Spectral Symmetry Iterative Frequency Estimation Algorithm

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ABSTRACT

In this paper, a new technique called the Spectral Symmetry Iterative (SSI) frequency estimation algorithm is disclosed. This algorithm can be used for estimating the frequency and phase of a wide class of signals that are bandpass in nature. This technique can precisely determine the centre frequency of this class of signal in noiseless condition and can yield frequency error variance performance that is comparable to the Cramer Rao lower bound (CRLB) on frequency estimation error of a single complex exponential in the additive white Gaussian noise (AWGN).

***Keywords* – Discrete Fourier transform, bandpass signal, OFDM, UltraWide Band, frequency estimation, data aided frequency estimation, phase estimation.**

1. INTRODUCTION

Frequency estimation of complex exponentials has long been a problem in the area of signal processing. In [1], Reisenfeld presented a DFT-based frequency estimation algorithm that can determine the frequency of a complex exponential signal very accurately. The estimation algorithm in [1] exploits the even symmetrical nature of the magnitude spectra of a complex exponential to iteratively search for its frequency, 'f_c'. In this paper, a new algorithm for estimating the frequency and phase of a complex bandpass signal is presented. This algorithm, called the spectral symmetry iterative (SSI) frequency estimation algorithm, is based on the same concept presented in [1] where the spectral symmetry of the resulting magnitude spectra of a manipulated bandpass signal is exploited to estimated its centre frequency, f_c.

The SSI frequency estimation algorithm is based on calculating DFT coefficients hence it is very computationally efficient. The performance and computational complexity of this algorithm enable real time digital signal processing implementation of a large number of applications requiring centre frequency determination. This particular frequency estimation algorithm can be applied to a number of wireless communication systems such as OFDM and UltraWide Band (UWB) and defence applications such as radar and sonar tracking.

The rest of the paper is organised as follows, section 2 will introduce the SSI frequency estimation algorithm, section 3 will provide simulated results of the algorithm and this will be followed by the conclusion.

2. THE SPECTRAL SYMMETRY ITERATIVE (SSI) FREQUENCY ESTIMATION ALGORITHM

Consider a general bandpass signal $s[n]$ which is represented by:

$$\begin{aligned} s[n] &= \frac{1}{N}\sum_{k=0}^{N-1} d_k \exp\left[j\left(2\pi n\left(\frac{k_c+k}{N}\right)+\theta_c\right)\right] \\ &= \frac{1}{N}\sum_{k=0}^{N-1} a_k \exp\left[j\left(2\pi\left(f_c+f_k\right)nT_s+\theta_c+\theta_k\right)\right], \end{aligned} \tag{1}$$

where d_k is the corresponding k^{th} discrete Fourier coefficients of the signal $s[n]$, which in general can be expressed as $d_k = a_k \exp(j\theta_k)$ where a_k and θ_k corresponds to the magnitude and phase of the k^{th} discrete Fourier coefficients respectively. k is the index of the discrete spectral distribution and has a range from 0 to $N-1$ where N corresponds to the number of sample points. f_k is the frequency corresponding to the k^{th} discrete Fourier coefficients of the signal $s[n]$. k_c is the value of k that corresponds to carrier frequency, f_c, of the signal $s[n]$. θ_c is the phase of the said carrier frequency.

Now the baseband representation of the signal $s[n]$, denoted as $x[n]$, can be expressed as:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} d_k \exp\left[j2\pi n \frac{k}{N} \right] \\ = \frac{1}{N}\sum_{k=0}^{N-1} a_k \exp\left[j\left(2\pi f_k n T_s + \theta_k\right)\right]. \tag{2}$$

