SCALE-INVARIANT DETECTION BY A RATIO OF GENERALIZED VARIANCES

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ABSTRACT

Motivated by the problem of designing a scale-invariant or self-normalizing test for detecting a multi-rank signal embedded in interference when no signal-free secondary data is available, a scale-invariant test based on a ratio of generalized variances is proposed in this paper. Computer simulations show that the prototype test works well. Remarkably, it is shown that the new ratio of generalized variances test can be interpreted as a ratio of the determinants of Wiener filter error covariance matrices, and in the case of a rank-1 signal, the new method is equivalent to a normalized version of Capon's method.

1. INTRODUCTION

The detection of a signal received on an array of sensors in the presence of noise is an important problem in passive sonar. When the noise and signal are complex multivariate Gaussian distributed with covariance matrices R and $\sigma_s^2 \vec{s}\vec{s}^H$ respectively, a likelihood-ratio test (LRT) statistic [1] is

$$z_{LRT} \equiv \left| \vec{s}^H R^{-1} \vec{x} \right|^2 \tag{1}$$

where $\vec{x}$ is an $m \times 1$ vector of measurements containing noise or noise plus signal. Out of all possible tests, the LRT achieves the greatest probability of detection for a fixed false alarm rate, that is, it is the optimum detector according to the Neyman-Pearson criterion [1].

Instrumentation of (1) is problematic since the noise covariance R is unknown. In radar and sonar, test statistic (1) or variants of the LRT are often implemented adaptively by replacing the unknown noise covariance matrix by the sample covariance matrix [2]

$$\hat{R} = \frac{1}{K} \sum_{n=1}^{K} \vec{x}_k \vec{x}_k^H \tag{2}$$

calculated from noise or signal plus noise measurements $\vec{x}_1, \ldots, \vec{x}_K$. As an example, one popular adaptive beamformer used in passive sonar is Capon's method (also known as the minimum variance distortionless response (MVDR) method [3]):

$$z(\theta)_{capon} = \frac{1}{\vec{s}(\theta)^H \hat{R}^{-1} \vec{s}(\theta)} \tag{3}$$

where θ is the look direction.

Implementation of a detector such as (3) in an actual sonar system also requires a means of determining and setting the test statistic threshold so that the detector operates at some pre-specified false alarm rate. For instance, under the null hypothesis it is well known from the classical result of Capon and Goodman [4] that (3) is chi-square distributed (if the data snapshots $\vec{x}_k$ are independent and identically distributed as complex Gaussian with zero-mean) as

$$z_{capon} \overset{D}{=} (\vec{s}^H R^{-1} \vec{s})^{-1} \cdot \mathrm{X}^2_{2(K-m+1)} \tag{4}$$

where K is the number of data snapshots used in estimating the covariance matrix. Unfortunately the distribution depends itself on the unknown covariance matrix R, so (4) cannot be used to calculate detector thresholds. Traditional passive sonar processing usually normalizes (3) by a local estimate of the background noise power obtained from adjacent directions or beams, like in a split window approach [5] and sets the threshold empirically. A major drawback of this approach is that the background noise level estimates are often biased by sidelobe leakage and contamination from nearby signals, resulting in poor normalization and performance. Threshold setting is greatly simplified if the adaptive test has true constant false alarm (CFAR) properties, i.e. the distribution of the test statistic under the null hypothesis does not depend on the unknown noise parameters, or at least is *invariant* to scale or *self-normalizing* in some aspect. Therefore it is desirable to design an adaptive detector that has some invariance properties.

An alternative approach is to design a detector from first principles that has invariance properties with respect to the unknown interference and noise parameters. In passive sonar, interference and signals from discrete monochromatic acoustic sources received on an array of hydrophones can be accurately modeled using a linear subspace model. That is,

$$\vec{x} \approx H\vec{\theta} + \mu S\vec{\phi} + \vec{n} \tag{5}$$

where the r columns of H are the interference modes or basis vectors and the p columns of S are the signal basis vectors plus an ambient noise component $\vec{n}$. The detection problem is to test the hypothesis H_0: $\mu = 0$ against the alternate hypothesis H_1: $\mu = 1$. If the elements of the signal amplitudes vector $\vec{\phi}$ are assumed unknown, but deterministic and the elements of $\vec{n}$ are modeled as independent and identically distributed (IID) complex Gaussian random variates with zero mean, then the uniformly most power invariant (UMPI) test statistic with respect to data scalings and rotations and translations in the interference subspace takes on the form (see Scharf [6])

$$z_{umpi} = \frac{\left\| P_{P_{H\perp}S}\vec{x} \right\|_F^2}{\left\| P_{HS\perp}\vec{x} \right\|_F^2} \tag{6}$$

