DIFFUSED DISTRIBUTION REASSIGNMENT

Julien GOSME, Cédric RICHARD

Institut des Sciences et Technologies de l'Information de Troyes - FRE CNRS 2732 UTT
12, rue Marie Curie - B.P. 2060 - 10010 Troyes cedex - France
tél : 03.25.71.80.39 fax : 03.25.71.56.99
julien.gosme@utt.fr

ABSTRACT

In this paper, we take a look at diffusion methods and reassignment techniques to process adaptive time-frequency distributions. We briefly recall both techniques. Next we reinterpret the reallocation of spectrogram in terms of homogeneous diffusion. This allows us the extend the classic reassignment process via adaptive diffusion. Finally, some simulations are proposed to illustrate the efficiency of our approach.

1. INTRODUCTION

Because time-frequency representations (TFR) illustrate evolutions of signals with respect to both time and frequency, they have been largely used to deal with non-stationary environment. Among the host of solutions that have been proposed, Cohen class encloses bilinear TFR that are covariant with respect to time shifts and frequency shifts. Such tools lead to representations with better localization properties, but at the cost of undesirable cross-terms [1]. One of the main goal of time-frequency smoothing is to improve readability by removing these cumbersome cross-terms while preserving localization of signal components.

In the context of homogeneous smoothing, a low-pass kernel is chosen such that the trade-off between readability and localization is optimum. The radial gaussian kernel proposed in [2] is such a kernel. Observing that the analyzed signals are nonstationary, many authors have proposed to use locally adaptive techniques. This paper deals with two of them, namely adaptive diffusion [3, 4, 5] and reassignment [6, 7]. We interpret spectrogram reassignment in terms of homogeneous diffusion. This allows us to extend the classic diffusion process via adaptive diffusion.

2. TIME-FREQUENCY DISTRIBUTION DIFFUSION

In this first part we briefly recall the principles of adapted diffusion. For a more thorough treatment, the reader shall refer to [3, 4, 5]. We start by a presentation of homogeneous diffusion and then review adaptive diffusion.

2.1. Homogeneous diffusion

Diffusion is the process by which matter is transported from areas of high concentration to areas of lower concentration as a result of the movement of an ensemble of molecules inside a region. This is mathematically formulated by a basic equation referred to as Fick's law [8] :

$$J = -C\,\nabla U, \tag{1}$$

where J is the flux of molecules, U their concentration, C the diffusion tensor, and ∇ the gradient operator. If C is a scalar-valued conductance function, which implies that J and ∇U are collinear, the diffusion process is called *isotropic*. Otherwise it is called *anisotropic*. It is said that the diffusion process is *homogeneous* if the diffusion tensor C is constant over the region of interest. Location-dependent diffusion is called *non-homogeneous* or *inhomogeneous*. The law of conservation of mass is expressed as :

$$\frac{\partial U}{\partial \tau} = -\mathrm{div}(J)\,, \tag{2}$$

where τ is the diffusion time, and div the divergence operator. Combining this relationship with Fick's law produces the law of diffusion, which states :

$$\frac{\partial U}{\partial \tau} = -\mathrm{div}(-C\,\nabla U)\,. \tag{3}$$

In this paper, this diffusion is said to be *linear* because the tensor C does not vary with τ. Let us now restrict our discussion to the partial differential equations (PDE's) in two-plus-one dimension

$$\begin{cases} U(v_1, v_2; \tau = 0) = U_0(v_1, v_2) \\ \frac{\partial U}{\partial \tau} = \mathrm{div}(\nabla U)\,, \end{cases} \tag{4}$$

where $U_0 \in \mathcal{L}^1(\mathbb{R}^2)$ denotes the initial spatial condition. It is well-known that the solution of (4) is

$$U(v_1, v_2; \tau) = \begin{cases} (G * U_0)(v_1, v_2) & (\tau > 0) \\ U_0(v_1, v_2) & (\tau = 0), \end{cases} \tag{5}$$

with $*$ the usual 2-D convolution, and $G(v_1, v_2; \tau) = (4\pi\tau)^{-1}\exp(-[v_1^2 + v_2^2]/4\tau)$ an isotropic Gaussian kernel which is referred to as the Green function of the PDE given above. This means that the solution $U(v_1, v_2; \tau)$ of the heat diffusion equation (4) at each time instant τ can simply be obtained by convolution of the initial spatial condition $U_0(v_1, v_2)$ with the Green function $G(v_1, v_2; \tau)$.