A new signal, denoted as $z[n]$, is formed by multiplying $s[n]$ in (1) and the complex conjugate of $x[n]$ in (2) which can be mathematically expressed as:

$$z[n] = \psi \times s[n] x^*[n]$$
$$= \frac{\psi}{N^2}\left(\sum_{k=0}^{N-1} d_k \exp\left[j\left(2\pi\left(f_k + f_c\right) nT_s + \theta_c\right)\right] \sum_{m=0}^{N-1} d^*{}_m \exp\left(-j2\pi f_m n T_s\right)\right)$$
$$= \frac{\psi}{N^2}\begin{pmatrix} \sum_{k=0}^{N-1} d_k d^*{}_0 \exp\left[j\left(2\pi f_c n T_s + \theta_c\right)\right] \exp\left[j\left(2\pi\left(f_k - f_0\right) nT_s\right)\right] + \\ \sum_{k=0}^{N-1} d_k d^*{}_1 \exp\left[j\left(2\pi f_c n T_s + \theta_c\right)\right] \exp\left[j\left(2\pi\left(f_k - f_1\right) nT_s\right)\right] + \\ . \\ . \\ . \\ \sum_{k=0}^{N-1} d_k d^*{}_{N-1} \exp\left[j\left(2\pi f_c n T_s + \theta_c\right)\right] \exp\left[j\left(2\pi\left(f_k - f_{N-1}\right) nT_s\right)\right] \end{pmatrix}. \tag{3}$$

where ψ can be any real value constant.

Denote the difference between any two θ_k's as $\theta_{\alpha,\beta} = \theta_\alpha - \theta_\beta = -\left(\theta_\beta - \theta_\alpha\right) = -\theta_{\beta,\alpha}$, then multiplying any two of the said d_k's yields:

$$d_\alpha d^*{}_\beta = a_\alpha a_\beta \exp\left(j\theta_{\alpha,\beta}\right) = \left(d_\beta d^*{}_\alpha \right)^* \tag{4}$$

Now, denote the difference between any two *k's* as

$$k_{\alpha,\beta} = k_\alpha - k_\beta = -\left(k_\beta - k_\alpha\right) = -k_{\beta,\alpha}. \tag{5}$$

First, consider the case where f_c is an integer multiple of $1/T$, where T is the signal observation period. k_c will then be of an integer value. Taking the magnitude of the DFT of $z[n]$ in (3) and substituting the definition in (4) and (5) yields:

$$\left|Z[k]\right| = \frac{\psi}{N^2}\begin{pmatrix} a_0{}^2\delta\left(k - k_e\right) + a_1 a_0 \delta\left(k - \left(k_e + k_{1,0}\right)\right) + \ldots\ldots + a_{(N-1)} a_0 \delta\left(k - \left(k_e + k_{(N-1),0}\right)\right) + \\ a_0 a_1 \delta\left(k - \left(k_e - k_{1,0}\right)\right) + a_1{}^2 \delta\left(k - k_e\right) + \ldots\ldots + a_{(N-1)} a_1 \delta\left(k - \left(k_e + k_{(N-1),1}\right)\right) + \\ . \\ . \\ . \\ a_0 a_{(N-1)} \delta\left(k - \left(k_e - k_{(N-1),0}\right)\right) + a_1 a_{(N-1)} \delta\left(k - \left(k_e - k_{(N-1),1}\right)\right) + \ldots\ldots + a_{(N-1)}{}^2 \delta\left(k - k_e\right) \end{pmatrix}. \tag{6}$$

where $\delta(k)$ is the Kronecker delta function which is defined as,

$$\delta(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}, \tag{7}$$

and $Z[k]$ is the k^{th} DFT coefficients of $\left\{z[n]\right\}_0^{N-1}$.

It can be seen the magnitude of the Z[k] in (6) is even symmetric about the frequency correspond to DFT bin k_c. In this example, k_c was chosen to be an integer value for illustrative purpose only. In general, k_c can be of an integer or non-integer value and the described even symmetric property of the magnitude of the discrete spectral distribution |Z[k]| in (6) the will hold whether k_c is of an integer or a non-integer value.

The recursive algorithm presented in [1] employs a discriminate that works on a contraction principle in minimizing the difference in the magnitude of two modified DFT coefficients which are plus and minus half a DFT bin away from the estimated frequency. It relies on the fact that the magnitude spectra of a complex exponential is even symmetrical about its frequency hence the difference in the magnitude of the two modified DFT coefficients will eventually reduced to zero in the noiseless case with the increasing number of recursions of the algorithm.

