NON-CENTRAL QUADRATIC FORMS ON COMPLEX RANDOM MATRICES AND APPLICATIONS

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ABSTRACT

In this paper, the densities of non-central quadratic forms on complex random matrices and their joint eigenvalue densities are derived for applications to information theory. These densities are represented by complex hypergeometric functions of matrix arguments, which can be expressed in terms of complex zonal polynomials and invariant polynomials. One of the special cases of studied quadratic forms is complex non-central Wishart matrices. We also show that the joint eigenvalue density of a complex non-central Wishart matrix can be expressed by an easily computable bounded density function. The derived densities are used to evaluate the capacity of multiple-input multiple-output (MIMO) Rician distributed channels.

1. INTRODUCTION

Let an $n \times m$ $(n \geq m)$ complex Gaussian (or normal) random matrix $\mathbf{X}$ be distributed as $\mathbf{X} \sim \mathcal{C}N(M, \Sigma_1 \otimes \Sigma_2)$ with mean $\mathcal{E}\{\mathbf{X}\} = M$ and covariance $\operatorname{cov}\{\mathbf{X}\} = \Sigma_1 \otimes \Sigma_2$, where $\Sigma_1 \in \mathbb{C}^{n\times n}$ and $\Sigma_2 \in \mathbb{C}^{m \times m}$ are positive definite Hermitian matrices. Here we read the symbol "$\sim$" as "is distributed as", $\mathcal{C}N$ denotes the complex normal distribution and $\otimes$ denotes the Kronecker product. The quadratic form on $\mathbf{X}$ associated with the positive definite Hermitian matrix A is defined by

$$\mathbf{S} = \mathbf{X}^H A \mathbf{X}.$$

Here, we study the distribution of $\mathbf{S}$, denoted by $\mathcal{C}Q_{n,m}(A, \Sigma_1, \Sigma_2, M)$, and its application to information theory. We also derive the joint eigenvalue densities of $\mathbf{S}$, which are represented by complex zonal polynomials and invariant polynomials. Complex zonal polynomials are symmetric polynomials in the eigenvalues of a Hermitian matrix, see [6], [11], and they enable us to represent the derived densities as infinite series. Invariant polynomials have two matrix arguments, which extend the single matrix argument of zonal polynomials, see [3].

The distributions of central quadratic forms (i.e., $M = 0$) on real and complex random matrices are studied in [5], [10] (and references therein) and [4], [12], [13] (and references therein), respectively. However, the general non-central quadratic forms on complex random matrices, $\mathbf{S}$, are not studied in the literature. Special cases of non-central quadratic forms are studied in [2] and [14].

This paper is organized as follows. Section 2 provides the necessary tools for deriving the non-central distribution theory. The complex non-central quadratic forms are studied in Section 3. The capacity of MIMO Rician channels and the computational methods are given in Section 4. Finally, concluding remarks are given in Section 5.

2. ESSENTIAL RESULTS

First, we define the complex multivariate hypergeometric coefficients $[a]_\kappa$, which frequently occur in integrals involving complex zonal polynomials. Let $\kappa = (k_1, \ldots, k_m)$ be a partition of an integer k with $k_1 \geq \cdots \geq k_m \geq 0$ and $k = k_1 + \cdots + k_m$. Then

$$[a]_\kappa = \prod_{i=1}^{m} (a - i + 1)_{k_i},$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ and $(a)_0 = 1$. The complex zonal polynomial of a Hermitian matrix $X \in \mathbb{C}^{m\times m}$ is defined in [6] as

$$C_\kappa(X) = \chi_{[\kappa]}(1)\chi_{[\kappa]}(X), \tag{1}$$

where $\chi_{[\kappa]}(1)$ is the dimension of the representation $[\kappa]$ of the symmetric group on k symbols given by

$$\chi_{[\kappa]}(1) = k! \, \frac{\prod_{i<j}^{m}(k_i - k_j - i + j)}{\prod_{i=1}^{m}(k_i + m - i)!},$$