In (6), $P_{HS\perp}$ and $P_{P_{H\perp}S}$ are projection operators on to the orthogonal complement of the joint interference and signal subspace <*H*,*S*> (the operator <·> will be used to denote subspaces) and the part of the signal subspace <S> which is orthogonal to the interference subspace <H> respectively. Mathematically, they are defined as

$$P_{H\perp S} = I - P_{H\perp}S(S^H P_{S\perp}S)^{-1}S^H P_{H\perp} \tag{7}$$

$$P_{HS\perp} = I - [H\,|\,S]([H\,|\,S]^H[H\,|\,S])^{-1}[H\,|\,S]^H \tag{8}$$

$$P_{H\perp} = I - H(H^H H)^{-1}H^H \tag{9}$$

Statistic (6) is the best possible constant false alarm rate (CFAR) detector with respect to the unknown background noise variance and interference. Of course, in actual passive sonar systems, we do not known the interference subspace <*H*> beforehand. Thus the optimum CFAR or UMPI detector cannot be implemented.

However, if a signal-free training or secondary data set is available, as is the case in radar applications, it possible to construct non-UMPI adaptive tests with desirable invariance properties and good performance (e.g. see [7,8,9]). The difficulty is that in passive sonar, signal-free secondary data is usually not available (e.g. non-stationary interference and/or signal contamination), thus necessitating that both *adaptation* and *detection* be done on the *same* data.

In the remainder of the paper it is assumed that K data snapshots

$$\begin{aligned} H_0 &: X = [\vec{x}_1\,|\cdots|\,\vec{x}_K] = H\theta + N \\ H_1 &: X = [\vec{x}_1\,|\cdots|\,\vec{x}_K] = H\theta + S\Phi^H + N \end{aligned} \tag{10}$$

are collected of either noise-only or noise plus signal measurements that follow the previous linear subspace model (5) from which detection is to be made. Under H_1, signals are assumed to be present in all K snapshots. The data measurements $\vec{x}_k$ are modeled as IID complex Gaussian distributed random vectors with zero-mean and covariance matrices $H\Lambda H^H$ and $H\Lambda H^H + S\Lambda_S S^H$ respectively under H_0 and H_1. Unless otherwise stated, a rank-1 signal is assumed, that is, $S\Lambda_S S^H = \sigma_S^2 \vec{s}\vec{s}^H$. The objective is to develop a scale-invariant adaptive test based on a ratio of generalized variances that does not require signal-free secondary training data.

2. GENERALIZED VARIANCE AND DETECTION

Intuitively, the presence or absence of a signal could be determined by detecting changes in the spread or dispersion of the multivariate data measurements (see fig. 1). Analogous to variance in the univariate case, the determinant of a covariance matrix is a measure of the hypervolume that a distribution of random vectors occupies in R^m and is referred to as the *generalized variance* (GV) by the multivariate statistics community [10].

It is well known from classical multivariate statistics that the volume of the concentration ellipsoid of a Gaussian probability density function is proportional to $|R|^{1/2}$ [10]. The connection of the sample generalized variance (SGV) to the dispersion of the multivariate sample is seen through the following theorem from Anderson [10, pg. 263] that states:

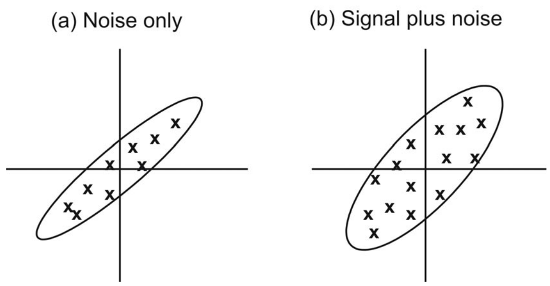

Fig 1: Multivariate dispersion.

