A Class of Quadratic FIR Filters with Applications to Spectral Shaping and Narrow Band Generation

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ABSTRACT

In signal processing problems involving quadratic performance functional and stationary signals, solutions often involve a first stage of filtering called whitening. The whitening filter is a characterization of the signal power spectrum. For this reason, whitening and spectral estimation problems are closely related. The whitening filter obtained by Yule-Walker equations uncorrelates N samples of the output with the past input samples (beyond the zero lag) as seen by the orthogonality principle. It does not constrain the output samples to be uncorrelated with each others or to have pre-defined autocorrelation values for finite number of samples. The latter is typically achieved by additional filtering process approximately. In this paper we consider finding the unique minimum order, minimum phase FIR filter $F_M(z)$ from the input-output autocorrelation equations directly such that for the colored input process $u(n)$ the output $y(n)$ has N pre-defined zero or nonzero autocorrelation samples. The input-output correlation equation is quadratic in the coefficients of $F_M(z)$ but is solved linearly. Existence and derivation of the $F_M(z)$ are addressed. Application and performance of such filters for spectral shaping and narrow band signal generation are shown and compared with the traditional whitening filters using Yule-Walker equations.

Keywords: Quadratic FIR filters, Semi-Whitening FIR filter, Spectral estimation, System identification.

1. INTRODUCTION

Power spectrum of the stationary signal $u(n)$ is given by $S_u(\omega) = \sum_{k=-\infty}^{\infty} r_u(k) e^{-j\omega k}$ where $r_u(k) = r_u(-k)$ represents the autocorrelation samples of $u(n)$. The ideal whitening filter $G_M(z)$ satisfies the spectral relationship given by $S_u(\omega) \cdot \left| G_M(e^{jw}) \right|^2 = 1$. In practice, one must resort to some sort of approximation of the whitening filter since not all values of $r_u(k)$ are available. When the sequence $\mathbf{r}_u[0, N]$ represents N+1 autocorrelation samples of the signal, the common approach is to let N=M and obtain coefficients of the minimum phase filter $F_M(z) = \sum_{m=0}^{M} f(m) z^{-m}$ by minimizing the quadratic functional

$$r_y(0) = E\{[(f * u)(n)]^2\} = \sum_{k=0}^{N} \sum_{m=0}^{N} f(k) f(m) r_u(k-m)$$

subject to the constraint $f(0) = 1$. This minimization leads to normal (Yule-Walker) equations which are linear in vector $\mathbf{f}$ representing the filter's coefficients. The relation to whitening is, however, indirect because the output $y(n) = f(n) * u(n)$ is uncorrelated with N immediate past samples of the input, and not necessarily with its own samples. This can easily be seen by the orthogonality principle $\mathbf{E}\{y(k)u(k-n)\} = 0$ for $1 \le n \le N$ which is equivalent to the normal equations [2, 3, and 4]. Furthermore, the autocorrelation samples of $y(n)$ beyond the zero lag are often not zero, especially when $u(n)$ is not a noiseless AR(N) process. Because of this, we choose another approach requiring the whitening process make the output samples of the filter uncorrelated with a finite number of its own previous samples. That is, we find the minimum phase, minimum order FIR filter $F_M(z)$ such that $r_y(l) = \mathbf{E}\{y(k) y(k-l)\} = 0$, for $1 \le l \le L$, and $L \le M$. This requirement seems more consistent with the spectral relationship defined for the ideal whitening filter $G_M(z)$. As a function of the filter's coefficients, however, these equations are quadratic in the coefficients of $F_M(z)$ since $r_y(l) = r_f(l) * r_u(l)$ or

$$r_y(l) = \sum_{k=0}^{M} \sum_{m=0}^{M} f(k) f(m) r_u(l+m-k) = 0, \ 1 \le l \le L \qquad (1)$$