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and $\chi_{[\kappa]}(X)$ is the character of the representation $[\kappa]$ of the linear group given as a symmetric function of the eigenvalues, $\mu_1, \ldots, \mu_m$, of X by

$$\chi_{[\kappa]}(X) = \frac{\det\left[\left(\mu_i^{k_j+m-j}\right)\right]}{\det\left[\left(\mu_i^{m-j}\right)\right]}.$$

Note that both the real and complex zonal polynomials are particular cases of Jack polynomials $C_\kappa^{(\alpha)}(X)$ for general α, where $\alpha = 1$ for complex and $\alpha = 2$ for real zonal polynomials, respectively. See [1] for details. In this paper we consider only the complex case; therefore, for notational simplicity we drop the superscript of Jack polynomials, as we did in (1), i.e., $C_\kappa(X) := C_\kappa^{(1)}(X)$.

The probability distributions of random matrices are often derived in terms of hypergeometric functions of matrix arguments. In the sequel, we need to use the following complex hypergeometric function of two Hermitian matrix arguments,

$$_0F_0^{(m)}(X,Y) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(X)C_\kappa(Y)}{k!C_\kappa(I_n)}, \qquad (2)$$

where $X \in \mathbb{C}^{m\times m}$, $Y \in \mathbb{C}^{n\times n}$ $(n \geq m)$ and $\sum_\kappa$ denotes summation over all partitions κ of k. The complex hypergeometric function of a Hermitian matrix argument is given by

$$_0F_1(n;X) = \sum_{k=0}^{\infty}\sum_{\kappa}\frac{1}{[n]_\kappa}\frac{C_\kappa(X)}{k!}. \qquad (3)$$

We have

$$C_\kappa(I_n) = k!\frac{\left[\prod_{i<j}^{n}(k_i - k_j - i + j)\right]^2}{\prod_{i=1}^{n}\Gamma(k_i+n-i+1)\prod_{i=1}^{n}\Gamma(n-i+1)},$$

where gamma function is define by $\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}\,dt$. The complex multivariate gamma function is define by

$$\mathcal{C}\Gamma_m(a) = \pi^{m(m-1)/2}\prod_{k=1}^{m}\Gamma(a-k+1), \quad \mathrm{Re}(a) > m-1.$$

Next, we describe a class of homogeneous polynomials $C_\phi^{\kappa,\tau}(X,Y)$ of degrees k and t in the elements of the $m\times m$ Hermitian matrices X and Y, respectively, see [3]. These polynomials are invariant under the simultaneous transformations

$$X \to E^HXE, \quad Y \to E^HYE, \quad E \in U(m),$$

where $U(m)$ denotes the group of unitary $m \times m$ matrices, i.e.,

$$U(m) = \{E \in \mathbb{C}^{m\times m}; E^HE = I_m\}.$$

Moreover, these polynomials satisfy the relationship

$$\int_{U(m)} C_\kappa(AE^HXE)C_\tau(BE^HYE)(dE) = \sum_{\phi\in\kappa.\tau}\frac{C_\phi^{\kappa,\tau}(A,B)C_\phi^{\kappa,\tau}(X,Y)}{C_\phi(I)}, \qquad (4)$$

where C_κ, C_τ, and C_ϕ are zonal polynomials indexed by the ordered partitions κ, τ, and ϕ of the nonnegative integers k, t, and $f = k+t$, respectively, into not more than m parts. If we let $Gl(m,\mathbb{C})$ denote the general linear group of $m\times m$ nonsingular complex matrices, then $\phi \in \kappa.\tau$ denotes the irreducible representation of $Gl(m,\mathbb{C})$ indexed by 2ϕ that occurs in the decomposition of the Kronecker product $2\kappa\otimes 2\tau$ of the irreducible representations indexed by 2κ and 2τ, see [3].

