SYNTHESIS MODELS FOR N-DIMENSIONAL DISCRETE-SPACE SELF-SIMILAR SIGNALS[1]

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ABSTRACT

New formulations and models are proposed for describing statistical self-similarity in general N-dimensional settings. By using a matrix scaling operator for defining statistical self-similarity, a wide class of continuous-space N-D processes can be characterized as self-similar with respect to specific matrix classes. Further, discrete-space versions of N-D statistical self-similarity are treated through a discrete-domain scaling operation. The mathematical basis for the approaches is provided along with 2-D synthesis examples.

Keywords: *Self-similarity, fractals, n-dimensional, synthesis model* .

1. INTRODUCTION

A *n-dimensional* random field in continuous-space(CS) is statistically self-similar (SS) in the wide-sense(WS) if its mean and autocorrelation functions are invariant with respect to spatial scaling. Statistical self-similarity of a *n-dimensional* random fields has been defined typically in terms of spatial scaling[1], which is a direct extension of the 1-D definition[1, 2]. This definition is widely accepted for CS self-similarity[3, 4] although it is restrictive in the sense that various random fields that demonstrate aspects of self-similarity are not covered by it. Approaches like digitizing continuous-space self-similar random fields (for e.g. Fractional Brownian Motion) has been the methodology to generate discrete-space self-similar random fields. Furthermore, in discrete-space(DS), SS random fields are usually characterized by indirect means such as stationary increments[5], power law form[3, 6] and wavelet transform of the fractional Brownian field[7]. The first approach to define DS self-similarity through scaling was proposed by Zhao and Rao[8] and subsequently models for image synthesis are proposed in [9, 10].

The key contributions of this paper are (1) it provides a definition for statistical self-similarity in *n-dimensional* CS that is more general than the current definition, (2) it develops a formalism for self-similarity based on *n-dimensional* scaling operator that works in DS rather than CS, and (3) provides synthesis examples of DS SS random fields.

The paper is organized as follows. Section 2 provides basic definition of the *n-dimensional* continuous-space self-similar random fields. Section 3 discusses discrete-space scaling in *n-dimension* and synthesis examples are provided in section 4. Conclusions and future work is drawn in section 5.

2. SELF-SIMILARITY IN A N-DIMENSIONAL CONTINUOUS SPACE

Definition 1: *A continuous-space n-dimensional random field* $x(\boldsymbol{t})$, $\boldsymbol{t}=[t_1,\ t_2,.....t_n]^T$ *is statistically self-similar in the strict-sense if*

$$x(\boldsymbol{t})\overset{d}{=}a^{-H}x(a\boldsymbol{t}),\ a>0 \qquad (1)$$

where $\overset{d}{=}$ denotes equality of the finite-dimension distributions and H is called the *Hurst parameter*. This definition is a direct extension of the 2-D self-similarity and characterizes the behavior of a random field when it is scaled isotropically. The relationship between the Hurst parameter and the fractal dimension D is $D=n+1-H$, which is the topological dimension of a *n-dimensional* fractional Brownian motion[11]. Suppose a SS random field $h(t_1,\ t_2,.....t_n)$ is the product of n independent 1-D random processes $f_1(t)$, $f_2(t)$,….., $f_n(t)$ with the same *Hurst parameter* H, that is $h(t_1,\ t_2,.....t_n)=f_1(t)f_2(t)......f_n(t)$. Then the process $h(t_1,\ t_2,.....t_n)$ has SS properties as shown below

$$\begin{aligned}h(t_1,\ t_2,.....t_n)&\overset{d}{=}(a_1a_2....a_n)^{-H}f_1(a_1t_1)f_2(a_2t_2)...f_n(a_nt_n)\\&=(a_1a_2....a_n)^{-H}h(a_1t_1,a_2t_2,.....a_nt_n)\end{aligned} \qquad (2)$$

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The process defined in (2) depicts self-similar properties when it is scaled anisotropically. However, this anisotropic scaling behavior cannot be captured by the definition in (1). Obviously, random fields that satisfy (2) also satisfy (1), but the converse is not true.