Theorem: *Let* $|\hat{R}|$ *be defined as by (2), where* $\vec{x}_1,\ldots,\vec{x}_K$ *are the K vectors of a sample. Then* $|\hat{R}|$ *is proportional to the sum of squares of the volumes of all different parallelotopes formed by using as principal edges m vectors with m of* $\vec{x}_1,\ldots\vec{x}_K$ *as one of the end points formed and the mean as the other end point.*

Thus the quantity $|\hat{R}|$ is measuring approximately the hypervolume occupied by the multivariate measurements $\vec{x}_1,\ldots,\vec{x}_K$ in the space.

Since $|R+\sigma_S^2\vec{s}\vec{s}^H| = |R|(1+\sigma_S^2\vec{s}^H R^{-1}\vec{s})$, the presence of a rank-1 signal must always increase the GV. Multi-rank signal covariances will also always increase the GV. It is noted that the term $\sigma_S^2\vec{s}^H R^{-1}\vec{s}$ is the output signal-to-noise ratio (SNR) of the optimum filter weights [2]. Therefore, the presence of a rank-1 signal results in an increase of the GV that is proportional to the output SNR, which is intuitively appealing. This suggests that another way to detect the signal, albeit sub-optimum, is to test the measurements $\vec{x}_1,\ldots\vec{x}_K$ against the hypotheses H_0: GV = $|R|$ and H_1: GV > $|R|$ using the GSV as a test statistic.

Eaton [11] shows that the SGV in the real multivariate Gaussian case is a maximal invariant (with respect to transformations of the form $\vec{y}=T\vec{x}+\vec{b}$ where T is an upper triangular matrix with positive diagonal elements and $|T|=1$) and possesses a monotone likelihood ratio as a function of the GV. The SGV is invariant to all linear transformations that preserve the hypervolume occupied by the multivariate data measurements in the space. Therefore the SGV is the uniformly most powerful invariant (UMPI) test for H_0: GV $\le c_1$ and H_1: GV > c_2. However, it does not possess scale-invariance and furthermore, inherently assumes that the subspace occupied by the signal in the measurement space R^m is unknown because of its invariance to transformations of the type $T\vec{x}$. This implies that the SGV-based UMPI test likely will perform poorly because it does not use any prior knowledge of the signal subspace.

3. DETECTION BY A RATIO OF GENERALIZED VARIANCES

When the sample size is large, it can be argued that the signal plus noise sample covariance matrix R_1 is roughly $R'+\hat{\sigma}_S^2\vec{s}\vec{s}^H$ where R' is the sample covariance matrix of only the noise components. In the approximation $|R_1|\approx|R'|(1+\hat{\sigma}_S^2\vec{s}^H R'^{-1}\vec{s})$ it is seen that the only term relevant to signal detection is the second one. Since $|R'|$ contains no signal information, random fluctuations of $|R'|$ due to finite sample size will degrade detection. These observations suggest that it might possible to construct an improved scale-invariant test by dividing or normalizing out the SGV by an estimate of the noise-only GV. Similar arguments can be made for the case when the signal is multi-rank. A noise-only biased estimate of the GV can be obtained using the part of the noise that lies in the orthogonal complement of the signal subspace.

3.1. Detector

Motivated by these discussions, the following ad hoc test is proposed: Since in the Gaussian case a sufficient statistic for detecting the signal are the projections of $\vec{x}$ on to $\langle H,S\rangle$, construct an $m \times (r+p)$ matrix Q with orthonormal columns that span $\langle H,S\rangle$ to reduce data dimensionality. The columns of Q can be interpreted as the signal and interference beams or steering vectors. Next construct an $m \times (r+1)$ signal-blocking matrix $Q_\perp$ whose orthonormal columns or beams span the column space of $[P_{S\perp}H\,|\,\vec{g}]$ ($P_{S\perp}=I-S(SS^H)^{-1}S^H$ is the projection operator on to the orthogonal complement of $\langle S\rangle$) such that $\vec{g}^H[H\,|\,S]=0$. $\vec{g}$ is an ancillary beam constructed to sample the background noise for use as a reference level. The columns of $Q_\perp$ represent beams that form a signal-free reference for the subspace interference and white noise. The proposed test statistic is then the ratio of generalized variances:

$$z=\frac{|Q^H\hat{R}Q|}{|Q_\perp^H\hat{R}Q_\perp|} \qquad (11)$$

Since $\vec{g}_k$, $k=1,\ldots,m-r-p$ vectors orthogonal to $\langle H,S\rangle$ can be constructed, an improved normalized test is obtained by averaging the signal-free GV estimates obtained from the beam matrices $Q_{\perp k}$ constructed using the $\vec{g}_k$:

$$z = \frac{|Q^H \hat{R} Q|}{(1/m-p)\cdot \sum_{k=1}^{m-r-p} |Q_{\perp k}^H \hat{R} Q_{\perp k}|} \qquad (12)$$

In effect, the signal is detected by changes in the GSV or the volume of the multivariate concentration ellipsoids relative to the noise-only reference GSV $|Q_{\perp}^H \hat{R} Q_{\perp}|$. By inspection we see that z is invariant to scale. However, no claims can be made regarding optimality.

4. EXPERIMENTAL RESULTS

4.1. Rank-1 signal

A scenario was computer simulated where a rank-1 monochromatic plane wave signal (Rayleigh distributed envelope and uniformly distributed phase) was to be detected using a twenty-element equi-spaced line array in the presence of three independent monochromatic jammers (Rayleigh distributed envelope and uniformly distributed phase) located in the far field plus white complex Gaussian noise. The jammers and signal were all separated in spatial wavenumber by ½ DFT bin. The jammer power to background white noise power ratio was set to 20 decibels. Twelve independent snapshots were used for detection. 3000 independent simulation trials were performed at each signal-to-noise ratio (SNR) comparing the new detector based on a ratio of the sample generalized variances (formula (12)) against the optimum LRT, matched filter, the optimum UMPI CFAR matched subspace detector (MSD) of Scharf (6), and the raw (unnormalized) sample generalized variance used as a test statistic. The probability of detection was evaluated with the test statistic threshold empirically set to obtain a false alarm probability of .02. The measured probability of detection curves in figure 2 show that the new method works remarkably well and compares favorably to the clairvoyant LRT detector and the MSD. Both the raw SGV and matched filter performed poorly, although the raw SGV detector did outperform the matched filter.

4.2. Multi-rank signal

The above experimental scenario and computer simulation was repeated for a rank-3 signal. The aggregate signal consisted of a cluster of three independent emitters spaced

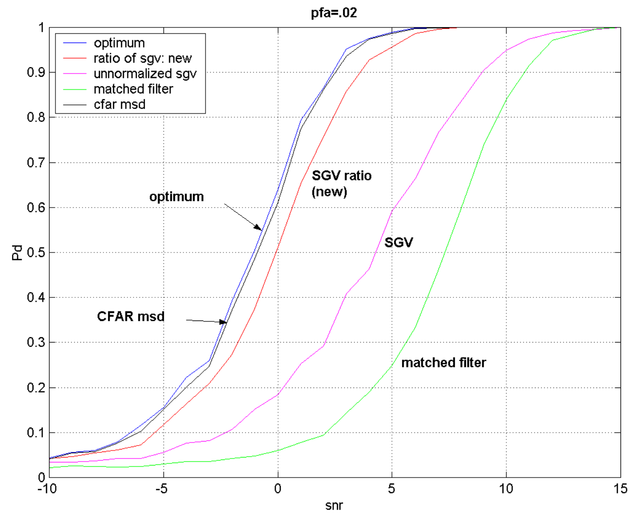

Fig 2: Rank-1 signal SNR vs. probability of detection (P_D). “SGV ratio” is the new detector and “SGV” is a detector based on using the raw sample generalized variance (unnormalized) as a test statistic. “CFAR msd” is the optimum CFAR UMPI matched subspace detector.