From (6), one can see the magnitude spectra of the DFT of z[*n*] is even symmetrical about $k = k_c$, hence it is possible to apply the frequency estimator in [1] to estimate f_c in s[*n*]. For general bandpass signal, evaluating the modified DFT coefficients at plus and minus half a DFT bin away as stated in [1] from the estimated frequency does not necessary guarantee optimal performance, hence the distance from the initial frequency estimate should be adjusted according to the spectral shape of the bandpass signal.

Summarizing the frequency estimation algorithm presented in [1]:

1) Perform a coarse frequency estimate such as the one described in Rife and Boorstyn [2], in which a N point complex DFT is performed on $\left\{z[n]\right\}_0^{N-1}$ as defined in (3) and a peak search is done on the magnitudes of the DFT output coefficients, to obtain the initial frequency estimate, $\hat{f}_0$. This estimate is obtained by, $\hat{f}_0 = k_{\max} F_s / N$, where $k_{\max}$ is the index of the maximum magnitude DFT output coefficient.

2) Calculate the modified DFT coefficients α_m and β_m defined by,

$$\alpha_m = \sum_{n=0}^{N-1} z[n] \exp\left(-j2\pi n\left(\hat{f}_m T_s - \omega\right)\right), \tag{8}$$

$$\beta_m = \sum_{n=0}^{N-1} z[n]\exp\left(-j2\pi n\left(\hat{f}_m T_s + \omega\right)\right). \tag{9}$$

where ω is the optimal DFT bin distance to evaluate the modified coefficients.

3) Calculate the discriminate ε_m defined as,

$$\varepsilon_m = \frac{1}{2N}\frac{|\beta_m| - |\alpha_m|}{|\beta_m| + |\alpha_m|}F_s. \tag{10}$$

4) Calculate the new adjusted frequency with the formula,

$$\hat{f}_{m+1} = \hat{f}_m + \varepsilon_m. \tag{11}$$

5) Perform steps 2 – 4 recursively for m = 0, 1, 2, 3…..

3. SIMULATION RESULTS

Figures 1, 2 and 3 illustrates an example of recovering a suppressed carrier from an UWB signal. Figure 1 shows a typical Gaussian monocycle pulse of an UWB signal in an AWGN channel with SNR equal to 0dB. For the purpose of the illustration, only the real component of the signal is shown and there are 24 samples point in each observation period. Figure 2 shows the same UWB signal as in figure 1 corrupted by carrier frequency offset equal to 1.5/T and phase offset equal to π/5. Figure 3 shows the recovered UWB signal from figure 2 using the frequency estimate obtained after 20 iterations of the proposed SSI frequency estimation algorithm and the phase estimate obtained using the following ML phase estimator [3]:

$$\hat{\theta}_{c,\hat{f}_m} = \tan^{-1}\left(\frac{\operatorname{Im}\left[\sum_{n=0}^{N-1} z[n]e^{-j2\pi\hat{f}_m nT_s}\right]}{\operatorname{Re}\left[\sum_{n=0}^{N-1} z[n]e^{-j2\pi\hat{f}_m nT_s}\right]}\right). \tag{12}$$

where $\hat{f}_m$ is the frequency estimate from the SSI frequency estimation algorithm.

The modified DFT coefficients were calculated at plus and minus 1 DFT bin from the previous frequency estimate. It is evident there is very difference between the recovered UWB signal in figure 3 compared to the original signal in figure 1 which proves the SSI frequency estimation algorithm was able to estimate the frequency offset very accurately.

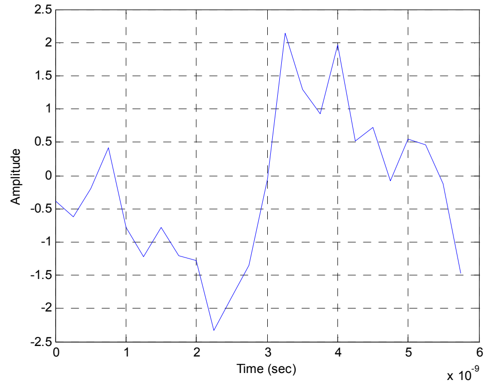

Fig. 1. Gaussian monocycle pulse in AWGN channel with SNR equal to 0dB.