Suppose a *n-dimensional random field* $h_1(t_1,\ t_2,\ldots..t_n)$ is the product of n independent random processes $f_1(t)$, $f_2(t)$,….., $f_n(t)$ and k of them are self-similar with degree *H*, then

$$\begin{aligned} h_1(t_1,\ t_2,..t_n) &\overset{d}{=} (a_1 a_2 \ldots a_k)^{-H} \\ & f_1(a_1 t_1) f_2(a_2 t_2) \ldots f_k(a_k t_k) f_{k+1}(a_{k+1} t_{k+1}) \ldots f_n(a_n t_n) \quad (3) \\ &= (a_1 a_2 \ldots. a_k)^{-H}\, h_1(a_1 t_1, a_2 t_2, ..a_k t_k, t_{k+1} \ldots. t_n) \end{aligned}$$

where $a_i = 1, k+1 \le i \le n$ and hence $h_1(t_1,\ t_2,\ldots..t_n)$ is *directionally self-similar* in the sense that self-similarity holds for scaling in the first *k* co-ordinates only. The problem with the definition in (1) is that the additional degrees of freedom is not made use off and hence we propose a new general definition for self-similarity based on matrix scaling that proves to be more general.

Definition 2: *A random field $x(t)$ is SS for a matrix class C with the index H if, for a non-singular matrix $A \in C$*

$$x(\boldsymbol{t}) \overset{d}{=} |\,D_A\,|^{-H/n}\ x(A\boldsymbol{t}) \qquad (4)$$

where $D_A = \det A$. It is now seen that the definition in (1) is a special case of the new definition and holds for the class of diagonal matrices with equal entries, i.e.,

$$A = \begin{bmatrix} a & 0\cdots\cdots 0 \\ 0 & a\cdots\cdots 0 \\ \vdots & \\ 0 & 0\cdots\cdots a \end{bmatrix},\quad a>0 \qquad (5)$$

According to the new definition, the separable *n-dimensional* random field $h(\boldsymbol{t})$ is self-similar with respect to matrices of the class given by

$$A = \begin{bmatrix} a_1 & 0\cdots\cdots 0 \\ 0 & a_2\cdots\cdots 0 \\ \vdots & \\ 0 & 0\cdots\cdots a_n \end{bmatrix},\quad \forall a_i>0 \qquad (6)$$

Furthermore, the *n-dimensional* random field $h_1(\boldsymbol{t})$ in (3) can be defined by a matrix class

$$A = \begin{bmatrix} a_1 & 0\cdots\cdots 0 \\ 0 & a_2\cdots\cdots 0 \\ \vdots & \\ 0 & 0\cdots\cdots a_n \end{bmatrix},\quad \forall a_i>0 \quad where\ \ a_i \in \{1,a\} \qquad (7)$$

Accordingly, using the new self-similarity definition in (4), it is possible to characterize much wider range of self-similarity.

3. DISCRETE-SPACE SCALING IN N-DIMENSION

We now formulate a scaling operator in DS that leads to developing a framework for treating self-similarity on lines similar to that for continuous space as in (4). With such a formulation, self-similarity in *n-dimension* can be addressed.

Definition 3: Let $f(\omega)$ be a 1-D warping function that transforms discrete-domain frequency $\omega \in [-\pi,\pi]$ to continuous domain frequency. Then the vector-valued function

$$\boldsymbol{\Omega} \triangleq f(\boldsymbol{\omega}) = \left[\Omega_1\ \Omega_2 \ldots. \Omega_n\right]^T = \left[f(\omega_1)\ f(\omega_2) \ldots f(\omega_n)\right]^T \qquad (8)$$

is a *n-dimension* frequency warping transform. The unwarping transform $f^{-1}(\boldsymbol{\Omega})$ maps $\boldsymbol{\Omega}$ into $\boldsymbol{\omega}$ where $\boldsymbol{\omega} = f^{-1}(\boldsymbol{\Omega}) \triangleq \left[f^{-1}(\Omega_1), f^{-1}(\Omega_2) \ldots.. f^{-1}(\Omega_n)\right]$.

Definition 4: *Let $\boldsymbol{m} = [m_1,\ m_2 \ldots.. m_n]^T$. A scaling operation $\Im_A$ by a non-singular matrix A in n-dimensional discrete-space associated with a n-dimensional warping transform f is defined by the following input-output relationship:*

$$y(\boldsymbol{m}) = \Im_A\left\{x(\boldsymbol{m})\right\} \triangleq \Phi^{-1}\left\{|D_A|\, X\left(\Lambda_A(\boldsymbol{\omega})\right)\right\} \qquad (9)$$

where $D_A = \det A$, $\Lambda_A(\boldsymbol{\omega}) \triangleq f^{-1}\left[A f(\boldsymbol{\omega})\right]$ *and* $X(\boldsymbol{\omega})$ is the Fourier transform of $x(\boldsymbol{m})$

$$X(\boldsymbol{\omega}) = \Phi[x(\boldsymbol{m})] \triangleq \sum_{\boldsymbol{m}} x(\boldsymbol{m}) e^{-j\boldsymbol{m}\boldsymbol{\omega}} d\boldsymbol{\omega}, \qquad (10)$$

Φ^{-1} *is the inverse discrete-space Fourier transform,*

$$x(\boldsymbol{m}) = \Phi^{-1}\left\{X(\boldsymbol{\omega})\right\} = \left(\frac{1}{2\pi}\right)^n \int_{\boldsymbol{\omega}} X(\boldsymbol{\omega}) e^{j\boldsymbol{m}\boldsymbol{\omega}} d\boldsymbol{\omega} \qquad (11)$$

The DS scaling operation $\Im_A$ by a matrix A is computed by the following procedures.