and where $r_f(l)$ represent autocorrelation samples of the filter's coefficients. Furthermore, we generalize the problem and require that $r_y(l)$ be a set of pre-defined values for $1 \le l \le L$. Formulation of this approach yields a new class of Quadratic Filter (QF) problems. The Semi-Whitening Filter (SWF) problem obtained by letting $r_y(l) = 0$ for $1 \le l \le L$ constitutes a key subset of this class of quadratic filters. Proving existence of a solution and finding the unique solution $F_M(z)$ to the QF or SWF problem are nontrivial. They have been discussed in detail in [1] for solving a system identification problem using a modified type of Levinson's algorithm. Contribution of this paper is two fold. First, we consider SW filter and introduce a modification to its derivation improving the filter's performance. The modification is carried out by suppressing the output autocorrelation samples. In [7] is discussed a method to obtain spectral estimate for input processes that has zeros close to the unit circle. In this paper we have discussed the spectral estimation for input processes that have zeros exactly on the unit circle. Second, we derive quadratic filters and show their applications for spectral shaping and narrow band signal generation. In these applications, we obtain the unique minimum-order, minimum-phase filter $F_M(z)$ using finite number of the input and output autocorrelation samples. The paper is organized as follows. In Section 2, we formulate the quadratic and semi-whitening problems. The input-output autocorrelation equation $r_y(l) = r_f(l) * r_u(l)$, which is linear in filter's autocorrelation sequence, is solved to find the minimum-length positive definite sequence $r_f(l)$. An algorithm is introduced to test existence of the solution and obtain the unique optimal solution $F_M(z)$ if the solution space is non-empty. Modification to the original QF or SWF problem is also introduced by suppressing samples of the output autocorrelation sequence beyond lag *L*. In Section 3, we present the simulation results and compare the performance of the SWF and QF filters with that of traditional whitening filters using Yule-Walker equations.

2. QUADRATIC FILTER PROBLEM

Consider figure 1 with the zero-mean stationary input process $u(k)$ representing an AR, MA, or ARMA process. We want to design a linear minimum phase FIR filter $F_M(z) = \sum_{k=0}^{M} f_k z^{-k}$ of minimum order *M* such that the output autocorrelation sequence is given by $r_y(l) = \gamma_l$ for $1 \le l \le L$. When $\gamma_l = 0$ we call the filter a Semi-whitening filter (SWF). The autocorrelation samples of the input, filter, and output are denoted by $r_u(k), r_f(k), r_y(k)$, respectively. For example, the input autocorrelation samples are computed by

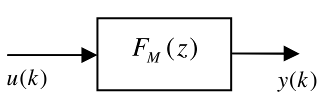

Figure 1: Single input-output model

$$r_u(l) = \sum_{k=0}^{M} u(k)u(k-l) = \sum_{k=0}^{M} u(k+l)u(k) = r_u(-l)$$

The equation $r_y(l) = r_f(l) * r_u(l)$ for the system can be written as

$$r_y(l) = r_f(0) r_u(l) + \sum_{m=1}^{M} r_f(m)[r_u(l-m) + r_u(l+m)] \qquad (2)$$

In matrix form Equation (2) is written as

$$\begin{pmatrix} r_y(0) \\ r_y(1) \\ \cdots \\ \cdots \\ r_y(L) \end{pmatrix} = \begin{pmatrix} r_u(0) & 2r_u(1) & \cdots & 2r_u(M) \\ r_u(1) & r_u(0)+r_u(2) & \cdots & r_u(M-1)+r_u(M+1) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ r_u(L) & r_u(L-1)+r_u(L+1) & \cdots & r_u(M-L)+r_u(M+L) \end{pmatrix} \begin{pmatrix} r_f(0) \\ r_f(1) \\ \cdots \\ \cdots \\ r_f(M) \end{pmatrix}$$

or $\mathbf{r}_y = \mathbf{R}_u \cdot \mathbf{r}_f$ (3)

$\mathbf{r}_y$ is $(L+1)\times\mathbf{1}$, $\mathbf{r}_f$ is $(M+1)\times\mathbf{1}$, $\mathbf{R}_u$ is $(L+1)\times(M+1)$, and $M \ge L \ge 1$. In order to solve the QF or SWF problem, we need to find $\mathbf{r}_f$ in (3) such that the real sequence $\mathbf{r}_f[0,M] = \mathbf{r}_f^T = [r_f(0), r_f(1), \ldots, r_f(M)]$ is positive-definite. Positive definiteness of $\mathbf{r}_f[0,M]$ guarantees the spectral factorization

$$R_f(z) = \sum_{m=-M}^{M} r_f(m) \cdot z^{-m} = F_M(z) \cdot F_M(z^{-1}),$$

where $r_f(m) = r_f(-m)$, and $F_M(z)$ is minimum phase and unique. It has been shown in [1] that inverse of $\mathbf{R}_u$ exists when *L*=*M*. Thus, Equation (3) can be written for $M \ge L$ as

$$\mathbf{r}_f = \boldsymbol{\alpha} + \boldsymbol{\beta}^T \cdot \mathbf{c} \qquad (4)$$

