A STAP ALGORITHM FOR RADAR TARGET DETECTION IN HETEROGENEOUS ENVIRONMENTS

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ABSTRACT

Traditional STAP processors for radar target detection, such as the GLRT and AMF, require an estimate of the noise covariance matrix. In practice, this estimate is obtained from a training data set that is usually constructed from range gates surrounding the test gate. The training data must be target free and statistically homogeneous with the test data. In heterogeneous and target rich environments, these assumptions do not necessarily hold and degradation in the detection performance results. In this paper, we propose a new detection algorithm, which we call the maximum likelihood estimation detector (MLED), and that operates only on the test data. We show that the new detector has the highly desirable CFAR property. We give the expressions for its probabilities of false alarm and detection and show that it has a performance that is comparable with the traditional algorithms.

1. INTRODUCTION

The detection of a signal in colored Gaussian noise is relevant to many fields such as radar, sonar and communications to name a few. In this paper, we focus on the detection of targets in the context of radar space-time adaptive processing (STAP), see [1] and [2]. Consider a linear antenna array consisting of M sensors that collects K_T data snapshots denoted by $\mathbf{x}_k$, $k = 1 \cdots K_T$. Then, the problem is that of detecting the presence of a signal with spatial and temporal steering vectors $\mathbf{s}_s$ and $\mathbf{s}_t$ respectively. The model we adopt here is

$$\mathbf{X} = \alpha \mathbf{s}_s \mathbf{s}_t^T + \mathbf{N} \tag{1}$$

where the received data snapshots have been arranged into an $(M \times K_T)$ matrix $\mathbf{X}$, and α is a complex magnitude. Note that the temporal and spatial notation of the steering vectors is for convenience and in general each of the steering vectors $\mathbf{s}_s$ and $\mathbf{s}_t$ is a space-time vector that is composed of interleaved temporal and spatial vectors, see [1]. The $(M \times K_T)$ matrix $\mathbf{N}$ consists of zero-mean circular complex Gaussian interference (clutter plus noise) with iid columns $\mathbf{n}_k \sim \mathcal{CN}_M(\mathbf{0}, \mathbf{C})$. The detection problem is usually treated as a hypothesis test with the null and alternative hypotheses given by

$$\begin{aligned} &\mathrm{H}_0: \quad \mathbf{X} = \mathbf{N} \\ \text{and} \qquad & \\ &\mathrm{H}_1: \quad \mathbf{X} = \alpha \mathbf{s}_s \mathbf{s}_t^T + \mathbf{N}. \end{aligned} \tag{2}$$

The optimum STAP filter, derived in [3], essentially consists of a whitening operation followed by a matched filter. The weights vector, $\mathbf{w}_{opt}$, is given by

$$\mathbf{w}_{opt} = \beta \mathbf{C}^{-1} \mathbf{s}_s, \tag{3}$$

β being an arbitrary constant. For K_T available test snapshots, the filter is applied to the coherently combined data. Let $\mathbf{g} = c\mathbf{X}\mathbf{s}_t^*$ with c being an arbitrary normalization constant, then the filter output is $y = \mathbf{w}_{opt}^H \mathbf{g}$. The output power is then compared to a threshold according to the test

$$|y|^2 \underset{H_0}{\overset{H_1}{\gtrless}} \gamma. \tag{4}$$

A highly desirable requirement of any radar target detector is the constant false alarm rate (CFAR) property which provides the ability to control the probability of false alarm. This can be achieved with a suitable choice of the normalization constant β. Putting $\beta = \left(\mathbf{s}_s^H \mathbf{C}^{-1} \mathbf{s}_s\right)^{-\frac{1}{2}}$ results in the required normalization and gives the processor denoted as the matched filter (MF).

In practice, the true covariance matrix is usually unknown and is estimated from a training data set that is drawn from neighboring range gates. The training data must be target free and statistically homogeneous with respect to the range gate under test. Assuming that such a secondary (or

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training) data set of size K_t is available, it is well known that the maximum likelihood covariance matrix estimate is

$$\hat{\mathbf{C}} = \frac{1}{K_t}\sum_{k=0}^{K_t}\mathbf{z}_k\mathbf{z}_k^H. \tag{5}$$