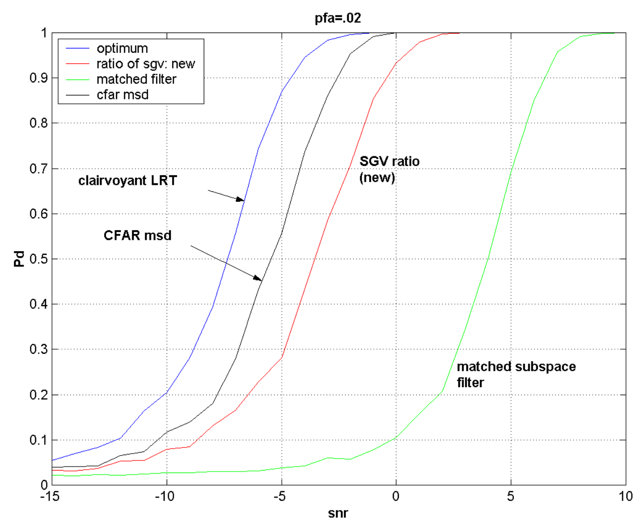

Fig 3: Rank-3 signal SNR vs. probability of detection (P_D). “SGV ratio” is the new detector and “matched subspace filter” is the sub-optimum detector $z_{MS} = \|P_S \vec{x}\|_F^2$. “CFAR msd” is the optimum CFAR UMPI matched subspace detector.

.3 DFT bin apart in spatial wavenumber and separated by ½ DFT bin from the jammer cluster. Thirty independent snapshots were now used for detection. The new detector was compared against the clairvoyant LRT detector for the rank-3 signal (assuming full knowledge of the signal covariance matrix) [1], the optimum UMPI CFAR matched subspace detector (6), and the sub-optimum matched subspace detector $z_{MS} = \|P_S \vec{x}\|_F^2$.

The measured probability of detection curves in figure 3 show that the new detector compares favorably to the UMPI CFAR matched subspace detector, with the new method only losing a few decibels of performance. The new detector is most fairly compared against the CFAR MSD since it also does not assume any knowledge of the signal amplitude covariances whereas the clairvoyant LRT detector does. Hence the performance of the clairvoyant LRT is overly optimistic.

5. CONNECTION TO WIENER FILTER AND CAPON'S METHOD

It is now demonstrated that the ratio of generalized variances test statistic (11) corresponds to a ratio of the mean-square errors of Wiener filters in estimating the subspace interference present in the signal and white noise reference beams from the orthogonal complement of the signal subspace. Furthermore, it is also equivalent to Capon's method.

Switching to a signal-based coordinate system, the matrix Q can always be written as $Q = [S \,|\, B]$ where the signal blocking matrix B is constructed such that $S^H B = 0$. Similarly, $Q_\perp$ can be written as $Q_\perp = [\vec{g} \,|\, B]$ where recall, $\vec{g}$ is the white noise reference beam. Using the above Q and $Q_\perp$ and the nominal covariance matrix R, the following block-partitioned covariance matrices are constructed

$$Q^H RQ = \begin{bmatrix} R_s & R_{s\perp}^H \\ R_{s\perp} & R_\perp \end{bmatrix} = \begin{bmatrix} S^H RS & S^H RB \\ B^H RS & B^H RB \end{bmatrix} \tag{13}$$

$$Q_\perp^H RQ_\perp = \begin{bmatrix} R_w & R_{w\perp}^H \\ R_{w\perp} & R_\perp \end{bmatrix} = \begin{bmatrix} \vec{g}^H R\vec{g} & \vec{g}^H RB \\ B^H R\vec{g} & B^H RB \end{bmatrix} \tag{14}$$

where $R_\perp$ is the covariance matrix of the outputs of the signal blocking beams, R_s is the covariance matrix of the signal beam outputs, $R_{s\perp}$ is the cross-covariance matrix between signal and signal blocking beams, R_w is the variance of the white noise reference beam , and $R_{w\perp}$ is the cross-covariance matrix between white noise reference and signal blocking beams. Using the formula for determinants of partitioned matrices [12], $|Q^H RQ|$ and $|Q_\perp^H RQ_\perp|$ can be written as

$$|Q^H RQ| = |R_\perp| \cdot |R_s - R_{s\perp}^H R_\perp^{-1} R_{s\perp}| \tag{15}$$

$$|Q_\perp^H RQ_\perp| = |R_\perp| \cdot |R_w - R_{w\perp}^H R_\perp^{-1} R_{w\perp}| \tag{16}$$

respectively. After plugging the nominal quantities (15) and (16) into (11) and simplifying, the test statistic can be re-written as

$$z = \frac{|R_s - R_{s\perp}^H R_B^{-1} R_{s\perp}|}{|R_w - R_{w\perp}^H R_B^{-1} R_{w\perp}|} \tag{17}$$