Let $M - L = \eta$, then $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are known with dimensions $(M+1)\times\mathbf{1}$ and $(\eta)\times(M+1)$, respectively. The $(\eta)\times 1$ vector $\mathbf{c}$ denotes the unknown parameters to be determined such that $\mathbf{r}_f$ is a positive definite sequence. Notice that the last η elements of $\mathbf{r}_f$ in (3) or (4) are the parameters, i.e. vector $\mathbf{c}$. Three nontrivial questions arise for finding parameter vector $\mathbf{c} \in \Re^{\eta}$ in (4). First, does a solution exist for finite value of $M \ge L$? Second, is the solution space $\mathbf{C}^{\eta} \subseteq \Re^{\eta}$ empty for a given *M*? Three, how to efficiently find the unique minimum order solution $\tilde{\mathbf{c}} \in \mathbf{C}^{\eta} \subseteq \Re^{\eta}$ if the solution space $\mathbf{C}^{\eta}$ is non-empty? These questions have all been answered favorably in [1] for the SWF problem where $r_y(l) = \gamma_l = 0$ for $1 \le l \le L$, and for the QF problem where $r_y(l) = \gamma_l$ are not all equally zeros. In this paper, we use the method provided in [1] and modify it to obtain new filter with improved performance for the SWF or QF problem. To find the desired $\mathbf{r}_f$, the entire solution space parameterized by $\mathbf{c}$ is searched for *M* = *L*, *L*+1, *L*+2, …, by a novel convergent positivity test algorithm for real finite parametric sequence $\mathbf{r}_f[0,M]$. Constraints to ensure positivity of the sequence are expressed as a convex set. When the solution set for a minimum value of *M* is non-empty, then a unique solution for $\mathbf{r}_f$ is obtained by minimizing the functional $r_y^2(0)$, which is quadratic in **c,** subject to the convex set of constraints. Minimization of $r_y^2(0)$ is consistent with the minimization of $r_y(0)$ with $f(0)=1$ in the traditional whitening method using Yule-Walker equations. In general, the traditional whitening filter does not belong to the solution space $\mathbf{C}^{\eta} \subseteq \Re^{\eta}$ except when $r_u(l)$ represents a noiseless AR process of order *M*.

The proposed method has an interesting frequency domain interpretation. Taking Fourier Transform of both sides of $\mathbf{r}_f = \boldsymbol{\alpha} + \boldsymbol{\beta}^T \cdot \mathbf{c}$ yields $R(\omega) = A(\omega) + \sum_{\upsilon=1}^{M-L} c_{\nu} B_{\nu}(\omega)$, where c_{ν} is a component of vector **c.** $B_{\nu}(\omega)$ is the Fourier-Transform of the column $\boldsymbol{\beta}_{\nu}$ of matrix $\boldsymbol{\beta}^T$. Notice that every column vector is assumed representing a finite sequence symmetric around the first element of the vector before taking its two-sided Fourier-Transform. Finding **c** such that $\mathbf{r}_f$ is positive is, therefore, equivalent to finding a linear combination of given spectrums $A(\omega)$ and $B_{\nu}(\omega)$ for minimum value of $M \geq L$ such that the spectrum $R(\omega)$ is positive definite for $-\pi \leq \omega \leq \pi$, i.e. $R(\omega)$ is power spectrum of an MA(*M*) process.

2.1 SOLVING THE SWF OR QF PROBLEM

Since we want to minimize $r_y^2(0)$ we treat $r_y(0)$ as a parameter and rearrange Equation (3) to obtain