(a) Compute the Fourier transform $X(\boldsymbol{\omega})$ of $x(\boldsymbol{m})$,

(b) Map a frequency vector $\boldsymbol{\omega}$ into a continuous frequency vector $\boldsymbol{\Omega}$ through the warping transform f to get $X_C(\boldsymbol{\Omega}) = X\left(f^{-1}(\boldsymbol{\Omega})\right) = \sum_{\boldsymbol{m}} x(\boldsymbol{m}) e^{-j\boldsymbol{m} f^{-1}(\boldsymbol{\Omega})}$,

(c) Transform $X_C(\boldsymbol{\Omega})$ by a matrix A to form $X_A(\boldsymbol{\Omega}) = |D_A| \sum_{\boldsymbol{m}} x(\boldsymbol{m}) e^{-j\boldsymbol{m} f^{-1}(A^T \boldsymbol{\Omega})}$,

(d) Apply the unwarping transform to compute $X_A(\boldsymbol{\omega}) = |D_A| \sum_{\boldsymbol{m}} x(\boldsymbol{m}) e^{-j\boldsymbol{m} f^{-1}(A^T f(\boldsymbol{\omega}))}$ and

(e) Obtain the scaled signal $x_A(\boldsymbol{m})$ by the inverse Fourier transform of $X_A(\boldsymbol{\omega})$.

The transform operator $\Im_A$ is linear, space varying, order dependent, and the inverse transform operator exists, i.e., if $y(\boldsymbol{m}) = \Im_A\{x(\boldsymbol{m})\}$, then $x(\boldsymbol{m}) = \Im_{A^{-1}}\{y(\boldsymbol{m})\}$. The scaling operation $\Im_A$ can be written as a kernel representation as

$$x_A(\boldsymbol{m}) = \Im_A\{x(\boldsymbol{m})\} = \sum_{\boldsymbol{k}} x(\boldsymbol{k}) g_A(\boldsymbol{m},\boldsymbol{k}) \tag{12}$$

where $g_A(\boldsymbol{m},\boldsymbol{k}) = \left(\frac{|D_A|}{(2\pi)^n}\right)\int_{\boldsymbol{\omega}} e^{-j[\boldsymbol{k}\Lambda_A(\boldsymbol{\omega})+\boldsymbol{m}\boldsymbol{\omega}]}d\boldsymbol{\omega}$ is the scaling operator kernel. The autocorrelation of the output signal $y(\boldsymbol{m})$ is given as,

$$\begin{aligned}R_{yy}(\boldsymbol{m},\boldsymbol{m}') &= E[y(\boldsymbol{m})y(\boldsymbol{m}')] \\ &= \sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'} E[x(\boldsymbol{k})y(\boldsymbol{k}')] g_A(\boldsymbol{m},\boldsymbol{k}) g_A(\boldsymbol{m}',\boldsymbol{k}') \quad (13) \\ &= \Im_{AA}[R_{xx}(\boldsymbol{m},\boldsymbol{m}')]\end{aligned}$$

where $\Im_{AA}$ is application of $\Im_A$ first with respect to $\boldsymbol{m}$ (or $\boldsymbol{m}'$) and then respect to $\boldsymbol{m}'$ (or $\boldsymbol{m}$). Taking the Fourier transform of both sides of (13), we have

$$P_{yy}[\boldsymbol{\omega},\boldsymbol{\omega}'] = |D_A|^n P_{xx}[\Lambda_A(\boldsymbol{\omega}),\Lambda_A(\boldsymbol{\omega}')] \tag{14}$$

where

$$P_{xx}[\boldsymbol{\omega},\boldsymbol{\omega}'] \triangleq \sum_{\boldsymbol{m}}\sum_{\boldsymbol{m}'} R_{xx}(\boldsymbol{m},\boldsymbol{m}')e^{-j(\omega\boldsymbol{m}+\omega'\boldsymbol{m}')} \tag{15}$$

Definition 5: *A n-dimensional discrete-space random field $x(\boldsymbol{m})$ is SS for a matrix class C with the index H in the wide-sense if, for a non-singular matrix $A \in C$*

$$\begin{aligned}E\{\Im_A\{x(\boldsymbol{m})\}\} &= |D_A|^{-H/n} E\{x(\boldsymbol{m})\} \\ \Im_{AA}\{R_{xx}(\boldsymbol{m},\boldsymbol{m}')\} &= |D_A|^{-H} R_{xx}(\boldsymbol{m},\boldsymbol{m}')\end{aligned} \tag{16}$$

where $R_{xx}(\boldsymbol{m},\boldsymbol{m}')$ is the autocorrelation of the random field $x(\boldsymbol{m})$, and D_A is the determinant of the matrix A.

