AN EFFICIENT FINITE POSITIVITY TEST ALGORITHM FOR STATISTICAL SIGNAL PROCESSING APPLICATIONS

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ABSTRACT

A real finite sequence $\mathbf{r}[0,M]$ is positive if and only if its Fourier Transform is positive over the entire frequency range [1-4]. Testing positivity of $\mathbf{r}[0,M]$ using its Fourier transform or its autocorrelation matrix requires an infinite or very lengthy algorithm. As an alternative, an efficient finite algorithm to test positivity, non-negativity or negativity of the real finite sequence $\mathbf{r}[0,M]$ is presented. The need to test the positivity of a real sequence arises in many practical situations. For instance, though theoretically non-negative, practical considerations (or missing data) result in a non-positive spectral estimate, e.g. unbiased autocorrelation lag estimates [6]. The algorithm described in this paper is a finite algorithm based on Sturm's theorem from the classical theory of equations. Performance analysis of the algorithm is presented together with simulation results for different positive and non-positive test cases.

***Key Words*:** Finite Positivity Test Algorithm, Real Finite Sequences, Minimum phase FIR filter

1. INTRODUCTION

A finite, non-zero, real sequence $\mathbf{r}[0,M]$ is called positive ($\mathbf{r}[0,M] > 0$), or non-negative ($\mathbf{r}[0,M] \geq 0$), if and only if [1] [2] [3],

(a) Its Fourier transform, given by,

$$R(e^{j\omega}) = \sum_{k=-M}^{M} r(|k|)\; e^{-j\omega k} \;, \qquad (1)$$

is positive (or non-negative) for $-\pi \leq \omega \leq \pi$. Where $r(k)$ is kth component of the Mth order sequence $\mathbf{r}[0,M]$.

(b) or, the infinite dimensional symmetric Toeplitz matrix $\mathbf{R}_\infty$ is positive (or non-negative), where the first row of $\mathbf{R}_\infty$ is $\mathbf{r}[0,\infty] = [r(0), r(1), \ldots, r(M), 0, 0, \ldots]$.

Condition (b) is the equivalent time-domain necessary and sufficient condition of (a) [1-4]. In practice, applying either one of the aforementioned methods will require lengthy computations to achieve a satisfactory result since $R(e^{j\omega})$ is indeed a continuous function of frequency. Testing positivity of matrix $\mathbf{R}$ using Schur or Levinson algorithms [5] also results in an infinite or lengthy algorithm. In this paper we take an indirect approach to verify (a) and (b). The alternative algorithm helps in testing positivity or non-negativity of the real finite sequence $\mathbf{r}[0,M]$. Unlike the methods mentioned above, the proposed algorithm is finite, computationally very efficient, and can conclusively determine whether the given sequence is positive-definite, positive semi-definite, or negative-definite [1]. Positive sequences arise in many statistical signal processing problems such as optimal (Wiener) filtering, estimation, spectral factorization and system identification with wide applications in communication, radar, noise cancellation, acoustic, speech, or geophysical signal processing. An important property of the positive real finite sequence $\mathbf{r}[0,M]$ is that it represents autocorrelation sequence of an MA process and as such it can uniquely determine an M-order minimum phase FIR filter using Yule-Walker equations. Another practical application that might require testing of positivity is when missing or corrupted data samples results in a non-positive sequence $\mathbf{r}[0,M]$ hindering spectral factorization or inverse filtering problem. Section 2 of this paper states the problem numerically. This section also describes a set of mathematical transformations that are utilized in the algorithm. Section 3 describes the positivity test algorithm. The computational efficiency of the new algorithm is discussed in Section 4. Section 5 elaborates the performance of the algorithm with an example. Section 6 and 7 comprises of Conclusion and References.