$$\mathbf{r}_f = \mathbf{a} + \mathbf{B}^T \cdot \mathbf{c} \tag{5}$$

The new unknown parameter vector **c** includes the old vector **c** and $r_y(0)$ as defined by $\mathbf{c}^T = [\mathbf{c} \quad r_y(0)]$ with dimension $1 \times (\eta+1)$. **a** is $(M+1)\times 1$ and matrix $\mathbf{B}^T$ is $(M+1)\times(\eta+1)$. The sequence $\mathbf{r}_f[0,M]$ is linearly parametric in **c**. Existence of the solution for finite *M* was shown in [1] and an efficient convergent algorithm testing positivity of such sequence was introduced. The algorithm linearly transforms the sequence represented by $R(\omega)$ into a set of auxiliary polynomials of order *M* whose coefficients are linear in **c**. The auxiliary polynomial is of the form $G_M(x,\mathbf{c}) = \sum_{k=0}^{M} g_k(\mathbf{c})\, x^k$, where $g_k(\mathbf{c}) = a(k) + \mathbf{b}^T(k)\cdot \mathbf{c}$, $a(k)$ is real scalar, and $\{\mathbf{b}(k), \mathbf{c}\} \in \Re^{\eta+1}$. The transformation is done so that positivity of $\mathbf{r}_f[0,M]$ is equivalent to positivity of the auxiliary polynomials over real positive axis. The algorithm creates a set of inequalities on the coefficients of the polynomials forming a set of convex constraints. These convex constraints are then used in minimizing the quadratic function $r_y^2(0)$ in order to find a unique solution. As the value of *M* is incremented from *M*=*L*, the algorithm searches the entire solution space to determine if $\mathbf{C}^{\eta+1}$ is empty and finds the unique solution $\tilde{\mathbf{c}} \in \mathbf{C}^{\eta+1} \subseteq \Re^{\eta+1}$ if $\mathbf{C}^{\eta+1}$ is non-empty for every *M*. The algorithm takes the following steps.

Step 1: *Initialization*- Define $r_y(l)$ for l=1,2,…,*L*, and $r_u(l)$ for l=0, 1, 2,…, *L*+*M*.
Let *p*=0. Form the $R(\omega)$ using Equation (5). Using the Chebyshev and bilinear transformations, obtain the auxiliary polynomials

$G_M^n(x,\mathbf{c}) = \sum_{k=0}^{M} g_k^n(\mathbf{c})\, x^k$, for $n=1$, from $R(\omega)$.

Step 2: Create a set of inequality constraints using coefficients of the auxiliary polynomials [1].

Step 3: Minimize $r_y^2(0)$ with respect to the set of inequality constraints. Stop if the solution $\tilde{\mathbf{c}}$ is found. Go to Step 6. If no solution exists for $M \leq M_{\max}$. then go to Step 4.

Step 4: Stop if $M > M_{\max}$. No solution exists. Otherwise, go to Step 5.

Step 5: Let *M*=*M*+1, *p*=*p*+1. Using the linear transformations $x = x+1$ and $x = \frac{1}{x+1}$, obtain two new polynomials from each previous auxiliary polynomials, i.e. obtain the polynomials

$G_M^n(x,\mathbf{c}) = \sum_{k=0}^{M} g_k^n(\mathbf{c})\, x^k$, for $n = 1,2,4,\ldots,2^p$.

Go to Step 2.

Step 6: Using Equation (5) and the solution $\tilde{\mathbf{c}}$, obtain the positive definite sequence $\mathbf{r}_f[0,M]$. Obtain the unique minimum-phase SWF or QF $F_M(z) = \sum_{k=0}^{M} f_k z^{-k}$, from the spectral-Factorization given by

$$R_f(z) = \sum_{m=-M}^{M} r_f(m)\cdot z^{-m} = F_M(z)\cdot F_M(z^{-1}).$$

It was shown in [1] that for every *p*, not more than *M* auxiliary polynomials would be needed for evaluation in Step 2 or 3. It was also shown that the algorithm would converge exponentially.

2.2 IMPROVING PERFORMANCE OF THE SWF

Excellent results are obtained when the SWF is used to estimate the spectrum of the process $u(n)$ derived as an MA, AR or ARMA process. Some examples are shown in figures 6, 7, and 8. When the MA model for $u(n)$ process has zeros on the unit circle small ripples show up in the estimated spectrum. The effect of the ripple is less when *L* is large. These ripples are caused by slightly higher than expected values of the output autocorrelation beyond lag L. Figure 2, 3 and 4 show such lag values beyond $r_y(L)$ for the MA processes generated by $1+0.8z^{-2}$, $1+z^{-1}$, and $(1+0.8z^{-1})(1+z^{-2})$ when L=13. To reduce the ripple effect, we propose cascading the SWF with a second FIR filter. The second filter is derived to suppress the output lags beyond lag L leaving smaller lags unaffected. The order P of this filter depends on the number of output autocorrelation lags that should be suppressed. Derivation of the second filter does not require any additional input autocorrelation lags beyond M+L+1 which was required to obtain the SWF. Figure 5 shows the use of the second filter $W_P(z)$. $W_P(z)$ is obtained using the autocorrelation equation