3. NON-CENTRAL QUADRATIC FORMS

In this section, the densities of non-central quadratic forms on complex random matrices are studied and their joint eigenvalue densities are derived. These densities are used in information theory, hypothesis testing, principal component analysis, canonical correlation analysis, multiple discriminant analysis, etc. The next theorem gives the density of non-central quadratic forms on complex random matrices $\mathbf{S} = \mathbf{X}^HA\mathbf{X}$. This theorem is one of the key contributions of this paper.

Theorem 1 *Let $\mathbf{X}$ be an $n\times m$ $(n \geq m)$ complex Gaussian random matrix distributed as $\mathbf{X} \sim \mathcal{C}N(M, \Sigma_1\otimes\Sigma_2)$, where $\Sigma_1 \in \mathbb{C}^{n\times n}$ and $\Sigma_2 \in \mathbb{C}^{m\times m}$ are positive definite Hermitian matrices and $M \in \mathbb{C}^{n\times m}$. Then the density function of $\mathbf{S} = \mathbf{X}^HA\mathbf{X}$ is given by*

$$f(S) = \frac{\mathrm{etr}\left(-\Sigma_2^{-1}M^H\Sigma_1^{-1}M\right)}{\mathcal{C}\Gamma_m(n)(\det\Sigma_1A)^m(\det\Sigma_2)^n}(\det S)^{n-m} \times {}_0F_0^{(m)}(B,-\Sigma_2^{-1}S)\,{}_0F_1(n;CC^HS), \qquad (5)$$

where $\mathrm{etr}(\cdot) = \exp(\mathrm{tr}(\cdot))$, $A \in \mathbb{C}^{n\times n}$ is a positive definite Hermitian matrix, $B = A^{-1/2}\Sigma_1^{-1}A^{-1/2}$ and $C = \Sigma_2^{-1}M^H\Sigma_1^{-1}A^{-1/2}$.

The distribution of the matrix $\mathbf{S}$ is denoted by $\mathcal{C}Q_{n,m}(A,\Sigma_1,\Sigma_2,M)$. From this generalized density we can easily derive other well-known densities. If $A = I_n$, $\Sigma_1 = I_n$ and $\Sigma_2 = \Sigma$, then $\mathbf{S} = \mathbf{X}^H\mathbf{X}$ is said to have a complex non-central Wishart distribution, denoted by $\mathcal{C}W_m(n,\Sigma,\Omega)$, with density

$$f(S) = \frac{\mathrm{etr}(-\Omega)(\det S)^{n-m}}{\mathcal{C}\Gamma_m(n)(\det\Sigma)^n}\,\mathrm{etr}(-\Sigma^{-1}S)\,{}_0F_1(n;\Omega\Sigma^{-1}S), \qquad (6)$$

where $\Omega = \Sigma^{-1} M^H M$. The eigenvalue densities of complex non-central quadratic forms cannot be solved in terms of hypergeometric functions or zonal polynomials. Here we derive these densities using invariant polynomials, which are proposed by Davis [3]. These invariant polynomials have two matrix arguments, which extend the signal matrix argument of zonal polynomials. The next theorem gives the joint eigenvalue density of non-central quadratic forms on complex random matrices $\mathbf{S} = \mathbf{X}^H A \mathbf{X}$. This theorem is another key contribution of this paper.

Theorem 2 *Consider the $m \times m$ positive definite Hermitian matrix $\mathbf{S} = \mathbf{X}^H A \mathbf{X} \sim \mathcal{C}Q_{n,m}(A, \Sigma_1, \Sigma_2, M)$, where $A \in \mathbb{C}^{n\times n}$ is a positive definite Hermitian matrix. Then the joint density of the eigenvalues, $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$, of $\mathbf{S}$ is*

$$g(\Lambda) = \frac{\pi^{m(m-1)} \operatorname{etr}\left(-\Sigma_2^{-1} M^H \Sigma_1^{-1} M\right)}{\mathcal{C}\Gamma_m(n)\mathcal{C}\Gamma_m(m)(\det \Sigma_1 A)^m (\det \Sigma_2)^n} \prod_{k=1}^{m} \lambda_k^{n-m}$$