The numerator and denominator of (17) are immediately recognized as the determinants of the covariance matrices of the conditional Gaussian means [12]. These are the covariance matrices of the Wiener filter errors in predicting the subspace interference leakage in the S and $\vec{g}$ channels from the signal-free noise reference measurements $B^H \vec{x}$, as in a generalized sidelobe canceller. Equivalently, they can be interpreted as the covariances of the residual noise and signal remaining after the subspace interference had been estimated and subtracted. Also, the determinants in (17) are the generalized variances of the Wiener filter error covariance matrices. The resultant ratio of GVs (17) is intriguing because of the connection of the SGV to the UMPI test discussed earlier and its invariance to transformations by matrices that preserve the population dispersion hypervolume. Finally, when R is replaced by the sample covariance matrix, (17) can be interpreted as the estimated signal channel residual GV normalized by an estimate of the background noise GV.

The equivalence to Capon's method in the rank-1 signal case is now discussed. Using the previous signal-based coordinate system and (13) with the partitioned matrix inversion formula, it is straightforward to show that

$$\frac{1}{\vec{e}^H (Q^H RQ)^{-1} \vec{e}} = R_s - R_{s\perp}^H R_\perp^{-1} R_{s\perp} = \frac{|Q^H RQ|}{|B^H RB|} \tag{18}$$

where $\vec{e} = [10\ldots 0]^T$ is the signal vector in the signal-based coordinate system. By inspection, formulas (17) and (18) are identical except for the normalization. Remarkably, Capon's method itself can be interpreted as a ratio of generalized variances with the normalization term

$|B^H RB|$ corresponding to the GV of the noise lying in <B>.

6. TEST STATISTIC DISTRIBUTION

By equivalence of the numerator and denominator to Capon's method in (11), the ratio of SGVs test statistic (11) is distributed as:

$$z \overset{D}{=} \frac{\vec{g}^H R^{-1} \vec{g}}{\vec{s}^H R^{-1} \vec{s}} \cdot \frac{\chi^2_{1,2(K-r)}}{\chi^2_{2,2(K-r)}} \tag{19}$$

Note that the numerator and denominator in (19) are not necessarily independent and the first term in (19), the ratio of SNRs, is a function of R, $\vec{s}$, and $\vec{g}$. However, when the interference and signal subspaces are almost orthogonal, it can be argued that in the large sample case the chi-square random variables in the numerator and denominator are approximately independent because $R_{s\perp} \approx 0$ and $R_{w\perp} \approx 0$ in (17). In addition, the first term in (19) is roughly equal to one from the above argument. Therefore it is argued that (19) can be approximated by a scaled F random variable.

Below is an F probability paper plot of the computer simulated GSV ratio test statistic under the null hypothesis using the previous experimental results. This empirically confirms that the test statistic is approximately F-distributed (if the measurements are truly F-distributed, the scatter plot should look like a straight line).

7. CONCLUSION

A new scale-invariant test based on a ratio of generalized variances has been developed. Computer simulations show that the prototype test works well for the both the rank-1 and multi-rank signal cases. Remarkably, it is shown that the new ratio of generalized variances test can be recast as a ratio of the determinants of Wiener filter error covariance matrices. Although in the case of a rank-1 signal the new method is equivalent to a normalized version of Capon's method, the alternative interpretation may nonetheless lead to new implementations and insights.

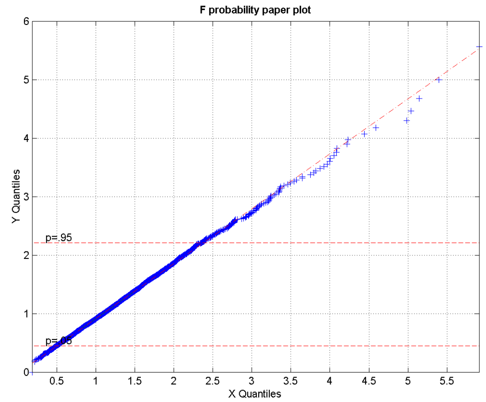

Figure 3: Computer simulated GSV ratio test statistic under the null hypothesis plotted on F probability paper.

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