$$\hat{r}_y(l) = r_y(l) * r_w(l) \tag{6}$$

Where $r_w(l)$ represents autocorrelation of $W_P(z)$ and

$$r_y(l) = r_f(l) * r_u(l) \tag{7}$$

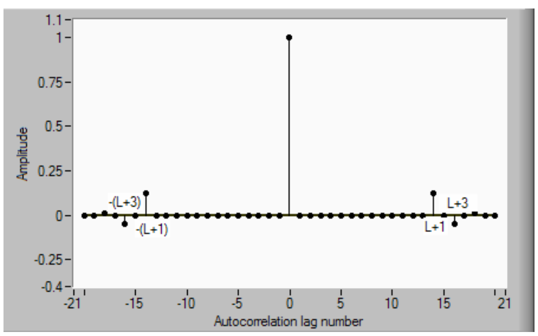

Figure 2: Output autocorrelation for $1+0.8z^{-2}$

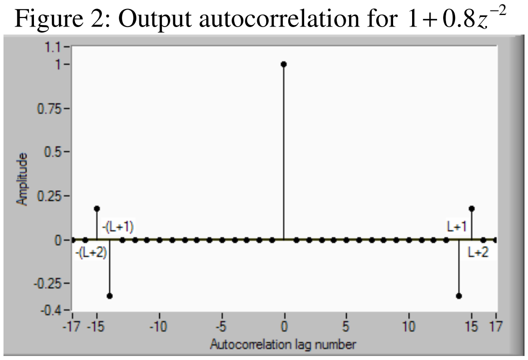

Figure 3: Output autocorrelation for $1+z^{-1}$

The MA process $u(n)$ has finite autocorrelation sequence of which L+M+1 samples were used to obtain the SWF $F_M(z)$. Thus, using Equation (7) we can obtain at least the first L+2M+1 samples of the finite autocorrelation sequence $r_y(l)$. To suppress $r_y(L+q)$ for $1 \le q \le M-(L/2)$ we obtain $r_w(l)$ in Equation (6) such that

$$\hat{r}_y(l) = \begin{cases} 1 & l = 0 \\ \mathbf{0} & 1 \le l \le L+q \end{cases} \tag{8}$$

Rewriting Equation (7) in matrix form, we obtain

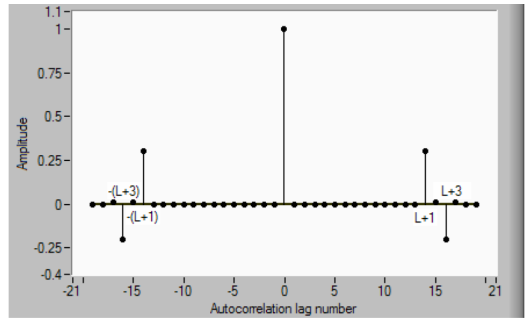

Figure 4: Output autocorrelation for $(1+0.8z^{-1})(1+z^{-2})$

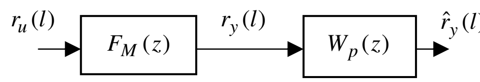

Figure 5: SWF cascaded with second FIR filter

$$\begin{pmatrix} \hat{r}_y(0) \\ \hat{r}_y(1) \\ \dots \\ \dots \\ \hat{r}_y(L+q) \end{pmatrix} = \begin{pmatrix} r_y(0) & 2r_y(1) & \dots & 2r_y(L+q) \\ r_y(1) & r_y(0)+r_y(2) & \dots & r_y(L+q-1)+r_y(L+q+1) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ r_y(L+q) & r_y(L+q-1)+r_y(L+q+1) & \dots & r_y(0)+r_y(2L+2q) \end{pmatrix} \begin{pmatrix} r_w(0) \\ r_w(1) \\ \dots \\ \dots \\ r_w(L+q) \end{pmatrix}$$

Or, $$\hat{\mathbf{r}}_y = \mathbf{R}_y \cdot \mathbf{r}_w \tag{9}$$

$\mathbf{R}_y$ is a square symmetric matrix and its inverse exists [1]. Thus,

$$\mathbf{r}_w = \mathbf{R}_y^{-1}\hat{\mathbf{r}}_y \tag{10}$$

The finite autocorrelation sequence represented by $\mathbf{r}_w$ is positive definite since $r_y(l)$ and $\hat{r}_y(l)$ are positive definite. The coefficients of the filter $W_P(z)$ are derived from the values of $\mathbf{r}_w$ by the means of Spectral factorization as discussed before. The cascade of $F_M(z)$ and $W_P(z)$ is a finite order, minimum phase FIR filter of order M+L+q satisfying the semi-whitening condition. Using $F_M(z)$ and $W_P(z)$ reduces the ripples and improves spectral estimate of the MA process $u(n)$ using M+L+1 autocorrelation lags of the process.

