THE NONCENTRAL WISHART DISTRIBUTION: PROPERTIES AND APPLICATION TO SPECKLE IMAGING

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ABSTRACT

Whereas central Wishart distributions have been extensively studied in the literature, the related noncentral distributions have received less attention. This paper studies interesting properties of these noncentral Wishart distributions such as closed form expressions for the moments. This work is motivated by the fact that this distribution plays a central part in the modeling of images corrupted by speckle. More precisely, the multivariate distribution of the intensity arising from a nonzero mean Gaussian wavefront amplitude is the diagonal of a noncentral Wishart distribution. Two examples which are interesting for synthetic aperture radar imaging and optical imaging through turbulence are discussed.

1. INTRODUCTION

Multiple Input Multiple Output (MIMO) systems are at the origin of a renewed interest in the study of "central" Wishart distributions, [1, 2]. This paper is devoted to the study of the broader family of noncentral Wishart distributions. It is organized as follows. Section 2 presents interesting results regarding noncentral Wishart distributions. The moment generating function of these distributions as well as moments of any order are derived. Section 3 shows that multivariate noncentral Wishart distributions play an important role for images corrupted by speckle, when the wavefront amplitude is non zero mean Gaussian. More precisely, we show that adjacent pixels of the image are the diagonal elements of a noncentral Wishart distribution. This result is interesting since multivariate noncentral distributions allow to model dependencies between adjacent pixels of the image. This model is useful in SAR imaging when a strong scatterer is embedded in weak clutter [3, p. 113]. The same model occurs for the direct imaging of exoplanets where the random nature of the wavefront amplitude is due to the residue of atmospheric turbulence (uncorrected by the adaptive optical system and the deterministic shift of the wavefront amplitude, the coronograph used to "cancel" the light diffracted by the star). Section 4 addresses the case where the image has been recorded using a photocounting camera. The image distribution is obtained by computing the Poisson-Mandel transform of a noncentral Wishart distribution. This section shows that the moments of this distribution can be expressed in closed form.

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2. NONCENTRAL WISHART DISTRIBUTIONS

This section reminds useful results on noncentral Wishart distributions (NWDs). Consider k independent random vectors of $\mathbb{R}^d$ denoted by $Z_1, \dots, Z_k$ distributed according to multivariate Gaussian distributions $\mathcal{N}(0, \mathbb{I}_d)$ (where $\mathbb{I}_d$ is the $d \times d$ identity matrix). Consider also k nonrandom vectors of $\mathbb{R}^d$ denoted by $m_1, \dots, m_k$. The distribution of the following $d \times d$ matrix $W = \sum_{i=1}^k (Z_i + m_i)(Z_i + m_i)^T$ depends only on $M = \sum_{i=1}^k m_i m_i^T$, where ^{T} denotes transposition (see below). It is called a noncentral Wishart distribution and it is denoted by $\mathcal{W}_k(M, \mathbb{I}_d)$.

2.1. Laplace Transform

The Laplace transform of W is classically computed by

$$
\begin{aligned}
L(\theta) &= \mathsf{E}\left(e^{-\mathrm{Tr}\{\theta W\}}\right) = \prod_{i=1}^k \mathsf{E}\left(e^{-\mathrm{Tr}\{\theta (Z_i+m_i)(Z_i+m_i)^T\}}\right), \\
&= \prod_{i=1}^k \mathsf{E}\left(e^{-\mathrm{Tr}\{(Z_i+m_i)^T)\theta(Z_i+m_i)\}}\right), \\
&= \prod_{i=1}^k e^{-m_i^T \theta m_i} \mathsf{E}\left(e^{-2Z_i^T\theta m_i - Z_i^T \theta Z_i}\right), \\
&= \prod_{i=1}^k \frac{e^{-m_i^T\theta m_i}}{(2\pi)^{n/2}} \int e^{-2Z_i^T\theta m_i - \frac{1}{2}Z_i^T(2\theta+I_n)Z_i} dZ_i,
\end{aligned}
$$