$$\prod_{k<l}^{m} (\lambda_k - \lambda_l)^2 \sum_{k,t=0}^{\infty} \sum_{\kappa,\tau;\phi\in\kappa.\tau} \frac{C_\phi^{\kappa,\tau}(-\Sigma_2^{-1}, CC^H) C_\phi^{\kappa,\tau}(\Lambda,\Lambda)}{k!\, t!\, [n]_\tau\, C_\kappa(I_n)\, C_\phi(I_m)},$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$, $C_\phi^{\kappa,\tau}$ is an invariant polynomial, indexed by the ordered partitions κ, τ and ϕ of the nonnegative integers k, t, and $f = k + t$, respectively, into not more than m parts, $B = A^{-1/2} \Sigma_1^{-1} A^{-1/2}$ and $C = \Sigma_2^{-1} M^H \Sigma_1^{-1} A^{-1/2}$.

From this generalized joint eigenvalue density we can easily derive the joint eigenvalue density of non-central Wishart matrix. If $A = I_n$, $\Sigma_1 = I_n$ and $\Sigma_2 = \Sigma$, then the joint eigenvalue density of the complex non-central Wishart matrix $\mathbf{S} = \mathbf{X}^H \mathbf{X} \sim \mathcal{C}W_m(n, \Sigma, \Omega)$ is given by

$$g(\Lambda) = \frac{\pi^{m(m-1)} \operatorname{etr}(-\Omega)}{\mathcal{C}\Gamma_m(n)\mathcal{C}\Gamma_m(m)(\det \Sigma)^n} \prod_{k=1}^{m} \lambda_k^{n-m} \prod_{k<l}^{m} (\lambda_k - \lambda_l)^2$$

$$\times \sum_{k,t=0}^{\infty} \sum_{\kappa,\tau;\phi\in\kappa.\tau} \frac{C_\phi^{\kappa,\tau}(-\Sigma^{-1}, \Omega\Sigma^{-1}) C_\phi^{\kappa,\tau}(\Lambda,\Lambda)}{k!\, t!\, [n]_\tau\, C_\phi(I_m)}.$$

Moreover, if $A = I_n$, $\Sigma_1 = I_n$ and $\Sigma_2 = I_m$, then the joint eigenvalue density of the complex non-central Wishart matrix $\mathbf{S} = \mathbf{X}^H \mathbf{X} \sim \mathcal{C}W_m(n, I_m, \Omega)$ is given by

$$\begin{aligned} g(\Lambda) &= \frac{\pi^{m(m-1)} \operatorname{etr}(-\Omega)}{\mathcal{C}\Gamma_m(m)\mathcal{C}\Gamma_m(n)} \operatorname{etr}(-\Lambda) \prod_{k=1}^{m} \lambda_k^{n-m} \\ &\quad \times \prod_{k<l}^{m} (\lambda_k - \lambda_l)^2 {}_0F_1^{(m)}(n; \Omega, \Lambda), \end{aligned} \tag{7}$$

where $\Omega = M^H M$. The joint eigenvalue density formula given in Theorem 2 is difficult to compute. This difficulty motivates us to consider the approximate density formula, or specifically, finding an upper bound on the density, which is studied next. The following lemma is required in the sequel [14].

Lemma 1 *Let X be an $n \times m$ $(n \geq m)$ complex matrix of rank m. Then the following inequality holds:*

$${}_0F_1(n; X^H X) \leq {}_0F_0\left(X^H X/n\right), \tag{8}$$

with equality as $n \to \infty$.

The joint eigenvalue density of a complex non-central Wishart matrix can be expressed by a bounded density function, which is given by the following theorem.