3. SIMULATION RESULTS

In this section is shown the simulation results obtained for the SWF and QF problem. First, the results obtained for the SWF as a spectral estimator is shown when the input has an MA, AR or ARMA model. The performance of our filter is compared with the traditional whitening filters using Yule-Walker equations. Second, we present application and performance of the QF problem. The minimum order, minimum phase FIR filter $F_M(z)$ is derived to shape an arbitrary input spectrum into a desired spectrum or into a narrow band signal using finite number of input and output autocorrelation

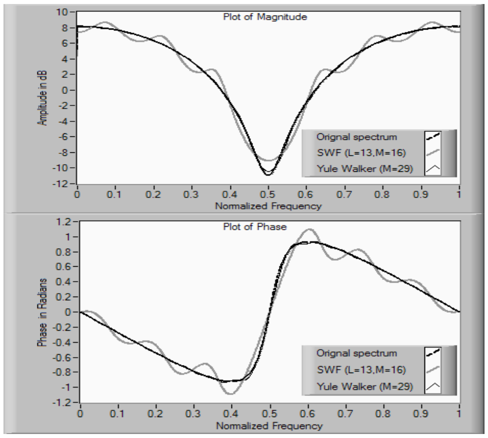

Figure 6: Spectral estimation of $1+0.8z^{-2}$

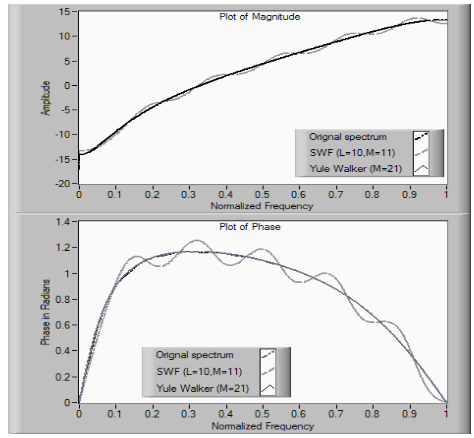

Figure 8: Spectral estimation of ARMA (1,1)

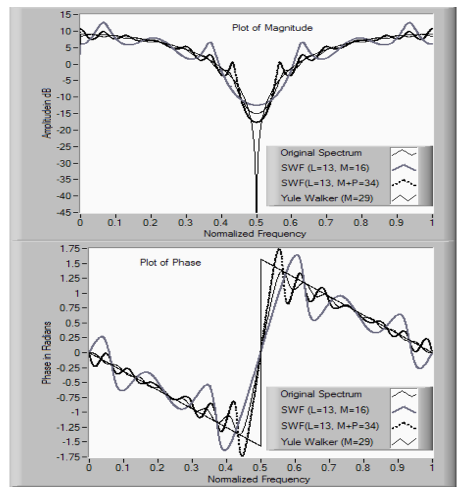

Figure 9: Spectral Estimation of $1+z^{-2}$

lags. Figure 6, 7 and 8 show the performance of the SWF as a spectral estimator for the MA model $1+0.8z^{-2}$, AR model $\dfrac{1}{1+0.6z^{-1}+0.8z^{-2}}$, and ARMA model $\dfrac{1+0.8z^{-1}}{1+0.45z^{-1}}$. Exact autocorrelation samples of the models were used. Performance of the SWF is quite comparable with that of the whitening filter using the same number of input autocorrelation samples. Unlike the whitening filter, the SWF preserves the zero values of the output lags for $1 \le l \le L$. Exception is when the AR model is used for $u(n)$. Table 1 shows numerical values of the output autocorrelation lags for the SWF and the traditional whitening filters. Figure 9 shows spectrum estimates of the MA model obtained by the filters. Although the SWF has a few ripples it tracks the true spectrum very well comparable with the conventional whitening filter. It is also seen that using the second filter with the SWF filter suppresses the ripples and improves the performance of the SWF. Figures 10 and