The spectrogram is a widely used tool that belongs to the Cohen class. As the square modulus of the short-time Fourier transform, it can also be written as a convolution between the Wigner distribution of the signal and that of the analysis window. Note that the Wigner distribution of a gaussian window is a 2D-gaussian kernel. Interpreting time-frequency representation as a heat distribution one can consider its diffusion as follows :

$$\begin{cases} D_x(t,f;\tau=0) \;=\; W_x(t,f) \\ \frac{\partial D_x(t,f;\tau)}{\partial\tau} \;=\; \mathrm{div}_{t,f}(\nabla_{t,f} D_x(t,f;\tau)), \end{cases} \tag{6}$$

where W_x is the representation to be processed, which plays the role of the initial state of the diffusion process. The diffused representation $D_x(t,f;\tau)$ denotes the energy distribution at the time instant τ. As we just stated, the solution of such classical heat diffusion equation is an isotropic gaussian function. Therefore the partial derivative equation (6) has the following solution :

$$D_x(t,f;\tau) = \frac{1}{4\pi\tau}\iint W_x(\eta,\nu)e^{-\frac{(t-\eta)^2+(f-\nu)^2}{4\tau}} d\eta d\nu. \tag{7}$$

Thus the use of the heat diffusion equation with a Wigner distribution is equivalent to convolving it with a gaussian kernel whose variance increases with the diffusion time τ. We will detail these results in the fourth section.

2.2. Adaptive diffusion

Adaptive diffusion was recently introduced in the context of time-frequency distributions smoothing [3]. In this paper, the authors propose a smoothing based on the link between heat diffusion and gaussian kernel smoothing. Next they locally tune the diffusion process in order to get a locally adaptive smoothing scheme that reads :

$$\begin{cases} D_x(t,f;\tau=0) \;=\; W_x(t,f) \\ \frac{\partial D_x(t,f;\tau)}{\partial\tau} \;=\; \mathrm{div}_{t,f}(c_x(t,f)\nabla_{t,f} D_x(t,f;\tau)), \end{cases} \tag{8}$$

where W_x is the to-be-smoothed distribution, D_x is the smoothed distribution up to time τ and c_x is the function that locally controls the amount of smoothing. This later is referred to as the conductance function.

In an analysis context, the aim is to remove cross terms while preserving the localization of signal terms. In [3], the authors have proposed a scalar valued function to achieve this goal. This diffusion is called *isotropic*. This technique relies on the spectrogram S_x. Although this distribution spreads out signal components in the time-frequency domain, it is approximately equal to zero over non-energetic areas where the interference terms of the WD are likely to be situated. Thus making the conductance function a decreasing function of S_x such as

$$c_{S_x}(t,f) = \left[1 + \left(\frac{S_x(t,f)}{\beta}\right)^{\alpha}\right]^{-1} (\alpha,\beta) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \tag{9}$$

allows to tune the diffusion rate over the time-frequency domain. In [4, 5], the authors have proposed a tensor valued conductance function that enables to act on both the local strength and orientation of the smoothing, called *anisotropic* diffusion. In a nutshell, diffusion methods start from a concentrated distribution with cross terms and selectively smooth them out while preserving localization of signal components.

3. TIME-FREQUENCY DISTRIBUTION REASSIGNMENT

Another very popular method tackles the problem from another perspective. The reassignment method starts from an interference free distribution, like the spectrogram for example, and increases its sharpness. The interference-free distribution results from the low-pass filtering of a Wigner distribution. This has two consequences : elimination of interferences and delocalization of signal-terms. For a spectrogram S_x based on a window $h(t)$,

$$S_x(t,f;h) = \iint W_x(\eta,\nu)W_h(\eta-t,\nu-f)d\eta\, d\nu, \tag{10}$$

the reassignment process operates as follows [6, 7] :