Here the training data vectors are denoted by $\mathbf{z}_k$ in order to distinguish them from the test data vectors. The matrix $\hat{\mathbf{C}}$, which is complex Wishart distributed, is non-singular with probability 1 provided that $K_t > M$, [4] p. 73. Using this estimate in place of the true covariance matrix gives the adaptive matched filter (AMF) detection test, [5],

$$\frac{\left|\mathbf{s}_s^H\hat{\mathbf{C}}^{-1}\mathbf{g}\right|^2}{\mathbf{s}_s^H\hat{\mathbf{C}}^{-1}\mathbf{s}_s} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma. \tag{6}$$

Kelly, on the other hand, applied the generalized likelihood ratio approach to the detection problem and derived the generalised likelihood ratio test (GLRT), [6],

$$\frac{\left|\mathbf{s}_s^H\hat{\mathbf{C}}^{-1}\mathbf{g}\right|^2}{\mathbf{s}_s^H\hat{\mathbf{C}}^{-1}\mathbf{s}_s\left(1+\frac{1}{K_t}\mathbf{g}^H\hat{\mathbf{C}}^{-1}\mathbf{g}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma. \tag{7}$$

Both the AMF and GLRT have been analyzed and shown to possess the CFAR property, [6] and [5].

It has been recognized for some time that the homogeneity assumptions are often not satisfied, [7]. Terrain variations lead to inhomogeneity in the clutter returns, even between adjacent range gates. Furthermore, high target densities can bias the covariance matrix estimate. Consequently, the detection performance is degraded. This has led to research in knowledge-based STAP (KB-STAP) aimed at utilizing a-priori information (such as digital terrain maps) to help in the training data or algorithm selection, [8], [9] and [10]. In such heterogeneous environments, "single data set" (SDS) STAP algorithms that avoid the need for training data from other range gates are clearly desirable. Inspired by the APES filter, [11], *we propose here an SDS algorithm that is "analogous" to the AMF. We analyze the algorithm and show that it possesses the CFAR property. We also give the theoretical probabilities of false alarm and detection*. The detailed derivations of the P_{fa} and P_d expressions is beyond the scope of this paper and are carried out in [12]. The paper is organized as follows: In the following section we present the APES filter and then proceed to obtain the MLED detection statistic from it. An alternative derivation of the MLED is also given in section 2.1. The new detector is analyzed in section 3 and shown in section 3.1 to possess the constant false alarm rate (CFAR) property. The theoretical expressions for the probabilities of false alarm and detection are given in section 3.2 and are verified by simulation in section 4. Finally some conclusions are given in section 5.

2. THE MAXIMUM LIKELIHOOD ESTIMATION DETECTOR

The Amplitude and Phase Estimation algorithm was suggested in the context of synthetic aperture radar imaging, [11]. The filter minimises the sum of the squares of the errors between the received signal and the temporal steering vector $\mathbf{s}_t$ subject to a gain constraint at the spatial steering vector $\mathbf{s}_s$. The filter design problem is stated as follows:

$$\min_{\mathbf{w},\alpha}\left(\mathbf{w}^H\mathbf{X}-\alpha\mathbf{s}_t^T\right)\left(\mathbf{w}^H\mathbf{X}-\alpha\mathbf{s}_t^T\right)^H \text{ s.t. } \mathbf{w}^H\mathbf{s}_s = 1, \tag{8}$$

The solution, obtained using the Lagrange multiplier method, is given by, [13],

$$\mathbf{w} = \frac{\mathbf{Q}^{-1}\mathbf{s}_s}{\mathbf{s}_s^H\mathbf{Q}^{-1}\mathbf{s}_s} \qquad \text{and} \qquad \hat{\alpha} = \mathbf{w}^H\mathbf{g}, \tag{9}$$

where $\mathbf{Q} = \frac{1}{K_T}\mathbf{X}\mathbf{X}^H - \mathbf{g}\mathbf{g}^H$ and $\mathbf{g} = \frac{1}{K_T}\mathbf{X}\mathbf{s}_t^*$. Noting that the filter output power, $P_o = \mathbf{w}^H\mathbf{Q}\mathbf{w}$, is an estimate of the residual noise power at the target steering vector $\mathbf{s}_s$, we see that the APES filter implicitly derives a measure of the output signal to noise ratio, ρ_o, at $\mathbf{s}_s$

$$\begin{aligned}\rho_o &= \frac{|\alpha|^2}{\mathbf{w}^H\mathbf{S}\mathbf{w}} \\ &= |\alpha|^2\mathbf{s}_s^H\mathbf{Q}^{-1}\mathbf{s}_s.\end{aligned} \tag{10}$$

We propose using this signal to noise ratio, which we call the MLED statistic, for target detection in the STAP context. Substituting the expression for the weights vector into (10) we can re-write the MLED test as

$$\frac{\left|\mathbf{s}_s^H\mathbf{Q}^{-1}\mathbf{g}\right|^2}{\mathbf{s}_s^H\mathbf{Q}^{-1}\mathbf{s}_s} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma. \tag{11}$$

Note the similarity between the MLED statistic above and that of the AMF shown in equation (6).