Theorem 3 *Let $\mathbf{S} \sim \mathcal{C}W_m(n, \Sigma, \Omega)$ with $n \geq m$. Then $\mathbf{S}$ is an $m \times m$ positive definite Hermitian matrix. The joint density of the eigenvalues, $\lambda_1 > \cdots > \lambda_m > 0$, of $\mathbf{S}$ satisfies the inequality*

$$\begin{aligned} g(\Lambda) &\leq \frac{\pi^{m(m-1)} \operatorname{etr}(-\Omega)}{\mathcal{C}\Gamma_m(m)\mathcal{C}\Gamma_m(n)(\det \Sigma)^n} \prod_{k=1}^{m} \lambda_k^{n-m} \\ &\quad \times \prod_{k<l}^{m} (\lambda_k - \lambda_l)^2 {}_0F_0^{(m)}(-\Psi, \Lambda), \end{aligned} \tag{9}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$, the diagonal elements of Ψ, $\psi_1, \ldots, \psi_m$, are the eigenvalues of the matrix $\left(\Sigma^{-1} - \Omega\Sigma^{-1}/n\right)$, and $\Omega = \Sigma^{-1} M^H M$.

Proof. From [[6], p. 488, equation (93)] we have

$$g(\Lambda) = \frac{\pi^{m(m-1)}}{\mathcal{C}\Gamma_m(m)} \prod_{k<l}^{m} (\lambda_k - \lambda_l)^2 \int_{U(m)} f(E\Lambda E^H)(dE). \tag{10}$$

The result follows by substituting (6) into (10) and noting that

$$\begin{aligned} &\int_{U(m)} \operatorname{etr}\left(-\Sigma^{-1} E\Lambda E^H\right) {}_0F_1\left(n; \Omega\Sigma^{-1} E\Lambda E^H\right)(dE) \\ &\quad \leq \int_{U(m)} \operatorname{etr}\left(-\Sigma^{-1} E\Lambda E^H\right) {}_0F_0\left(\Omega\Sigma^{-1} E\Lambda E^H/n\right)(dE) \\ &\quad \leq \int_{U(m)} \operatorname{etr}\left(-\left(\Sigma^{-1} - \Omega\Sigma^{-1}/n\right) E\Lambda E^H\right)(dE) \\ &\quad \leq {}_0F_0^{(m)}(-\Psi, \Lambda). \end{aligned} \tag{11}$$

The first inequality follows from Lemma 1 and the last inequality follows from [[6], p. 488, equation (92)]. □

4. MIMO RICIAN CHANNEL CAPACITY

Let us denote the number of inputs (or transmitters) and the number of outputs (or receivers) of the MIMO wireless communication system by n_t and n_r, respectively, and assume that the channel coefficients are distributed as complex Gaussian and correlated at both the transmitter and the receiver ends. Then the MIMO Rician flat fading channel can be represented by an $n_r \times n_t$ complex random matrix $\mathbf{H} \sim \mathcal{C}N(M, \Sigma_r \otimes \Sigma_t)$, where Σ_r and Σ_t are positive definite Hermitian matrices which represent the channel correlation at the receiver and transmitter ends. This means the covariance matrix of the columns and rows of $\mathbf{H}$ are denoted by Σ_r (same for all columns) and Σ_t (same for all rows), respectively. If $\Sigma_r = I_{n_r}$ (or $\sigma^2 I_{n_r}$) and $\Sigma_t = \sigma^2 I_{n_t}$ (or I_{n_t}), then the channel is called an *uncorrelated Rician distributed channel*, and otherwise it is called a spatially correlated Rician distributed channel. The Rician channel model is used when there is a strong direct signal path (line-of-sight) between the transmitter and receiver. Hence, the complex channel coefficients are modeled by non-zero mean. In [7], [8] and [9], the authors studied the capacity of the uncorrelated MIMO Rician channel.

The complex signal received at the jth output can be written as

$$y_j = \sum_{i=1}^{n_t} h_{ij} x_i + v_j, \qquad (12)$$

where h_{ij} is the complex channel coefficient between input i and output j, x_i is the complex signal at the ith input and v_j is complex Gaussian noise with unit variance. The signal vector received at the output can be written as

$$\begin{bmatrix} y_1 \\ \vdots \\ y_{n_r} \end{bmatrix} = \begin{bmatrix} h_{11} & \cdots & h_{n_t 1} \\ \vdots & \vdots & \vdots \\ h_{1 n_r} & \cdots & h_{n_t n_r} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n_t} \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_{n_r} \end{bmatrix},$$