2. PROBLEM DEFINITION

An MA stationary process is the output of a linear, shift invariant FIR filter $H(z)$ with zero mean, unit variance white input noise. Let $H(z)$ be defined as:

$$H(z) = \sum_{k=0}^{M} h(k) \cdot z^{-k} , \quad (2)$$

where $h(k)$ is real for $0 \le k \le M$. The autocorrelation lags of the output process are thus real and are given by:

$$r(k) = \sum_{t=0}^{M} h(t).h(t+k) , \quad (3)$$

where $r(k) = r(-k)$ and $0 \le k \le M$. Similarly the power spectrum is given by Fourier transform of (3) as:

$$R(\omega) = r(0) + 2\sum_{k=1}^{M} r(k)\cos(k\omega) , \text{ for } 0 \le \omega \le \pi \quad (4)$$

and $$R(z) = \sum_{k=-M}^{M} r(k) z^{-k} = H(z)H(z^{-1}) \quad (5)$$

Recall that the sequence $\mathbf{r}[0, M]$ is positive if and only if $R(\omega) > 0$ for $0 \le \omega \le \pi$ and is non-negative if and only if $R(\omega) \ge 0$, for $0 \le \omega \le \pi$. Polynomial $R(z)$ has the following properties. Let $z_0 = \rho e^{j\omega}$ and $z_0^* = \rho e^{-j\omega}$, then

(1) $R(z_0) = 0 \Leftrightarrow R(z_0^*) = 0$, and

(2) $R(z_0) = 0 \Leftrightarrow R\left(\frac{1}{z_0}\right) = R\left(\frac{1}{z_0^*}\right) = 0$.

Thus if $z_0 = e^{j\omega}$ is a root of $R(z) = H(z)H(z^{-1})$ with multiplicity n, then we must have,

$$R(z) = (z - z_0)^{2n} (z - z_0^*)^{2n} \hat{R}(z) , \; \hat{R}(z_0) \ne 0 .$$

Also, $R(e^{j\omega}) = R(z)|_{z=e^{j\omega}}$ is real even function for $-\pi \le \omega \le \pi$. Using the above properties we have the following propositions:

Proposition I

The sequence $\mathbf{r}[0, M]$ is positive if and only if:

(1) $r(0) > 0$ and $R(z) = \sum_{k=-M}^{M} r(|k|) z^{-k}$, has no root on the unit circle,

(2) and, $\mathbf{r}[0, M] \ne 0$ is non-negative if and only if $r(0) > 0$ and roots of $R(z)$ on the unit circle are of even multiplicity.

In order to develop some techniques that will provide information about the roots of $R(z)$ in terms of its coefficients we next consider some polynomial transformations.

Chebyshev Transformation

Let $\cos\omega = x$, $(|x| \le 1)$, and $V_l(x) = \cos[l \cos^{-1} x]$, then recursive equations that generate the polynomials $V_l(x)$ for all values of l are as follows:

$$\begin{cases} V_0(x) = 1, \quad V_1(x) = x \\ V_l(x) = 2xV_{l-1}(x) - V_{l-2}(x) \text{ for all } l. \end{cases} \quad (6)$$

Applying this transformation to $R(\omega)$, we obtain:

$$R(x) = \mathbf{r}^T \mathbf{V}(x) = \mathbf{V}^T(x).\mathbf{r} , \quad (7)$$

where, $\mathbf{r}^T = [r(0) \quad 2r(1) \quad 2r(2) \cdots 2r(M)]$ and

$$\mathbf{V}^T(x) = [V_0(x) \quad V_1(x) \quad V_2(x) \cdots V_M(x)]$$

From the recursive equations (6) we have,

$V(x) = \mathbf{A} \cdot \mathbf{x}$, $\mathbf{x}^T = (1 \quad x \quad x^2 \cdots x^M)$, and (8)

$$\mathbf{A} = \begin{bmatrix} +1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ -1 & 0 & 2 & \cdots \\ 0 & -3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{(M+1)\times(M+1)} \quad (9)$$

Thus $R(x)$ can now be written as,

$R(x) = \mathbf{a}^T \cdot \mathbf{x}$, $\mathbf{a}^T = [a(0) \quad a(1) \quad a(2) \cdots a(M)]$, and

$$\mathbf{a}^T = \mathbf{r}^T \cdot \mathbf{A} \quad (10)$$

The above transformation converts $R(\omega)$ into M-order polynomial $R(x)$ so that the coefficients $a(m)$, for m=1, 2,…, M are linear in **r**. Then Proposition I can be restated as follows.