2.1. Alternative Derivation of The MLED

We will now present an alternative perspective of the MLED statistic to that of the previous section. Recall that the MF assumes that the true noise covariance matrix is known. In practice, this is usually not the case and Robey *et al.* in [5] used an estimate of the covariance matrix in the MF expression to give the AMF. The estimate is obtained from a set of noise only training data. Since the noise only data is zero mean under both hypotheses, it is well known that the maximum likelihood estimate of the noise covariance matrix is given by equation (5). In the absence of a training data set, the noise covariance matrix estimate must be obtained from the test data itself. It is evident from our signal model that under both hypotheses the target signal is represented by the sample mean of the test data whereas its covariance matrix gives the noise subspace (with the target amplitude $\alpha = 0$ under H_0). Therefore, the detection test is equivalent to one of testing for the data mean with the covariance matrix unknown. Assuming that K_T test data vectors $\{\mathbf{x}_k\}$ are available, we obtain the maximum likelihood estimates of the mean and covariance matrix of the data and substitute them into the expression of the MF. These estimates are

$$\mathbf{g} = \frac{1}{|\mathbf{s}_t|^2} \sum_{k=1}^{K_T} \mathbf{x}_k \mathbf{s}_t^*(k) = \frac{1}{|\mathbf{s}_t|^2} \mathbf{X}\mathbf{s}_t^* \tag{12}$$

and

$$\begin{aligned} \mathbf{Q} &= \frac{1}{|\mathbf{s}_t|^2} \sum_{k=1}^{K_T} (\mathbf{x}_k - \mathbf{g}\mathbf{s}_t(k))(\mathbf{x}_k - \mathbf{g}\mathbf{s}_t(k))^H \\ &= \frac{1}{|\mathbf{s}_t|^2} \mathbf{X}\mathbf{X}^H - \mathbf{g}\mathbf{g}^H. \end{aligned} \tag{13}$$

The use of these expressions in the MF statistic gives the MLED statistic with arbitrary temporal and spatial steering vectors. If the temporal steering vector is taken to be a complex sinusoid, then $|\mathbf{s}_t|^2 = K_T$ and the MLED statistic reduces to the form given in the previous section.

3. THEORETICAL ANALYSIS

In this section we proceed to analyze the MLED test and show that it has the required CFAR property. We also give the probabilities of false alarm, P_{fa}, and detection, P_d, in forms that are analogous to those of the GLRT and AMF.

Using our assumed model, the data snapshots $\mathbf{x}_k$ follow a multivariate complex normal distribution. That is $\mathbf{x}_k \sim \mathcal{CN}_M(\alpha\mathbf{s}_s\mathbf{s}_t(k), \mathbf{C})$. Furthermore, assume that they statistically independent and set, without loss of generality, $|\mathbf{s}_t|^2 = 1$. Now the vector $\mathbf{g}$ is a linear combination of independent Gaussian vectors and is itself Gaussian distributed. Its mean and covariance matrix are easily shown to be $\alpha\mathbf{s}_s$ and $\mathbf{C}$ respectively. The matrix $\mathbf{Q}$, on the other hand, has a complex Wishart distribution independently of $\mathbf{g}$, see [12]. In summary we have that

$$\mathbf{g} \sim \mathcal{CN}_M(\alpha\mathbf{s}_s, \mathbf{C}) \quad \text{and} \quad \mathbf{Q} \sim \mathcal{CW}_M(\mathbf{C}, K_T - 1) \tag{14}$$

where $\mathcal{CW}_M(\mathbf{C}, K-1)$ is the complex Wishart distribution with scale matrix $\mathbf{C}$.