i.e., in vector notation,

$$y = Hx + v, \qquad (13)$$

where $y, v \in \mathbb{C}^{n_r}$, $H \in \mathbb{C}^{n_r \times n_t}$, $x \in \mathbb{C}^{n_t}$ and $v \sim \mathcal{C}N(0, I_{n_r})$. It should be noted that the noise v is independent of the input signal x and channel matrix H. The total input power is constrained to ρ,

$$\mathcal{E}\{x^H x\} \le \rho \quad \text{or} \quad \operatorname{tr} \mathcal{E}\{x x^H\} \le \rho.$$

We assume that the realization of $\mathbf{H}$ is known to the receiver, or equivalently, the channel output consists of the pair $(y, \mathbf{H})$. Note that the transmitter does not know the channel and its statistics (i.e., M, Σ_r and Σ_t) and the input power is distributed equally over all transmitting antennas, which is a natural thing to do in this case. Moreover, if we assume a block-fading model and coding over many independent fading intervals, then the Shannon or ergodic capacity of the random MIMO channel is given in [15] by

$$\begin{aligned} C &= \mathcal{E}_{\mathbf{H}} \left\{ \log\det \left(I_{n_t} + \frac{\rho}{n_t} \mathbf{H}^H \mathbf{H} \right) \right\} \\ &= \mathcal{E}_{\mathbf{S}} \left\{ \log\det \left(I_{n_t} + \frac{\rho}{n_t} \mathbf{S} \right) \right\}, \end{aligned} \qquad (14)$$

where $\mathbf{S} = \mathbf{H}^H \mathbf{H}$ and expectation is evaluated using the density of the non-central quadratic forms on complex random matrices. The capacity can also be computed using the joint eigenvalue density $g(\Lambda)$, i.e.,

$$C = \mathcal{E}_{\Lambda} \left\{ \log \left(\prod_{k=1}^{n_t} \left[1 + \frac{\rho}{n_t} \lambda_k \right] \right) \right\}. \qquad (15)$$

Next, a numerical capacity evaluation of a Rician $n_r \times 2$ channel matrix with correlated at the transmitter end is given. Thus, we have a two-input ($n_t = 2$) n_r-output communication system operating over a Rician fading environment. Let $n_t = 2$ and $\Psi = \operatorname{diag}(\psi_1, \psi_2)$. Then we have

$$\begin{aligned} {}_0F_0^{(2)}(-\Psi, \Lambda) &= \frac{1}{(\psi_2 - \psi_1)(\lambda_1 - \lambda_2)} \\ &\quad [\exp\{-(\psi_1\lambda_1 + \psi_2\lambda_2)\} - \exp\{-(\psi_1\lambda_2 + \psi_2\lambda_1)\}]. \end{aligned}$$

The following theorem gives an upper bound for the correlated Rician channel capacity for an $n_r \times 2$ channel matrix.

Theorem 4 *Consider a two-input Rician channel, i.e.,* $\mathbf{H} \sim \mathcal{C}N(M, I_{n_r} \otimes \Sigma_t)$, *with* $n_r \ge 2$. *If the input power is constrained by* ρ, *then the capacity* C *satisfies the inequality*

$$\begin{aligned} C \le \frac{(\det \Sigma_t)^{-n_r} \operatorname{etr}(-\Omega)}{(\psi_2 - \psi_1)} \times \\ \Bigg[\frac{\psi_2^{-n_r+1}}{\Gamma(n_r)} \int_0^\infty \log\left[1 + \frac{\rho}{2}\lambda_1\right] \lambda_1^{n_r-1} e^{-\psi_1\lambda_1} d\lambda_1 - \\ \frac{\psi_1^{-n_r+1}}{\Gamma(n_r)} \int_0^\infty \log\left[1 + \frac{\rho}{2}\lambda_1\right] \lambda_1^{n_r-1} e^{-\psi_2\lambda_1} d\lambda_1 - \\ \frac{\psi_2^{-n_r}}{\Gamma(n_r-1)} \int_0^\infty \log\left[1 + \frac{\rho}{2}\lambda_1\right] \lambda_1^{n_r-2} e^{-\psi_1\lambda_1} d\lambda_1 + \\ \frac{\psi_1^{-n_r}}{\Gamma(n_r-1)} \int_0^\infty \log\left[1 + \frac{\rho}{2}\lambda_1\right] \lambda_1^{n_r-2} e^{-\psi_2\lambda_1} d\lambda_1 \Bigg], \end{aligned} \qquad (16)$$