where $\mathrm{Tr}(A)$ is the trace of the matrix A and θ is an appropriate symmetric $d \times d$ matrix. It is well known that the following identity $\mathsf{E}(e^{h^T Z}) = e^{\frac{1}{2}h^T\Sigma h}$ holds, when Z is a zero mean Gaussian vector with covariance matrix Σ (denoted as $Z \sim \mathcal{N}(0,\Sigma)$). By using this result with $\Sigma = (2\theta + \mathbb{I}_d)^{-1}$ and $h = -2\theta m_i$, the following expression can be obtained:

$$L(\theta) = \frac{1}{[\det(\mathbb{I}_d + 2\theta)]^{k/2}} \prod_{i=1}^{k} \exp\left(-m_i^T \theta(\mathbb{I}_d + 2\theta)^{-1} m_i\right), \tag{1}$$

where $\det(A)$ is the determinant of matrix A. Standard results on matrix traces allow to rewrite (1) as follows

$$L(\theta) = \frac{\exp\left(-\text{Tr}\left[\theta(\mathbb{I}_d + 2\theta)^{-1}M\right]\right)}{[\det(\mathbb{I}_d + 2\theta)]^{k/2}}, \tag{2}$$

where $M = \sum_{i=1}^{k} m_i m_i^T$.

2.2. Marginal Distributions

The distributions of principal submatrices of W are also non central Wishart distributions. We also describe the joint distribution of the diagonal $\widetilde{W} = (W_{11}, \ldots, W_{dd})^T$ of W. The Laplace transform of $\widetilde{W}$ can be obtained from (1) by setting to zero the off-diagonal elements of θ:

$$\begin{aligned} L(\widetilde{\theta}\,) &= \mathsf{E}\left(e^{-\sum_{i=1}^{k}\theta_{ii}W_{ii}}\right), \\ &= \exp\left(-\sum_{i=1}^{k} \frac{\theta_{ii}M_{ii}}{1+2\theta_{ii}}\right) \prod_{i=1}^{k} [1+2\theta_{ii}]^{-k/2}. \end{aligned} \tag{3}$$

2.3. Extension to correlated Gaussian vectors

Consider k independent random vectors $X_1, \ldots, X_k$ of $\mathbb{R}^d$ distributed according to multivariate Gaussian distributions $\mathcal{N}(0,\Sigma)$, Σ being a positive definite symmetric matrix. Consider also k non random vectors of $\mathbb{R}^n$ denoted by $\mu_1, \ldots, \mu_k$. The distribution of

$$U = \sum_{i=1}^{k} (X_i + \mu_i)(X_i + \mu_i)^T$$

is a noncentral Wishart distribution denoted as $U \sim \mathcal{W}_k(M, \Sigma)$, where A is the square root of Σ (i.e. $\Sigma = A^2$, with A symmetric positive definite) and $M = A^{-1}\left(\sum_{i=1}^{k} \mu_i \mu_i^T\right) A^{-1}$. The Laplace transform of this distribution can be easily obtained by noting that

$$A^{-1}X_i = Z_i \sim \mathcal{N}(0, \mathbb{I}_d).$$

Indeed, by denoting $m_i = A^{-1}\mu_i$, U can be expressed as

$$U = AWA,$$

where $W = \sum_{i=1}^{k}(Z_i + m_i)(Z_i + m_i)^T \sim \mathcal{W}_k(M, \mathbb{I}_d)$ and $M = A^{-1}\left(\sum_{i=1}^{k} \mu_i \mu_i^T\right) A^{-1}$. As a consequence, the Laplace transform of U can be written:

$$L(S) = \mathsf{E}\left[\exp\left(-\text{Tr}\{SU\}\right)\right] = \mathsf{E}\left[\exp\left(-\text{Tr}\{\theta W\}\right)\right],$$

where $\theta = ASA$. By using (2), the following result can be obtained:

$$L(S) = \frac{\exp\left(-\text{Tr}\left[ASA(\mathbb{I}_d + 2ASA)^{-1}M\right]\right)}{[\det(\mathbb{I}_d + 2ASA)]^{k/2}}.$$

By using standard results on matrices, the Laplace transform of a noncentral Wishart distribution $\mathcal{W}_k(M, \Sigma)$ can be written:

$$L(S) = \frac{\exp\left(-\text{Tr}\left[S\left(\mathbb{I}_d + 2\Sigma S\right)^{-1}\mu\right]\right)}{[\det(\mathbb{I}_d + 2\Sigma S)]^{k/2}}, \tag{4}$$

where $\mu = \sum_{i=1}^{k} \mu_i \mu_i^T$. The Laplace transforms of U_{ii} and $\widetilde{U} = (U_{11}, \ldots, U_{dd})$ can be easily computed from (4) by choosing appropriate values of S, similarly to section (2.2). The Laplace transform (4) can also be used to derive the moments of noncentral Wishart distributions, as shown in the next section.