Proposition II

For a given sequence $\mathbf{r}[0, M]$, let $R(x)$ be the transformed polynomial derived from (4) by applying Chebyshev transformation. Then it can easily be shown that:

(a) $r[0, M] > 0$ if and only if $R(x) > 0$ for $x \in (-1, +1)$.

(b) $r[0, M] \ne 0$ is non-negative if and only if $R(x) \ge 0$ for $x \in (-1, +1)$ and for $x_0 \in (-1, +1)$, where $R(x_0) = 0$, then $R(x) = (x - x_0)^{2n} G(x)$, where $G(x_0) > 0$ and, $n = 1, 2, \ldots < \frac{M}{2}$.

If $R(-1) > 0$, then the transformation $x = \dfrac{1-s}{1+s}$ applied to $R(x)$ yields the equation,

$$F(s) = \sum_{k=1}^{M} f(k) s^k = 0 , \quad (11)$$

and we have the following corollary.

Corollary

For a given sequence $\mathbf{r}[0, M]$, let $F(s)$ be derived from $R(\omega)$ given in (4), by the transformation $\cos(\omega) = \dfrac{1-s}{1+s}$ then,

(a) $\mathbf{r}[0, M] > 0$ if and only if $F(s) > 0$, for $s \in (0, \infty)$.

(b) $\mathbf{r}[0, M] \ne 0$ is non-negative if and only if $F(s) \ge 0$, for $s \in (0, \infty)$ and positive real roots of $F(s)$ are of even multiplicities.

Polynomial $F(s)$ is obtained by applying the above bilinear transformation to polynomial $R(x)$ given by (7). If we define the following vectors and matrices,

$$\mathbf{f}^T = [f(0) \quad f(1) \quad f(2)\cdots f(M)],$$

$$\mathbf{r}^T = [r(0) \quad 2r(1) \quad 2r(2)\cdots 2r(M)].$$

Then $F(s)$ is parameterized by:

$$\mathbf{f} = \mathbf{T}\cdot\mathbf{A}^T\cdot\mathbf{r}, \tag{12}$$

where $\mathbf{T}$ is a square nonsingular matrix, constant for a given order [1].

3. FINITE ALGORITHM BASED ON STURM'S THEOREM

With the method in section 2 the problem of determining the positivity of a finite real sequence was transformed into a problem of finding the real roots between 0 and ∞ and its multiplicity. In this section we discuss a finite algorithm that can specify the number of real roots of a given polynomial on the interval (a,b). This method along with the propositions in Section 2 is then used to develop a finite algorithm for testing positivity of $\mathbf{r}[0,M]$.

Cauchy index [1] gives number of real roots of a polynomial $f(x)$ in the real interval (a,b) is denoted as $\mathrm{I}_a^b[f'(x)/f(x)] = q$ where $f'(x)$ is derivative of $f(x)$. Sturm's theorem offers a simple way to compute the Cauchy index using Sturm's chain as follows.

Sturm chain in the interval (a,b) [1][4]:
Consider a sequence of real polynomials $f_1(x), f_2(x),\ldots, f_m(x)$ over the interval (a,b)**,** where $a < b$ are real and could be $-\infty \le a,b \le +\infty$. Then such a sequence is called Sturm's chain in the interval (a, b) when the following properties hold;

(1) if $f_k(x) = 0$ for $a < x < b, k = 2,3,\ldots, m-1$, then $f_{k-1}(x)\cdot f_{k+1}(x) < 0$, and

(2) $f_m(x) \neq 0$, for all x such that $a < x < b$.