3.1. CFAR Property

In order to show that the test is CFAR, we first apply a whitening operation followed by a unitary transformation. Since $\mathbf{C}$ is positive definite, we can take its square root. Let $\tilde{\mathbf{X}} = \mathbf{C}^{-\frac{1}{2}}\mathbf{X}$. The columns of $\tilde{\mathbf{X}}$ are now distributed as $\tilde{\mathbf{x}}_k \sim \mathcal{CN}_M(\alpha\tilde{\mathbf{s}}_s\mathbf{s}_t(k), \mathbf{I})$, where $\tilde{\mathbf{s}}_s = \mathbf{C}^{-\frac{1}{2}}\mathbf{s}_s$. Let $\mathbf{U}$ be the unitary transformation that rotates the whitened spatial steering vector into the first elementary vector. That is $\mathbf{U}^H\tilde{\mathbf{s}}_s = \mathbf{e}$, where $\mathbf{e}^T = [1, 0, \cdots, 0]$. Since the unitary transformation does not alter the statistical properties of the random vectors it is applied to, we retain the same notation for the rotated quantities. Substituting the transformed quantities into the detection test, we re-write it in the equivalent form, [12],

$$\frac{\left|\mathbf{e}^T\tilde{\mathbf{Q}}^{-1}\tilde{\mathbf{g}}\right|^2}{\mathbf{e}^T\tilde{\mathbf{Q}}^{-1}\mathbf{e}} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma. \tag{15}$$

Under the null hypothesis, the received data consists solely of noise and the transformed quantities $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{Q}}$ are distributed as

$$\tilde{\mathbf{g}} \sim \mathcal{CN}_M(\mathbf{0}, \mathbf{I}), \text{ and } \tilde{\mathbf{Q}} \sim \mathcal{CW}_M(K-1) \tag{16}$$

Note that $\mathcal{CW}_M(K_T - 1)$ is a complex Wishart distribution with the scale matrix being the identity. Since both $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{Q}}$ have standard distributions that do not depend on the noise covariance matrix $\mathbf{C}$, and as $\mathbf{C}$ itself does not appear in the test statistic or the threshold in equation (15), we conclude that the test is independent of the true noise covariance and is consequently CFAR.

3.2. Probabilities of False Alarm and Detection

In this section we present the theoretical expressions for the probabilities of false alarm and detection. In [12], we show that the detection test of (15) is equivalent to

$$\zeta \underset{H_0}{\overset{H_1}{\gtrless}} L\eta\gamma \tag{17}$$

where $L = K_T - M$. The random variables ζ and η are mutually independent with η having a type I beta distribution with $m = L + 1$ and $n = M - 1$ degrees of freedom. Its probability density function is

$$f_{\beta,m,n}(x) = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} x^{m-1}(1-x)^{n-1}. \tag{18}$$

The variable ζ, on the other hand, has a complex singly non-central F distribution with 2 and $2L$ degrees of freedom and a non-centrality parameter $\lambda = K_T M\eta\rho$, ρ being the signal to noise ratio. The probabilities of false alarm and detection are then given by, [12],

$$P_{fa}(\gamma) = \int_0^1 (1+\eta\gamma)^{-L} f_{\beta,L+1,M-1}(\eta)\, d\eta. \tag{19}$$

and

$$P_d = \int_0^1 h(\eta) f_\beta(\eta; L+1, M-1) d\eta, \tag{20}$$

where

$$h(\eta) = 1 - (1+\eta\gamma)^{-L} \sum_{l=1}^{L} \binom{L}{l} (\eta\gamma)^l G_l\left(\frac{\lambda}{1+\eta\gamma}\right) \quad (21)$$

and

$$G_l(y) = e^{-y} \sum_{n=0}^{l} \frac{y^n}{n!}. \quad (22)$$

These expressions show that there is a loss of 1 complex degree of freedom with respect to the AMF. This loss is incurred in the denominator of the F distribution. The numerator has 1 complex degree of freedom in both cases. Since in the case of the AMF the covariance matrix is estimated from a separate noise-only training data set all of the available degrees of freedom are employed in the noise subspace estimation. The MLED, on the other hand, estimates both the signal subspace (of dimensionality 1) and the noise subspace from the same data set. Therefore, 1 complex degree of freedom is used in the estimation of the signal subspace and as a result, the degrees of freedom that are available for the noise subspace estimation are decreased by 1.