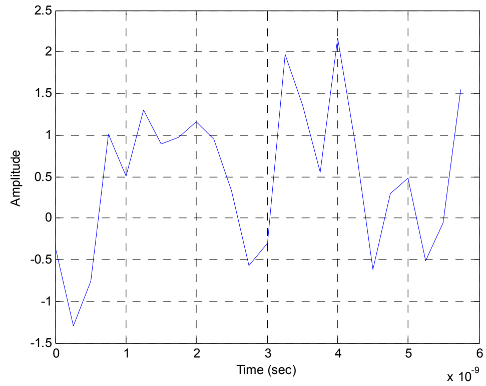

Fig. 2. Frequency and phase offset corrupted version of the signal in figure 1.

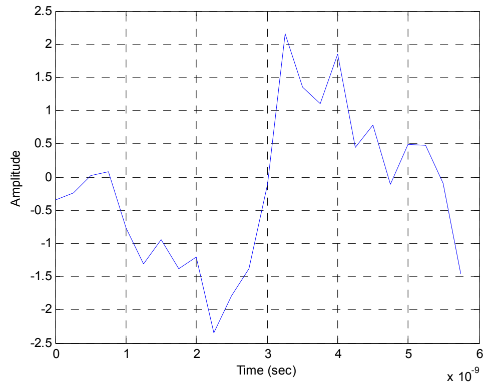

Fig. 3.The recovered signal.

Figure 4 and 5 shows the simulated results of the frequency and phase error variance performance as a function of SNR for the proposed SSI frequency estimation algorithm compared and the ML phase estimator as described in (12) to the respective error variance CRLBs for a single complex exponential. The number of samples points used for each simulation trial equals to 24, the frequency offset is generated in each trial from an uniform distribution with a range of $-fs/2$ to $fs/2$ and the phase offset is generated in each trial from an uniform distribution with a range of 0 to 2π . The results shown were averaged across 5000 trials at each SNR.

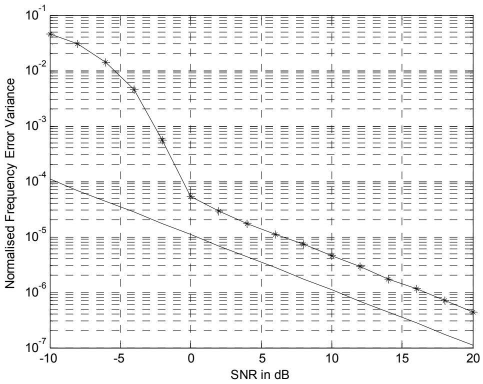

Fig. 4. The frequency error variance performance of the SSI frequency estimation algorithm as a function of SNR.

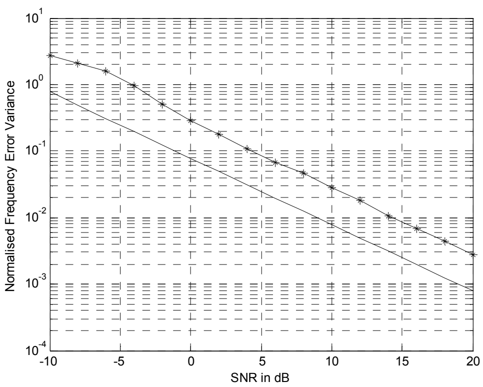

Fig. 5. The phase error variance performance of the ML phase estimator as described in (12) as a function of SNR.

4. CONCLUSION

In this paper, it was shown that if the baseband representation of the transmitted signal was known such as in the case of transmitting training signals, it is possible to acquire the frequency offset exist in the received signal using the proposed SSI frequency estimation algorithm. It was shown via the illustration that the SSI frequency estimation algorithm is effective in estimating the frequency offset in the received signal even at very low SNR.

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