2.4. Moments

The moments of a noncentral Wishart distribution can be classically obtained by differentiating the Laplace transform (4). For this, it is convenient to express $L(S)$ as follows:

$$L(S) = \exp\left[-\text{Tr}\,(B)\right]\nu\left(2\Sigma S - \mathbb{I}_d\right),$$

where $B = \frac{\mu}{2}\Sigma^{-1}$ and

$$\nu(S) = \frac{1}{\det\left[(-S)\right]^{k/2}} \exp\left(\text{Tr}\left[B(-S)^{-1}\right]\right).$$

It is interesting to note that $\nu(S)$ is the Laplace transform of a positive measure which generates an exponential family of non central Wishart distributions. From this remark, [4] designs a method of computation of the moments of W of any order. We borrow to this paper the following computation of the moments of order 1 and 2. The dth differential of $\nu(S)$ can be computed by using the following Leibnitz formula

$$fg)^{(d)}(S)(h_1, \ldots, h_n) = \sum_{T \subset \{1,\ldots,d\}} f^{|T|}(S)(h_T) g^{|T'|}(S)(h_{T'}), \tag{5}$$

where

$$\begin{aligned} f(S) &= \exp\left(\text{Tr}\left[B(-S)^{-1}\right]\right), \\ g(S) &= \exp\left(\frac{k}{2}\log\det[(-S)^{-1}]\right). \end{aligned}$$

Note that we have used the following notations in (5): $h_1, ..., h_d$ are d matrices of size $d \times d$, the sum in the right hand side of (5) covers all subsets T of $\{1, ..., d\}$, $T' =$

$\{1,...,d\}\backslash T$ and $|T|$ denotes the cardinal of the set T. The differentials of f and g have been derived in [5]. Denoting by $\mathcal{S}_d$ the group of permutations π of $\{1,\ldots,d\}$, by $C(\pi)$ the set of cycles of the permutation π and by $m(\pi)$ the number of cycles of π, the differential of g can be obtained as follows

$$g^{(d)}(S)(h_1,\ldots,h_d) = g(S)\sum_{\pi\in\mathcal{S}_d}\left(\frac{k}{2}\right)^{m(\pi)} r_\pi(\sigma)(h_1,\ldots,h_d),$$

where $\sigma=(-S)^{-1}$ and

$$r_\pi(\sigma)(h_1,\ldots,h_d)=\prod_{c\in\, C(\pi)}\mathrm{Tr}\left(\prod_{j\in\, c}\sigma h_j\right).$$

To obtain the differential of f, we consider a quantity similar to $r_\pi(\sigma)(h_1,\ldots,h_d)$. For a given integer d, we introduce a set of objects denoted as P_d. Each element P of P_d is defined by

- a partition $T=(T_1,\ldots,T_q)$ of $\{1,\ldots,d\}$ into non void subsets (the order of the sequence $T_1,\ldots,T_q$ does not matter).
- A permutation π_j of T_j for each $j=1,\ldots,q$.

Thus the information about P is q and the q pairs (T_j,π_j). For instance, the set $\{1,2,3\}$ has 5 partitions:

$$T^{(1)}=(\{1\},\{2\},\{3\}),\ T^{(2)}=(\{1,2\},\{3\}),$$
$$T^{(3)}=(\{1\},\{2,3\}),\ T^{(4)}=(\{2\},\{1,3\}),$$
$$T^{(5)}=(\{1,2,3\}).$$

Consequently, $T^{(1)},\ldots,T^{(5)}$ generate respectively 1,2,2,2,6 elements of P_3 and P_3 has 13 elements.

