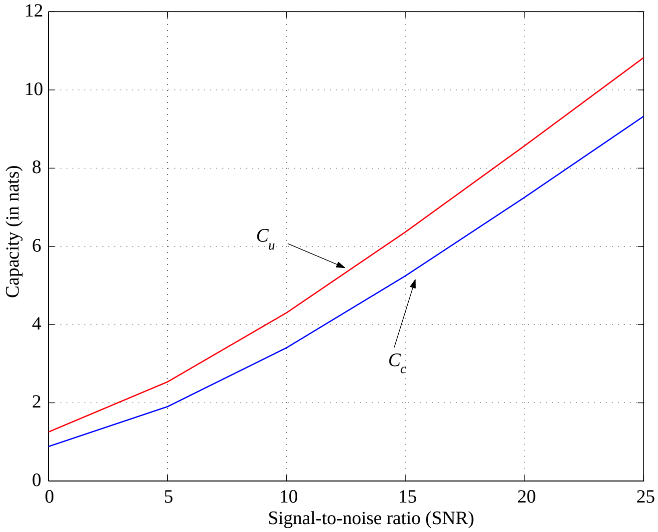

Fig. 1. Capacity vs SNR for $n_t = 4$ and $n_r = 2$, i.e., 2×4 channel matrix. C_u and C_c denote the capacity of uncorrelated and correlated Rayleigh channels, respectively.

5. CONCLUSION

In this paper, we studied the complex singular Wishart distribution and its application. In particular, we derived the complex singular Wishart density and joint eigenvalue density of a complex singular Wishart matrix. Using these distributions, both correlated and uncorrelated MIMO Rayleigh channel capacity formulas were obtained. The capacity of 2×4 MIMO Rayleigh channel matrices were computed for both correlated and uncorrelated channels. It was also shown how the channel correlation degrades the capacity of the communication systems.

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