A *generalized* Sturm-chain is obtained if the sequence of polynomials in Sturm-chain is multiplied by an arbitrary polynomial. For a given x, we define,

$v(x)$ = The number of variations of sign in $\{ f_1(x), f_2(x),\ldots, f_m(x) \}$. (13)

Note that if a is finite, then $v(a)$ is interpreted as $v(a+\varepsilon)$, where ε is a positive number sufficiently small so that in the half-closed interval $(a, a+\varepsilon]$ none of the functions $f_k(x)$ vanishes. Similar discussion applies for b. Based on the Sturm's theorem, we then have,

$$\mathbf{I}_a^b \frac{f_2(x)}{f_1(x)} = v(a) - v(b). \tag{14}$$

For any two polynomials $f(x)$ and $g(x)$, where the degree of $f(x)$ is not less than that of $g(x)$, we can always construct a (generalized) Sturm chain beginning with $f_1(x) = f(x)$ and $f_2(x) = g(x)$ by means of Euclid's algorithm as follows:

$$\begin{cases} f_1(x) = q_1(x)f_2(x) + f_3(x) \\ f_2(x) = q_2(x)f_3(x) + f_4(x) \\ \vdots \\ f_{k-1}(x) = q_{k-1}(x)f_k(x) + f_{k+1}(x) \\ \vdots \\ f_{m-1}(x) = q_{m-1}(x)f_m(x) \end{cases} \tag{15}$$

where $f_{k+1}(x)$ is the remainder on dividing $f_{k-1}(x)$ by $f_k(x)$ with the sign changed for all k. If $f_m(x)$ is not identically zero then it is the greatest common divisor of all the polynomials $f_k(x)$ in (15). If $f_m(x) \neq 0$ for $a < x < b$, then the sequence of $f_k(x)$ is a Sturm-chain. If $f_m(x)$ has any real root on (a,b), then $f_k(x)$ is a generalized Sturm-chain, i.e., $f_k(x)$, for $k = 1,2,\ldots,m$, can be divided by $f_m(x)$ resulting in a Sturm-chain. The steps below present a method to determine the number of positive real roots of an arbitrary real polynomial $f(x)$ by Euclidean division.

Method .1

Step1. Let $f_1(x) = f(x) = \sum_{k=0}^{M} a(k)x^k$, and $f_2(x) = \frac{d}{dx}(f(x)) = f'(x)$. Let $i = 0$.

Step 2. Let $i = i+1$, Using Equations (15), derive $f_k(x)$ for $k = 3,4,\ldots,m$,
Let $d_i = v(0) - v(+\infty)$, where $v(x)$ is given by (13),
If $d_i = 0$, or $f_m(x)$ is constant, then stop. Go to Step 4. Otherwise continue,

Step 3. Let $f_1(x) = f_m(x)$, and $f_2(x) = f_1'(x)$, Go to Step 2.

Step 4. Let $N = i + 1$, and $d_N = 0$, then:

d_1 = Number of *distinct* positive real roots of $f(x)$.

$c_i = d_i - d_{i+1}$ = Number of positive real roots of $f(x)$ *with multiplicity of i,* for $i = 1, 2, \ldots, N-1$.

A finite positivity test algorithm:

We use equations (15) and the above method to obtain a finite algorithm for testing positivity or non-negativity of $\mathbf{r}[0, M]$ as follows.

Step 1. If $r(0) \le 0$, stop. $\mathbf{r}$ is non-positive. Otherwise,

Step 2. Let $f_1(x) = F(s)|_{s=x}$. Apply method 1 to $f_1(x)$.