Here are now the functions s_P indexed by $P\in\mathrm{P}_n$ which imitate r_π. For a given P defined by q and the (T_j,π_j) for $j=1,\ldots,q$, we define

$$s_P(\sigma)(h_1,\ldots,h_d)=\prod_{j=1}^{q}\mathrm{Tr}\left(\sigma B\prod_{i\in T_j}\sigma h_{\pi_j(i)}\right).$$

The differential of g can then be obtained as follows:

$$g^{(d)}(S)(h_1,\ldots,h_d)=g(S)\sum_{P\in\mathrm{P}_d}s_P(\sigma)(h_1,\ldots,h_d).$$

We now apply this to the computation of first order and second order moments. For simplicity, we write $B'=\sigma B$ and $h'_j=\sigma h_j$. The previous propositions allow to obtain the following results

$$\begin{aligned}
\frac{f'(S)(h_1)}{f(S)} &= \mathrm{Tr}(B'h'_1),\\
\frac{f''(S)(h_1,h_2)}{f(S)} &= \mathrm{Tr}(B'h'_1)\mathrm{Tr}(B'h'_2)+\\
&\qquad \mathrm{Tr}(B'h'_1h'_2)+\mathrm{Tr}(B'h'_2h'_1),\\
\frac{g'(S)(h_1)}{g(S)} &= p\mathrm{Tr}(h'_1),\\
\frac{g''(S)(h_1,h_2)}{g(S)} &= p^2\mathrm{Tr}(h'_1)\mathrm{Tr}(h'_2)+p\mathrm{Tr}(h'_1h'_2).
\end{aligned}$$

These two examples are then combined with the Leibnitz formula (5) to obtain the first two differentials of fg. The moments of the non central Wishart random variable with Laplace transform (1) can then be obtained as follows:

$$\begin{aligned}
E[U_{ij}] &= L'(S)(h_{ij})|_{S=0},\\
&= \exp\left[-\mathrm{Tr}\ (B)\right]\nu'\left(2\Sigma S-\mathbb{I}_d\right)\left(2\Sigma h_{ij}\right)|_{S=0},\\
E[U_{ij}U_{kl}] &= L''(S)(h_{ij},h_{kl})|_{S=0},\\
&= \exp\left[-\mathrm{Tr}\ (B)\right]\nu''\left(2\Sigma S-\mathbb{I}_d\right)\left(2\Sigma h_{ij},2\Sigma h_{kl}\right)|_{S=0},
\end{aligned}$$

where h_{ij} is a $n\times n$ matrix whose elements are 0 except the element located at the ith line and the jth column which equals 1.

3. APPLICATION TO SPECKLE IMAGING

Increasing interest has been shown in the astronomical community for the problem of imaging extrasolar planets. More than hundred planets have already been discovered using indirect techniques. However, the direct imaging is the only method which should answer questions such as the study of planet formation and the search for life. We think that detection methods used in statistical signal processing might be useful for exoplanet detection. For instance, these methods allow to define appropriate signal to noise ratio associated to the planet and the background. The extrasolar planet detection problem can be formulated as a binary hypothesis testing problem. Different strategies can then be applied, depending on our knowledge regarding the parameters of the distributions under both hypotheses. In any case, it is important to derive the statistical properties of the observations as precisely as possible.
As explained in [6], the complex amplitude of a wave in the focal plane of a telescope, at a position (x,y), can be written as follows:

$$\psi(x,y)=C(x,y)+S(x,y),$$

where $C(x,y)\in\mathbb{C}$ is a deterministic term proportional to the wave amplitude in absence of turbulence and $S(x,y)\in\mathbb{C}$ is the wavefront amplitude (associated to the speckles) distributed according to a zero mean complex Gaussian distribution. We assume here that the telescope aperture has central symmetries which imply $C(x, y) \in \mathbb{R}$. In this case, by denoting as $I_s(x, y) = \mathsf{E}(|S(x, y)|^2)$ the variance of the complex amplitude, the real and imaginary parts of $\psi(x, y)$ denoted by $\psi_r(x, y)$ and $\psi_i(x, y)$) are independent Gaussian variables:

$$\psi_r(x, y) \sim \mathcal{N}(C(x, y), I_s(x, y)/2),$$
$$\psi_i(x, y) \sim \mathcal{N}(0, I_s(x, y)/2).$$