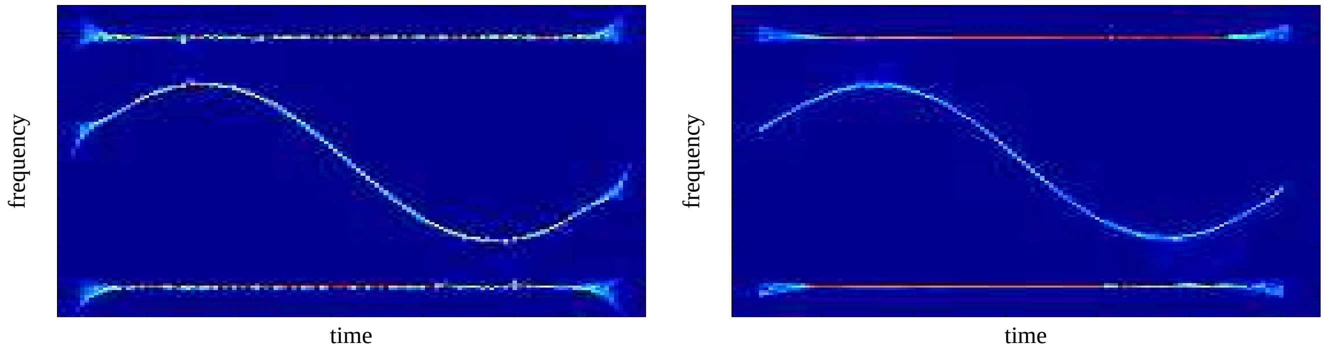
$$\hat{S}_x(t,f;h) = \iint S(\eta,\nu;h)\delta(t-\hat{t}_x^h(\eta,\nu), f-\hat{f}_x^h(\eta,\nu))d\eta\, d\nu, \tag{11}$$

where $\hat{t}_x$ and $\hat{f}_x^h$ indicates the location where the energy has to be moved to. They are given by $\hat{f}_x^h(t,f) = \overline{f}_x^h(t,f)/S_x(t,f;h)$ and $\hat{t}_x^h(t,f) = \overline{t}_x^h(t,f)/S_x(t,f;h)$ with

$$\overline{t}_x^h(t,f) \;=\; \iint \eta W_x(\eta,\nu)W_h(\eta-t,\nu-f)d\eta\, d\nu \tag{12}$$

and

$$\overline{f}_x^h(t,f) = \iint \nu W_x(\eta,\nu)W_h(\eta-t,\nu-f)d\eta\, d\nu. \tag{13}$$



(a) Reassigned spectrogram (b) Reassigned diffused distribution

Fig. 1. Comparison between the reassignment of a spectrogram and the reassignment of a distribution processed by adaptive diffusion.

In the next section we present an alternative reassignment formula based on diffusion process for gaussian spectrograms and distributions resulting from a separable smoothing. We then extend these results to propose a reassigned version of a locally adaptive distribution.

4. DIFFUSED DISTRIBUTION REASSIGNMENT

4.1. Alternative Cohen distribution reassignment

As hinted in the first part, one can use the Green function of time-frequency diffusion to obtain equivalences with members of the Cohen class. More precisely, as the diffusion

$$\begin{cases} D_x(t,f;\tau=0) &= W_x(t,f) \\ \frac{\partial D_x(t,f;\tau)}{\partial \tau} &= \mathrm{div}_{t,f}(\nabla_{t,f} D_x(t,f;\tau)), \end{cases} \quad (14)$$

is equivalent to the convolution

$$D_x(t,f;\tau) = \frac{1}{4\pi\tau} \iint W_x(\eta,\nu) e^{-\frac{(t-\eta)^2+(f-\nu)^2}{4\tau}} d\eta d\nu, \quad (15)$$

one can put forward the two following propositions.

Proposition 4.1. *Isotropic homogeneous diffusion smoothing $D_x(t,f)$ of the Wigner distribution of a signal x up to time τ is equivalent to the separable smoothing of the Wigner distribution :*

$$W_x(t,f) * [g(t)\ H(-f)], \quad (16)$$

of the same signal x using gaussian windows such as

$$g(t) = (4\tau\pi)^{-1/2}\ e^{-t^2/(4\tau)} \quad (17)$$

$$H(f) = (4\tau\pi)^{-1/2}\ e^{-f^2/(4\tau)}. \quad (18)$$

This proposition can be easily verified as

$$g(t)\ H(-f) = \frac{1}{4\pi\tau} e^{-\frac{(t)^2+(f)^2}{4\tau}}. \quad (19)$$

As the spectrogram with a gaussian window is a special case of separable smoothing of the Wigner distribution with $g(t) = h(t)$, we also have :

Proposition 4.2. *Isotropic homogeneous diffusion smoothing $D_x(t,f)$ of the Wigner distribution of a signal x up to time $\tau = \frac{1}{8\pi}$ is equivalent with the spectrogram of that signal using a gaussian window h as follows :*

$$h(t) = 2^{1/4} e^{-\pi t^2}. \quad (20)$$

Choosing $h(t) = 2^{1/4} e^{-\pi t^2} = g(t)$ implies that

$$g(t)\ H(-f) = 2e^{-2\pi\left(\alpha t^2 + \frac{f^2}{\alpha}\right)}. \quad (21)$$

Then (16) and (15) are equal for $\tau = \frac{1}{8\pi}$.