where λ_1 *is an unordered eigenvalue of* $\mathbf{S} = \mathbf{H}^H \mathbf{H}$, (ψ_1, ψ_2) *are the eigenvalues of* $\left(\Sigma_t^{-1} - \Omega\Sigma_t^{-1}/n_r\right)$ *and* $\Omega = \Sigma_t^{-1} M^H M$.

Figure 1 shows capacity in nats[1] versus SNR for $n_r = 2$ and 4 and $n_t = 2$, where the solid lines represent the upper bounds given in (16), and the dashed lines represent the simulation results. It can be seen that the derived capacity upper bound is quite tight with the simulation results for the entire range of SNRs, and this illustrates the accuracy of the bound. Note that here we assumed a correlated Rician channel $\mathbf{H} \sim \mathcal{C}N(M, I_{n_r} \otimes \Sigma_t)$, where the covariance matrix and the mean matrix are

$$\Sigma_t = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}, M = \begin{bmatrix} 0.25 + 0.25i & 0.25 + 0.25i \\ 0.25 + 0.25i & 0.25 + 0.25i \end{bmatrix},$$

[1] If we use $\log_e$ in equation (16), then the capacity is measured in nats. If we use $\log_2$, then the capacity is measured in bits. Thus, one nat is equal to $1/\log_e(2)$ bits/sec/Hz ($e = 2.718\ldots$).

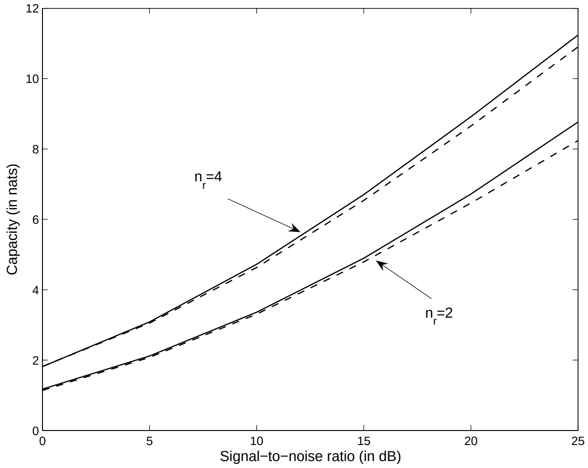

Fig. 1. Capacity versus SNR for $n_t = 2$ and $n_r = 2$ and 4; i.e., $\mathbf{H}$ is an $n_r \times 2$ Rician fading channel matrix. The solid lines represent the upper bounds given in (16), and the dashed lines represent the simulation results.

respectively. The Rician K-factor is define as the ratio of deterministic power to scattered power. This choice gives the $K-$factor$= 0.125/1$.

5. CONCLUSION

In this paper, we derived the densities of non-central quadratic forms on complex random matrices and their joint eigenvalue densities. These densities play an important role in information theory, numerical analysis and statistical hypothesis testing. One of the special cases of studied quadratic forms is complex non-central Wishart matrices. We show that the joint eigenvalue density of a complex non-central Wishart matrix can be expressed by an easily computable bounded density function. Using these densities, formulas for the so-called ergodic channel capacity (the most important information-theoretic measure) for MIMO Rician channel are derived. Specifically, exact and easily computable tight upper bound formulas for the ergodic capacity is given for both spatially correlated and uncorrelated MIMO Rician channels. Numerical results are also given, which show how the channel correlation degrades the capacity of the communication system.

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