The instantaneous intensity of the wave in the focal plane at a position (x, y), denoted as $\Lambda(x, y)$, is related to its complex amplitude as follows

$$\Lambda(x, y) = |\psi(x, y)|^2 = \psi_r^2(x, y) + \psi_i^2(x, y). \quad (6)$$

Consequently, $2\Lambda(x, y)/I_s(x, y)$ is distributed according to a noncentral χ^2 distribution with 2 degrees of freedom and noncentrality parameter $2C^2(x, y)/I_s(x, y)$. By omitting the notation (x, y) for brevity, the probability density function (pdf) of Λ, sometimes referred to as the modified Rician pdf, can then be expressed as [6], [7], [8]:

$$p(\Lambda) = \frac{1}{I_s} \exp\left(-\frac{\Lambda + I_c}{I_s}\right) I_0\left(\frac{2\sqrt{\Lambda I_c}}{I_s}\right),$$

where $I_c = C^2(x, y)$ is the intensity of the deterministic part of the wave and $I_0(.)$ is the modified Bessel function of the first kind [9, p. 376].

The generalization of this analysis to the multidimensional case is straightforward. To simplify notations, we rearrange the intensities $\Lambda(x, y)$ and the amplitudes $\psi(x, y)$, $\psi_r(x, y)$ and $\psi_i(x, y)$ into vectors of length $d = n^2$, denoted by :

$$\Lambda = [\Lambda(1), \dots, \Lambda(d)]^T, \psi = [\psi(1), \dots, \psi(d)]^T,$$
$$\psi_r = [\psi_r(1), \dots, \psi_r(d)]^T, \psi_i = [\psi_i(1), \dots, \psi_i(d)]^T.$$

Equation (6) shows that **the intensity vector Λ consists of the diagonal elements of the $d \times d$ matrix $\psi_r\psi_r^T + \psi_i\psi_i^T$**. In order to model correlations between the complex amplitudes at different positions, this paper proposes to model the two random vectors ψ_r and ψ_i as follows:

$$\psi_r \sim \mathcal{N}(C, \Sigma), \qquad \psi_i \sim \mathcal{N}(0, \Sigma),$$

where $C = [C(1), \dots, C(d)]^T$ and Σ is a covariance matrix whose diagonal elements are $I_s(x, y)/2$ (according to (6)). The distribution of $\psi_r\psi_r^T + \psi_i\psi_i^T$ is a noncentral Wishart distribution

$$\psi_r\psi_r^T + \psi_i\psi_i^T \sim \mathcal{W}_2(A^{-1}CC^TA^{-1}, \Sigma).$$

As a consequence, the components of the intensity vector Λ are the diagonal elements of a noncentral Wishart distribution (which can be also defined as a "multivariate noncentral chi-squared" distribution). The Laplace transform of this distribution can be obtained from (4), with $M = A^{-1}CC^TA^{-1}$ and $\theta = ASA$. Note that a similar analysis conducted in [10] for $C = 0$ led to a multivariate chi-squared distribution [11].

It is interesting to note that the marginal distributions of the intensities $\Lambda(i)$ for $i = 1, ..., d$ can be obtained as a byproduct of (4). Denote as diag$(x_1, ..., x_d)$ a diagonal matrix whose diagonal elements are $x_1, ..., x_d$. By setting $S = \text{diag}(0, ..., 0, u, 0, ..., 0)$, where u is at the ith position (associated to a particular position (x, y)), the Laplace transform of U can be written as follows (see appendix for details):

$$L(u) = \frac{1}{1 + I_s u} \exp\left(-\frac{I_c u}{1 + I_s u}\right),$$

with $I_c = C^2(i) = C^2(x, y)$. This result coincides with equation (10) of [6].

The first and second order moments of light intensities can be obtained by setting $d = n^2$,$k = 2$, $\mu = CC^T$ and $\Sigma = (\Sigma_{ij})$ in the results of section 3.4. The following results are then obtained

$$\begin{aligned} E[\Lambda(i)] &= C^2(i) + 2\Sigma_{ii}, \\ E[\Lambda(i)\Lambda(j)] &= C^2(i)C^2(j) + 2C^2(i)\Sigma_{jj} + 2C^2(j)\Sigma_{ii} \\ &\quad + 4\Sigma_{ij}^2 + 4C(i)C(j)\Sigma_{ij} + 4\Sigma_{ii}\Sigma_{jj}. \end{aligned}$$

This result coincide with the expressions obtained after replacing $\Lambda(i)$ by (6) and expanding the fourth order moments of a complex circular Gaussian variable as a function of its second and first order moments.