Step 3. $r[0, M]$ is positive if and only if $d_1 = 0$. Furthermore, $r[0, M] \ne 0$ is non-negative if and only if $d_1 \ne 0$ and $c_i = 0$ for odd values of *i*.

4. COMPUTATIONAL EFFICIENCY

The algorithm described in Section 3 requires $M(M + 1) - 2$ real multiplications for testing the positivity (or non-negativity) of a sequence of length $M + 1$. Direct application of FFT to the sequence can be used as a test of positivity. To best describe a sequence in frequency domain at least a 128-point FFT is required [6]. A 128-point radix-2 FFT requires $1792 = 2 \times 7 \times 128$ multiplications. Comparison with direct application of FFT for testing positivity of $R(e^{j\omega})$ in frequency domain shows the efficiency and superior advantages of the proposed algorithm. For example, when $M = 10$, the 128-point radix-2 FFT will require 1792 real multiplications whereas the proposed algorithm would only require 108 real multiplications. It is also not difficult to notice that $N = 128$ equally spaced frequency components can miss a zero value of the spectrum located at the frequency ω_0 where $\frac{2\pi k}{N} < \omega_0 < \frac{2\pi(k+1)}{N}$ and $0 \le k \le N-1$, thus making the FFT frequency domain testing erroneous.

Evaluation of the autocorrelation matrix $\mathbf{R}$ using Schur or Levinson algorithms may also require far more computations than 108 real multiplications. Depending on the sequence $\mathbf{r}[0,10]$, matrix $\mathbf{R}$ of dimensions beyond 11 may need to be evaluated to assert positivity (non-negativity) of the sequence . To that end, we recall that Levinson algorithm (prediction error filtering process) requires n^2 real multiplications for an inversion of an $n \times n$ real symmetric Toeplitz matrix $\mathbf{R}$ in order to compute reflection coefficients, i.e. to compute an *n*-order minimum phase FIR filter coefficients.

5. SIMULATION RESULTS

Many different examples were taken and results were analyzed. All the examples showed favorable results consistent with the analytical developments. Here we present application of the algorithm developed in Section 3 to an autocorrelation sequence obtained from the real coefficients of a 9-order FIR filter which is not minimum-phase. The reason for using an autocorrelation sequence to test the performance of the algorithm is that we know for sure that the given sequence is positive (semi) definite. Zeros of the filter are $-1, -1, -1, 0.5e^{\pm j\frac{\pi}{3}}, e^{\pm j\frac{\pi}{2}}, e^{\pm j\frac{5\pi}{6}}$. Autocorrelation sequence of the filter's coefficients was computed. Fourier transform of the sequence is shown in the Figure 1.a. This autocorrelation sequence is fed to the algorithm. The results are shown in Table 1. The results clearly show the ability of the algorithm to determine non-negativity of the given sequence. The next step was to test the performance of the algorithm with a negative input sequence. Few samples of the same autocorrelation sequence were corrupted to make the resulting sequence negative. Fourier transform of the corrupted sequence is shown in the Figure1.b. Negativity of the sequence was detected by algorithm as shown in Table 2. Furthermore, the algorithm also accurately specified the total number of distinct real roots and multiplicity of every repeated real root of polynomial $F(s)$. Note that the zeros at $-\pi$ and π are not shown by the algorithm. This is due to the fact that the transformed polynomial $F(s)$ is reduced in order when there is a zero at $-\pi$, π, or zero. As per Proposition II and its corollary, the given sequence $\mathbf{r}[0, M]$ is positive definite if $F(s)$ has no positive real root. The sequence $\mathbf{r}[0, M]$ is positive semi-definite if any positive real root of $F(s)$ is of only even multiplicity. And, $\mathbf{r}[0, M]$ is negative if $F(s)$ has any positive real root of odd multiplicity.