These results can be used to propose alternative implementations of spectrograms and of distributions smoothed via separable kernels.

Equations (12) and (13) can be interpreted as the convolution of $tW_x(t,f)$ and $fW_x(t,f)$ with the smoothing kernel employed in (10). Building on the equivalence between (14) and (15) one can use the following diffusions :

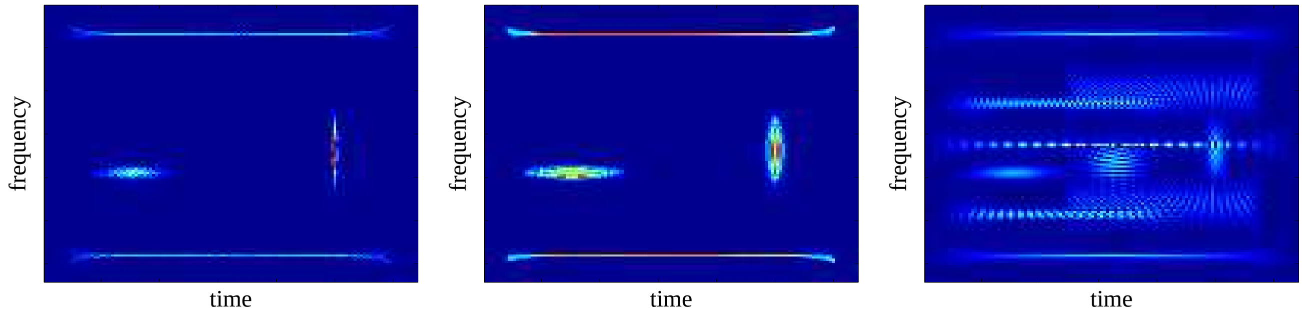
$$\begin{cases} \bar{t}_x(t,f;\tau=0) &= tW_x(t,f) \\ \frac{\partial \bar{t}_x(t,f;\tau)}{\partial \tau} &= \mathrm{div}_{t,f}(\nabla_{t,f} \bar{t}_x(t,f;\tau)) \end{cases} \quad (22)$$

and

$$\begin{cases} \bar{f}_x(t,f;\tau=0) &= fW_x(t,f) \\ \frac{\partial \bar{f}_x(t,f;\tau)}{\partial \tau} &= \mathrm{div}_{t,f}(\nabla_{t,f} \bar{f}_x(t,f;\tau)) \end{cases} \quad (23)$$

to compute time and frequency locations for reassignment. The reassignment process is then defined as

$$\hat{D}_x(t,f;\tau) = \iint D_x(\eta,\nu;\tau)\delta(t-\hat{t}_x(\eta,\nu;\tau), f-\hat{f}_x(\eta,\nu;\tau)) d\eta d\nu$$



(a) Reassigned spectrogram (b) Reassigned diffused distribution (c) Wigner distribution

Fig. 2. Comparison between the reassignment of a spectrogram and the reassignment of a distribution processed by adaptive diffusion. Isotropy of the diffusion kernel yields a similar treatment to both atoms. Entropy was used as a stopping criterion. The Wigner distribution is provided as a reference.

with $\hat{f}_x = \overline{f}_x / D_x$ and $\hat{t}_x = \overline{t}_x / D_x$.

This provides an alternative way to obtain reassigned distributions. Using stable diffusion numerical procedures, such as the additive operator splitting method presented in [9], this can be a very efficient technique. It can also be used in situations were the iterative nature of the diffusion would be an advantage. We now propose to leverage this into signal-adapted distribution reassignment by the use of adaptive diffusion instead of homogeneous ones.

4.2. Signal adaptive distribution reassignment

In order to obtain signal adaptive distributions, one can use adaptive diffusions such as :

$$\begin{cases} D_x(t,f;\tau=0) \;=\; W_x(t,f) \\ \frac{\partial D_x(t,f;\tau)}{\partial \tau} \;=\; \mathrm{div}_{t,f}(c_x(t,f)\nabla_{t,f} D_x(t,f;\tau)). \end{cases} \tag{24}$$

In that case we assume that the Green function exists and we denote it as $G(t,f;\tau,x)$ to emphasize its local dependency on the signal. Reassignment of this signal-adapted distribution requires to compute the time and frequency locations using :

$$\overline{t}_x(t,f;\tau) = \iint \eta W_x(\eta,\nu) G(\eta - t, \nu - f; \tau, x) d\eta\, d\nu \tag{25}$$

and

$$\overline{f}_x(t,f;\tau) = \iint \nu W_x(\eta,\nu) W_h(\eta - t, \nu - f; \tau, x) d\eta\, d\nu. \tag{26}$$