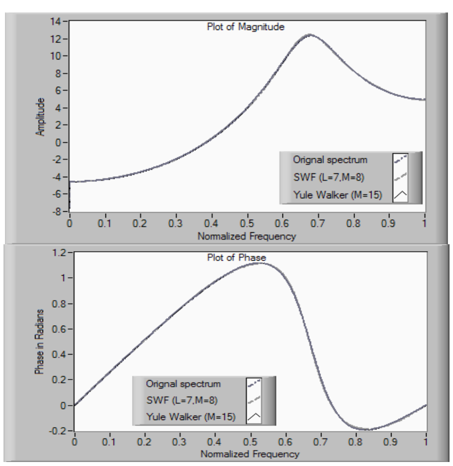

Figure 7: Spectral estimation of AR (2)

11 show applications of the QF problem in shaping spectrum of an input process. A key constraint is that the given L output autocorrelation lags be matched exactly. In the first example we have shaped spectrum of an input ARMA(5,5) process into the desired ARMA(7,5) as shown in Figure 10. In the second case the desired narrow band output spectrum approximating a 10th order Butterworth filter is obtained from an input ARMA(5,5) process as shown in Figure 11. Selected values for L and the order M of filter $F_M(z)$ in each example are shown in the figures. The true output spectrums for both examples are also shown indicating performance of the proposed QF method.

4. CONCLUSION

In this paper we discussed a new class of Quadratic FIR filtering (QF) problem and its special case called the Semi whitening filter (SWF). A novel algorithm which solved this problem was briefly discussed. The algorithm solved the input–output autocorrelation equation directly finding a unique minimum order, minimum phase FIR filter. The application of the SWF as a spectral estimator was discussed. QF problem was applied to spectral shaping application and narrow band generation. Performance of the proposed SWF was compared with the traditional whitening filter obtained by Yule-Walker equations. Simulations results were presented verifying the analytical developments.

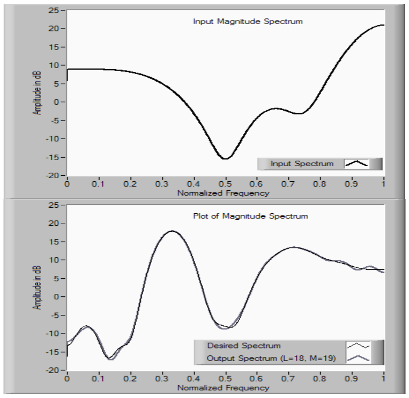

Figure 10: Spectral Shaping

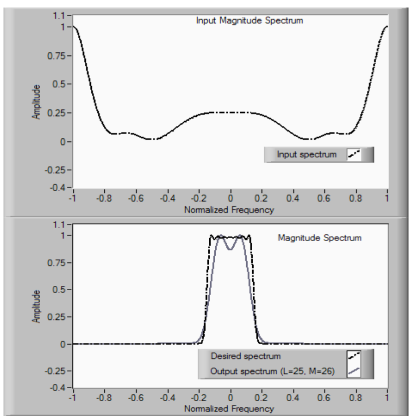

Figure 11: Narrow Band generation

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Table 1: Output Autocorrelation samples for $1+z^{-1}$ MA (1) input process.

SWF (*L*=4, *M*=5)	**SWF Plus 2nd filter (*L*=4, *M*=5, *P*= 11)**	**Yule Walker (*M*=9)**	**Yule Walker (*M*=5)**
1	1	1	1
-2.22872798090176e-015	-3.96512338778716e-015	0.009090909	0.023809524
-3.95625876035672e-015	-1.58316811046937e-015	-0.018181818	-0.047619048
-1.57823799604527e-015	-1.84399657311375e-015	0.027272727	0.071428571
4.6920589071616e-016	4.82229366982051e-016	-0.036363636	-0.095238095
0.342857142857143	1.06757696338165e-015	0.045454545	0.119047619
-0.157142857142857	-7.2789338412385e-017	-0.054545455	-0.142857143
	0.0383285174744588	0.063636364	
	-0.00408837519727759	-0.072727273	
	0.0267602740185294	0.081818182	
	-0.19524308473831	-0.090909091	
	0.187956109317542		
	-0.0476236392538862		