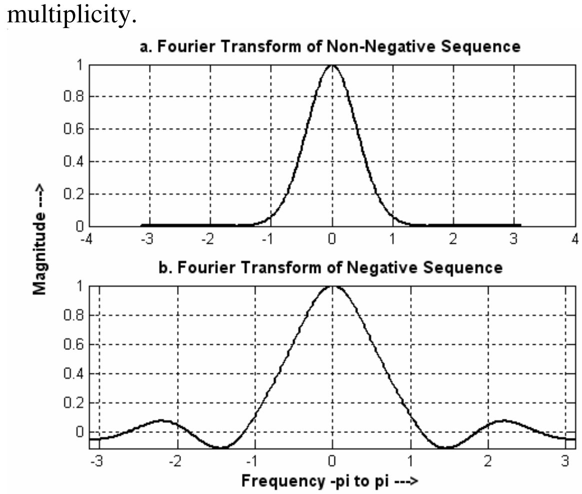

Figure 1. Fourier Transform of the given sequence $r[0, M]$

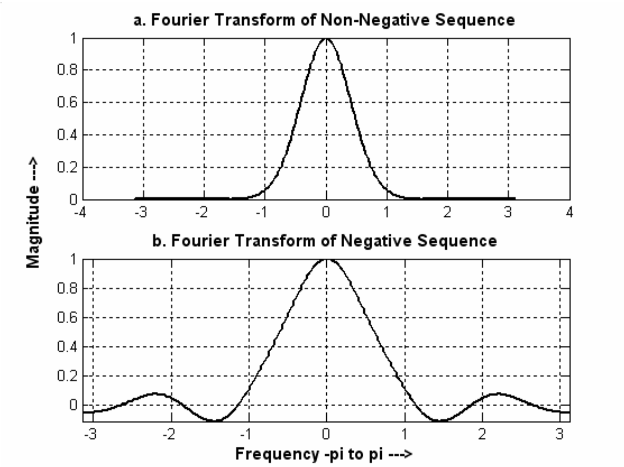

Figure 1. Fourier Transform of the given sequence $r[0, M]$

Table.1. Output of the algorithm

The given sequence is	Non-negative
Distinct positive real roots (except at $-\pi$ and π)	2
Real roots with multiplicity 1 (except at $-\pi$ and π)	0
Real roots with multiplicity 2 (except at $-\pi$ and π)	2

Table.2. Output of the algorithm

The given sequence is	Negative
Distinct positive real roots (except at $-\pi$ and π)	2
Real roots with multiplicity 1 (except at $-\pi$ and π)	1

6. CONCLUSION

An efficient positivity test algorithm for finite real sequence $\mathbf{r}[0, M]$ was presented. The algorithm was developed based on the Sturm's theorem. It was simple, conclusive, and finite. The algorithm was capable of specifying positivity, non-negativity, or negativity of the sequence $\mathbf{r}[0, M]$. Performance of the algorithm versus FFT was discussed. It was shown that the algorithm was fast and robust. Simulation results verified the analytical development and showed accuracy and capability of the algorithm.

7. REFERENCES

[1] Issa M.S. Panahi, "Positivity Tests and a class of FIR Filters: Applications in Spectral Estimation and System Identification", Ph.D Thesis, Department of Electrical Engineering, University of Colorado at Boulder, 1988.

[2] S. M. Kay, "Modern Spectral Estimation, Theory and Application", Prentice-Hall, 1988.

[3] A. Papoulis, S.U. Pillai, "Probability, Random Variables and Stochastic Processes", 4th Edition, Mc Graw Hill, 2002.

[4] W.S. Burnside, A.W. Panton, "The Theory of Equations", Dublin University Press, 1892.

[5] J. G. Proakis, D.G. Manolakis, "Digital Signal Processing, Principles, Algorithms, and Applications", 3rd Edition, Prentice-Hall, 1996.

[6] J.A. Cadzow, Y. Sun, "Sequences with Positive Semi-definite Fourier Transforms", IEEE Trans. On ASSP, VOL. 6, pp1502-1510, Dec 1986.