Theorem 1: *If the input to the n-dimensional DS scaling operator $\Im_A$ is a zero mean, WSS discrete random field with a power spectrum $P_x(\boldsymbol{\omega})$, the output is also WSS with the power spectrum given by*

$$P_y(\boldsymbol{\omega}) = \frac{|D_A|^n P_x[\Lambda_A(\boldsymbol{\omega})]}{\left|\det\left[\frac{d[\Lambda_A(\boldsymbol{\omega})]}{d\boldsymbol{\omega}}\right]\right|} \tag{17}$$

where $\Lambda_A(\boldsymbol{\omega}) = \begin{bmatrix}\Lambda_1(\omega_1,\omega_2...\omega_n) \\ \Lambda_2(\omega_1,\omega_2...\omega_n) \\ \vdots \\ \Lambda_n(\omega_1,\omega_2...\omega_n)\end{bmatrix}$, $\boldsymbol{\omega} = [\omega_1,\omega_2....\omega_n]^T$ and

$$\frac{d[\Lambda_A(\boldsymbol{\omega})]}{d\boldsymbol{\omega}} \triangleq \begin{bmatrix}\frac{d[\Lambda_1(\boldsymbol{\omega})]}{d\omega_1} & \frac{d[\Lambda_2(\boldsymbol{\omega})]}{d\omega_1}\cdots & \frac{d[\Lambda_n(\boldsymbol{\omega})]}{d\omega_1} \\ \frac{d[\Lambda_1(\boldsymbol{\omega})]}{d\omega_2} & \cdots\cdots & \frac{d[\Lambda_n(\boldsymbol{\omega})]}{d\omega_2} \\ \vdots & & \\ \frac{d[\Lambda_1(\boldsymbol{\omega})]}{d\omega_n} & \cdots\cdots & \frac{d[\Lambda_n(\boldsymbol{\omega})]}{d\omega_n}\end{bmatrix}.$$

Proof: Let $R_{xx}(\boldsymbol{m},\boldsymbol{m}')$ and $R_{yy}(\boldsymbol{m},\boldsymbol{m}')$ be the autocorrelation of input $x(\boldsymbol{m})$ and output $y(\boldsymbol{m})$ n-dimensional signals respectively. Since the input is WSS, the discrete Fourier transform of $R_{xx}(\boldsymbol{m},\boldsymbol{m}')$ can be given as

$$P_{xx}(\boldsymbol{\omega},\boldsymbol{\omega}') = (2\pi)^n P_x(\boldsymbol{\omega})\delta(\boldsymbol{\omega}+\boldsymbol{\omega}') \tag{18}$$

where $P_x(\boldsymbol{\omega})$ is the power spectrum density. The *1-D* proof is given in [12]. The Fourier transform of $R_{yy}(\boldsymbol{m},\boldsymbol{m}')$ is given by (14) and hence,

$$P_{yy}(\boldsymbol{\omega},\boldsymbol{\omega}') = (2\pi)^n |D_A|^n P_x[\Lambda_A(\boldsymbol{\omega})]\delta(\Lambda_A(\boldsymbol{\omega})+\Lambda_A(\boldsymbol{\omega}')) \tag{19}$$

From [13],

$$\delta(\Lambda_A(\boldsymbol{\omega})+\Lambda_A(\boldsymbol{\omega}')) = \frac{\delta(\boldsymbol{\omega}+\boldsymbol{\omega}')}{\left|\det\left[\frac{d\Lambda_A(\boldsymbol{\omega})}{d\boldsymbol{\omega}}\right]\right|} \tag{20}$$

Therefore,

$$P_{yy}(\boldsymbol{\omega},\boldsymbol{\omega}') = (2\pi)^n |D_A|^n P_x[\Lambda_A(\boldsymbol{\omega})]\frac{\delta(\boldsymbol{\omega}+\boldsymbol{\omega}')}{\left|\det\left[\frac{d\Lambda_A(\boldsymbol{\omega})}{d\boldsymbol{\omega}}\right]\right|} \tag{21}$$