It is interesting to note that the proposed model is also interesting for synthetic aperture radar images. The marginal distributions of SAR images corrupted by speckle can be found in many textbooks such as [3]. The speckle model leads to an intensity which is marginally distributed according to a noncentral χ^2 distribution, when a strong scatterer is embedded in weak clutter [3, p. 113]. The extension to a set of intensities associated to several pixels of the image is similar to the previous analysis. It leads to the diagonal of a multivariate noncentral Wishart distribution.

4. MULTIVARIATE PHOTON STATISTICS

The Wishart distributions studied in the previous section assume that the image has been recorded under a high flux assumption. However, for low-flux objects or short exposure time, the photocounting effect has to be considered. This section studies the statistical properties of low-flux images. The photon statistics at any point of the image is obtained by computing the Poisson-Mandel transform of $p(\mathbf{\Lambda})$. More precisely, denote as N_i the number of photons recorded at a given point of the image. The random variables $N_i, i = 1, \dots, d$ are independent and distributed according to Poisson distributions with means $\Lambda(1), \dots, \Lambda(d)$, conditioned upon the vector of intensities $\mathbf{\Lambda} = (\Lambda(1), \ldots, \Lambda(d))$. In this case, the probability masses of $\boldsymbol{N} = (N_1, \ldots, N_d)$ are defined as

$$\Pr(\boldsymbol{N} = \boldsymbol{k}) = \int_{(\mathbb{R}^+)^d} \cdots \int \prod_{\ell} \frac{(\Lambda(l))^{k_\ell}}{k_\ell!} \exp\left(-\Lambda(l)\right) p(\mathbf{\Lambda}) d\mathbf{\Lambda}, \quad (7)$$

where $p(\mathbf{\Lambda})$ is the diagonal of a noncentral Wishart distribution defined in section 3. Note that the distribution of $\boldsymbol{N}$ is fully characterized by the pdf $p(\mathbf{\Lambda})$. Tractable expressions of $\Pr(\boldsymbol{N} = \boldsymbol{k})$ defined in (7) are obviously difficult to obtain. However, many interesting properties regarding the distribution of $\boldsymbol{N}$ can be derived [12]. Some of these properties are recalled below:

- **Moment Generating Function**. The moment generating function of $\boldsymbol{N}$ expresses as:

$$\begin{aligned} \mathsf{E}\left(\prod_{k=1}^{d} z_k^{N_k}\right) &= \mathsf{E}\left(\prod_{k=1}^{d} \mathsf{E}\left(z_k^{N_k} | \Lambda(k)\right)\right), \\ &= \mathsf{E}\left(\prod_{k=1}^{d} e^{\Lambda(k)(z_k - 1)}\right), \\ &= L(z_1 - 1, \ldots, z_d - 1), \end{aligned} \quad (8)$$

  where $L(\boldsymbol{z})$ is the Laplace transform of the intensity vector $\mathbf{\Lambda}$ obtained from (4).

- **Moments**. Multivariate factorial moments yield much simpler expressions than classical joint moments as in the univariate case. By denoting $N^{[r]} = N(N-1)(N-r+1)$, the following results can be obtained:

$$\begin{aligned} \mathsf{E}\left(\prod_{k=1}^{d} N_k^{[r_k]}\right) &= \mathsf{E}\left(\prod_{k=1}^{d} \mathsf{E}\left(N_k^{[r_k]} | \Lambda(k)\right)\right), \\ &= \mathsf{E}\left(\prod_{k=1}^{d} \Lambda(k)^{r_k}\right). \end{aligned} \quad (9)$$