4. SIMULATION RESULTS

The algorithm presented above was implemented and simulated along with the MF, GLRT and AMF. The results are shown in figs. 1 to 4. Throughout the simulations, M was set to 10 and a P_{fa} of 10^{-2} was used. As expected the MF performs best and its curves form the bounds on the performance of the other algorithms. Figs. 1 and 2 show the performances of the algorithms for a test data set size $K_T = 2M$, whereas the training data set size, K_t, was increased from $2M$ to $5M$. It is clear that this increase in the size of the training data affects the GLRT and AMF algorithms only. The performance of the MLED algorithm is seen in fig. 1 to be slightly worse than the AMF due to the loss of degree of freedom mentioned in the previous section. In fig. 2, the traditional algorithm show a significant improvement in performance with respect to the MLED since the increase in K_t improves the noise covariance matrix estimation. Fig. 3 and 4 show the performances for an increased test data set size to $5M$. The increase in K_T results in an increase in the effective SNR for all algorithms. However, we see that it also has an additional effect on the performance of the MLED due to the increased number of degrees of freedom available for the noise subspace estimation. Thus, fig. 3 shows that the MLED outperforms the other two algorithms. In fig. 4, the test and training data set sizes are both equal to $5M$. In this case we find that all three algorithms have almost identical performances. This is due to the fact that as K_T increases (and $K_t = K_T$), the loss in the degrees of freedom, i.e. the loss of one degree of freedom for the MLED, becomes insignificant and the performances become almost identical. Note that in a heterogeneous environment the performances of the AMF and GLRT algorithms degrade whereas that of the MLED remains unchanged. This effect, however, is not demonstrated in this paper.

It is important to clarify that the MLED algorithm differs from the GLRT and AMF with respect to one important aspect. This is the fact that the MLED is essentially a spectral estimator with the important CFAR property. Whereas the other two algorithms obtain an estimate of the noise subspace over the entire spectrum and thus filter it, the MLED eliminates the interference in the bin of interest due to the rest of the spectrum.

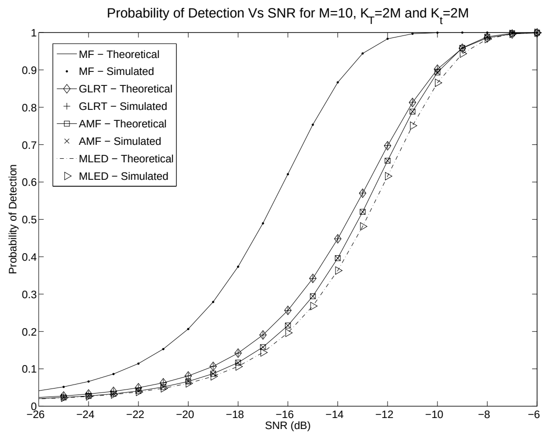

Fig. 1. Performance of the detection algorithms with both K_T and K_t set to $2M$ and $P_{fa} = 10^{-2}$.

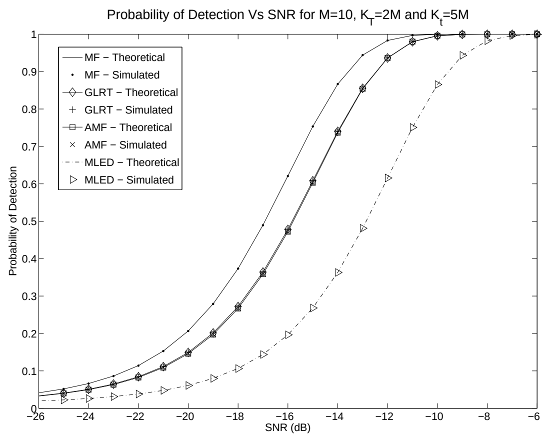

Fig. 2. Performance of the detection algorithms for $K_T = 2M$, $K_t = 5M$ and $P_{fa} = 10^{-2}$.

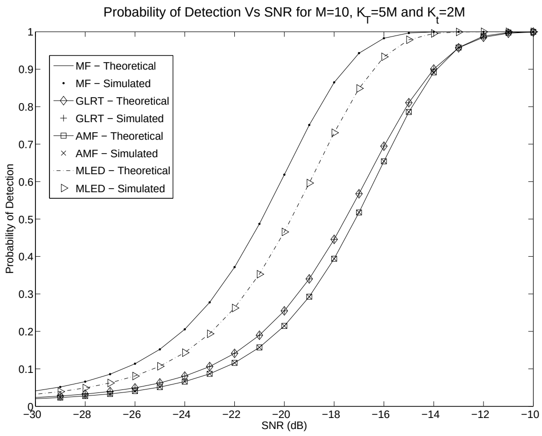

Fig. 3. Performance of the detection algorithms for $K_T = 5M$, $K_t = 2M$ and $P_{fa} = 10^{-2}$.