Note that it requires a formal knowledge of the signal-dependant Green function. This analytical formulation, up to the authors knowledge, is still an open question. However according to the definition of the Green function, it is equivalent to use either the convolution or the diffusion. We therefore propose to use the diffusion, bypassing the need to compute the Green function $G(t,f;\tau,x)$, as follows :

$$\begin{cases} \overline{t}_x(t,f;\tau=0) \;=\; tW_x(t,f) \\ \frac{\partial \overline{t}_x(t,f;\tau)}{\partial \tau} \;=\; \mathrm{div}_{t,f}(c_x(t,f)\nabla_{t,f}\overline{t}_x(t,f;\tau)), \end{cases} \tag{27}$$

and

$$\begin{cases} \overline{f}_x(t,f;\tau=0) \;=\; fW_x(t,f) \\ \frac{\partial \overline{f}_x(t,f;\tau)}{\partial \tau} \;=\; \mathrm{div}_{t,f}(c_x(t,f)\nabla_{t,f}\overline{f}_x(t,f;\tau)). \end{cases} \tag{28}$$

The reassignment process is done as previously.

4.3. Stopping criterion and illustrations

Without constraints on the diffusion time, D_x would converge to a uniform distribution over the time-frequency domain regardless of the analyzed signal. Hence, this reassigned diffusion process is to be stopped when some criterion is achieved. The Rényi entropy, defined as [10]

$$H_{\hat{D}}(\tau) = -\frac{1}{1-\alpha} \log \iint \hat{D}_x^{\alpha}(t,f;\tau) dt\, df,$$

is a natural candidate for measuring the concentration of TFRs[1]. While in previous studies, e.g. [4, 5], we used the entropy of the diffused distribution as a stopping criteria, we propose here to use the entropy of the reassigned distribution. While the first iterations of the diffusion are smoothing out interference terms, the later ones slowly regularize the signal. Too much diffusion produces a uniform distribution that cannot be reassigned properly anymore. This yields a

[1]We suggest the reader to refer to [10] for detailed study of the Rényi's entropies as time-frequency information measures. We also invite him to consult [11, 12] for full details on the Rényi's entropies as a means of extracting information from images during diffusion.

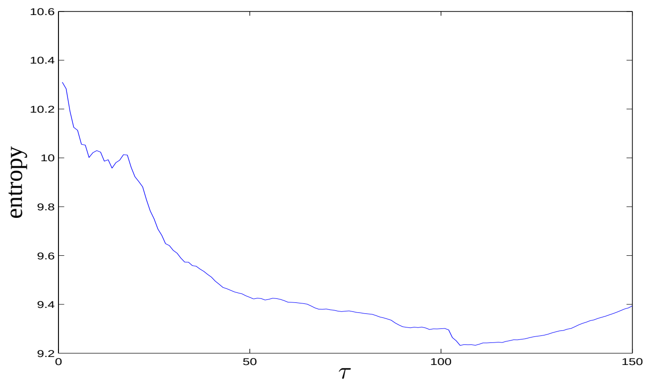


Fig. 3. Evolution of the entropy, $\alpha = 3$, of the reassigned diffusion as a function of τ. Experimentally, we observe that interferences are smoothed out during the first iterations of the process, while the next ones operate on the regularity of the distribution. Chosen stopping time is here $\tau^* = 105$.

minimum in the entropy curve that we use as a stopping point for the diffusion process, see figure 3 for an illustration.

As we can see on the figure 1, both representations are interference free and very concentrated. We note that using a diffused distribution yields a more regular result combining the good properties of both techniques. This denotes a smaller entropy. Note that the entropy of the reassigned spectrogram (see figure 1.a) is 9.4 whereas the entropy of the reassigned diffused distribution (see figure 1 .b) is 9.2. Figure 2 illustrates this technique with another signal. One can note that the shape of components is better preserved by our technique than by the reassigned spectrogram. The Wigner distribution is also provided as a reference of the signal of interest.

5. CONCLUSION

In this paper, we proposed an extension of the reassignment technique. First we reviewed diffusion and reassignment techniques. We interpreted the reassignment of spectrogram in terms of homogeneous diffusion and proposed an alternative way to compute time and frequency localization of reassignment. This new formulation is based on two diffusion equations which led us to propose an extension of the conventional reassignment process via adaptive diffusion.

One should note that our approach can be easily extended to include the recent developments of diffusion processing such as anisotropy and processing of time-scale representations.

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