  The last equality has been obtained from the factorial moments of a Poisson distribution. Joint moments can be derived by substituting in $\mathsf{E}(\prod_{k=1}^{d} N_k^{r_k})$ each $N_k^{r_k}$ by its expression as a function $N_k^{[r]}$, $r \leq r_k$, (see for example [13, p. 44]). Expanding the products and using (9) yields:

$$\mathsf{E}\left(\prod_{k=1}^{d} N_k^{r_k}\right) = \sum_{j_1=0}^{r_1} \cdots \sum_{j_d=0}^{r_d} \prod_{k=1}^{d} S(r_k, j_k) \mathsf{E}\left(\prod_{k=1}^{d} \Lambda(k)^{j_k}\right),$$

  where $S(j,k)$ are the Stirling numbers of the second kind [13]. Note that this last expression can also be obtained by derivation of the moment generating function (8). In particular, the following relations between the covariance matrices of $\boldsymbol{N}$ and $\mathbf{\Lambda}$ can be obtained:

$$\begin{aligned} \text{Cov}(\boldsymbol{N}) &= \text{Cov}(\mathbf{\Lambda}) + \text{Diag}(\mathsf{E}(\Lambda(1)), \ldots, \mathsf{E}(\Lambda(d))), \\ &= \text{Cov}(\mathbf{\Lambda}) + \text{Diag}(\mathsf{E}(N_1), \ldots, \mathsf{E}(N_d)). \end{aligned}$$

This section showed that the moments of photon statistics can be easily computed from the moments of the intensities.

5. CONCLUSIONS

This paper studied some properties of noncentral Wishart distributions. These results are interesting for modeling high flux images corrupted by speckle in two cases: optical imaging through turbulence and synthetic aperture radar imaging. Closed form expressions for the moments of noncentral Wishart distributions were derived. These moments can be useful for several image processing applications including image restoration (by matched filter). The statistical properties as well as the moments of low flux images were also discussed. The next step in our study will be to use these models for high-flux and low-flux astronomical data and exploit them for estimation purposes. An example of estimation procedure (based on the maximization of a composite likelihood criterion) which might be useful in this context is studied in [14].

6. APPENDIX

This appendix derives the Laplace transform of the ith intensity $\Lambda(i)$ by setting $S = \text{diag}(0, ..., 0, u, 0, ..., 0)$ (u is at the ith position) in (4). By denoting $A = (A_{ij})$ and by expanding the matrix ASA, we obtain:

$$ASA = (ua_{1i}\text{col}_i, ..., ua_{di}\text{col}_i),$$

where col_i is the ith column of the matrix A. Thus, $\det(\mathbb{I}_d + 2ASA)$ is an affine polynomial with respect to u, which can be expressed as

$$\det(\mathbb{I}_d + 2ASA) = 1 + 2\left(\sum_{j=1}^{d} a_{ji}^2\right) u = 1 + I_s u.$$

Indeed, $\sum_{j=1}^{d} a_{ji}^2$ is the element Σ_{ii} of the matrix $\Sigma = A^2$, which equals $I_s/2$. Moreover, by denoting $B = A(\mathbb{I}_d +$

$2ASA)^{-1}A^{-1} = (\mathbb{I}_d + 2u\Sigma)^{-1}$, we obtain

$$\begin{aligned}\text{Tr}\left[ASA(\mathbb{I}_d + 2ASA)^{-1}M\right] &= \text{Tr}\left[ASBCC^T A^{-1}\right] \\ &= \text{Tr}\left[SBCC^T\right] \\ &= \text{Tr}\left[CC^T SB\right] \\ &= uC_i \sum_{j=1}^{d} C_j B_{ij}.\end{aligned}$$

By using the standard result $(I - X)^{-1} = \sum_{j=0}^{\infty} X^j$ (valid for a matrix X with norm $\|X\| < 1$), straightforward computations show that $B_{ij} = 0$ if $i \neq j$ and

$$B_{ii} = \sum_{j=0}^{\infty} (-1)^j (2u\Sigma)^j = \frac{1}{1 + 2u\Sigma_{ii}} = \frac{1}{1 + uI_s},$$

hence

$$\text{Tr}\left[ASA(\mathbb{I}_d + 2ASA)^{-1}M\right] = \frac{uC_i^2}{1 + uI_s} = \frac{uI_c}{1 + uI_s}.$$

The Laplace transform of the ith intensity $\Lambda(i)$ can then be obtained:

$$L(u) = \frac{1}{1 + I_s u} \exp\left(-\frac{I_c u}{1 + I_s u}\right).$$

7. REFERENCES

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