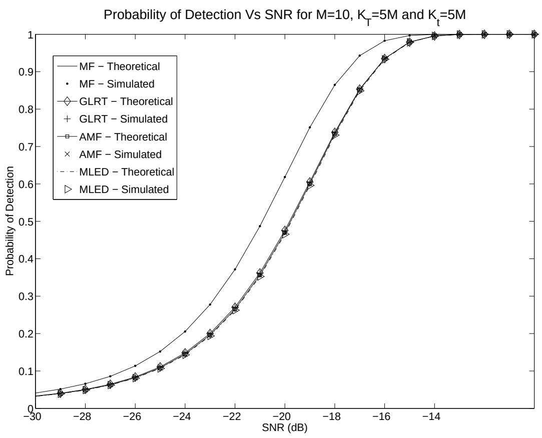

Fig. 4. Performance of the detection algorithms for $K_T = 5M$, $K_t = 5M$ and $P_{fa} = 10^{-2}$.

5. CONCLUSIONS

In this paper we have proposed an alternative single data set (SDS) algorithm for the detection of radar targets using space-time adaptive processing in heterogeneous environments. The new approach, called the MLED algorithm, operates solely on the test data and carries out a maximum likelihood separation of the noise and signal subspaces. The resulting mean vector and covariance matrix are then used in the expression of the MF to give a practical SDS implementation. We have analyzed the algorithm and shown that it enjoys the highly desirable constant false alarm property. We also presented its theoretical probabilities of false alarm and detection and the theoretical results were verified by simulations. It was demonstrated to have a performance that is comparable to those of the traditional two data sets (TDS) algorithms, namely the GLRT and AMF. The difference is due to the loss of one degree of freedom in the case of the MLED as the signal and noise subspaces are estimated from the same data set. This difference, however, reduces as the size of the test data set increases since the 1 degree of freedom becomes less significant as the total number of degrees of freedom increases. The performance of the MLED then approaches that of the TDS algorithms.

6. REFERENCES

[1] W.L. Melvin, "A stap overview," *IEEE Aerospace and Electronic Systems Magazine*, vol. 19, no. 1, pp. 19–35, Jan. 2004.

[2] R. Klemm, *Principles of space-time adaptive processing*, The Institution of Electrical Engineers, 2002.

[3] L.E. Brennan and L.S. Reed, "Theory of adaptive radar," *IEEE Transactions on Aerospace and Electronic Systems*, vol. AES-9, no. 2, pp. 237–251, March 1973.

[4] K.V. Mardia, J.T. Kent, and J.M. Bibby, *Multivariate Analysis*, Probability and Mathematical Statistics. Academic Press, 1979.

[5] F.C. Robey, D.R. Fuhrmann, E.J. Kelly, and R. Nitzberg, "A CFAR adaptive matched filter detector," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 28, no. 1, pp. 208–216, Jan. 1992.

[6] E.J Kelly, "An adaptive detection algorithm," *IEEE Transactions on Aerospace and Electronic Systems*, vol. AES-22, no. 1, pp. 115–127, Mar. 1986.

[7] W.L. Melvin, M.C. Wicks, and R.D. Brown, "Assessment of multichannel airborne radar measurements for analysis and design of space-time processing architectures and algorithms," in *IEEE National Radar Conference*, May 1996, pp. 130–135.

[8] R. Adve, P. Antonik, W. Baldygo, C. Capraro, G. Capraro, T. Hale, R. Schneible, and Wicks M., "Knowledge-Base Application to Ground Moving Target Detection," Tech. Rep. AFRL-SN-RS-TR-2001-185, Air Force Research Laboratory, Sept. 2001.

[9] W. Melvin, M. Wicks, P. Antonik, Y. Salama, Ping Li, and H. Schuman, "Knowledge-based space-time adaptive processing for airborne early warning radar," *IEEE Aerospace and Electronic Systems Magazine*, vol. 13, no. 4, pp. 37–42, Apr. 1998.

[10] M. Rangaswamy, J.H. Michels, and B. Himed, "Statistical analysis of the non-homogeneity detector for stap

applications," *Digital Signal Processing: A Review Journal*, vol. 14, no. 3, pp. 253–257, May 2004.

[11] J. Li and P. Stoica, "An adaptive filtering approach to spectral estimation and sar imaging," *IEEE Transactions on Signal Processing*, vol. 44, no. 6, pp. 1469–1483, June 1996.

[12] E. Aboutanios and B. Mulgrew, "Detection of signals with known templates in coloured gaussian interference," *Submitted to the IEEE Transactions on Signal Processing*.

[13] P. Stoica, H. Li, and J. Li, "A new derivation of the APES filter," *IEEE Signal Processing Letters*, vol. 6, no. 8, pp. 205–206, Aug. 1999.