According to the above equation the output is WSS and its spectral density is

$$P_y(\boldsymbol{\omega}) = \frac{|D_A|^n P_x[\Lambda_A(\boldsymbol{\omega})]}{\left|\det\left[\frac{d[\Lambda_A(\boldsymbol{\omega})]}{d\boldsymbol{\omega}}\right]\right|} \tag{22}$$

Suppose a SS random field is zero-mean, then the definition in (16) is written in the frequency domain as

$$|D_A|^n P_{xx}[\Lambda_A(\boldsymbol{\omega}),\Lambda_A(\boldsymbol{\omega}')] = |D_A|^{-H} P_{xx}(\boldsymbol{\omega},\boldsymbol{\omega}') \tag{23}$$

Furthermore, if the random field is WSS, by *Theorem 1* (23) can be written as,

$$\frac{P_x[\Lambda_A(\boldsymbol{\omega})]}{\left|\det\left[\frac{d[\Lambda_A(\boldsymbol{\omega})]}{d\boldsymbol{\omega}}\right]\right|} = |D_A|^{-H-n} P_x(\boldsymbol{\omega}) \tag{24}$$

Therefore a zero-mean WSS discrete random field with the power spectrum satisfying (24) is wide-sense SS with the *Hurst parameter H*. One example of such a self-similar random field is a stationary random field $x_1(\boldsymbol{m})$ with the power spectrum

$$P_1(\omega) = \frac{\|f(\omega)\|^r}{\left|\det\left[\frac{df(\omega)}{d\omega}\right]\right|} \quad (25)$$

where, $\|.\|$ represents Euclidian norm with $H = -r/2 - 1$. With the bilinear warping transform (BLT) $f(\omega) = \left[2\tan(\omega_1/2),\, 2\tan(\omega_2/2)....2\tan(\omega_n/2)\right]^T$, (25) can be simplified as

$$P_1(\omega) = \frac{2^r\left[\sum_{i=1}^{n}\tan^2(\omega_i/2)\right]^{r/2}}{\left|\prod_{i=1}^{n}\sec^2(\omega_i/2)\right|} \quad (26)$$

Unlike continuous SS random fields, the discrete SS random field with the power spectrum as in (26) is stationary. Hence, it behaves like the *n-dimensional* stationary fractional Gaussian noise rather than the non-stationary fractional Brownian motion.

Theorem 2: *For BLT, when* $\omega \to 0$, *the power spectrum of a stationary random filed* $x_1(\boldsymbol{m})$ *defined in (25) is given as* $P_1(\omega) \to \|\omega\|^{-2H-n}$.

Proof: Using the Taylor series expansion for $\tan(x)$ and $\sec(x)$ and with the approximation at $\omega \approx \mathbf{0}$, we have $\tan(\omega/2) \approx \omega/2$ and $\sec(\omega/2) \approx 1$, then (26) simplifies to

$$P_1(\omega) \to \|\omega\|^{-2H-n} \quad (27)$$

Although fractional Brownian Motion (fBm) is non-stationary its increment process are assumed to be stationary with $1/f^{\beta}$ spectrum. An n-dimensional fBm is generated by passing an n-dimensional uncorrelated white noise through an n-dimensional fractal filter. The fractal filter is of the form[11]

$$G(\boldsymbol{k}) = k^{1-H-(n/2)} \quad (28)$$

where $\boldsymbol{k} = (k_1, k_2, ...k_n)$, $k = \|\boldsymbol{k}\|$ and $n \geq 2$.

Theorem 3[11]*: For a fractal filter in (28) and* $0.5 < H < 1$ *,the autocorrelation of the n-dimensional homogenous process* $W_H(\boldsymbol{x})$ *is*

$$\begin{aligned} R_{W_H}(\boldsymbol{x}_1, \boldsymbol{x}_2) &= R_{W_H}(\boldsymbol{x}_2 - \boldsymbol{x}_1) = R_{W_H}(x) \\ &= 2H(H - 1 + n/2)V'_H\, x^{2H-2} \end{aligned} \quad (29)$$

where $x = \|\boldsymbol{x}_2 - \boldsymbol{x}_1\|$ *and* $V'_H = \frac{2^{1-n-2H}\Gamma(1-H)}{\pi^{n/2} H\Gamma(n/2 - H)}$. For an n-dimensional fBm, the correlation between $B_H(\boldsymbol{x})$ and $B_H(\boldsymbol{y})$ of any two position-vector points $\boldsymbol{x}$ and $\boldsymbol{y}$ is given as

$$\begin{aligned} R_{B_H}(\boldsymbol{x}, \boldsymbol{y}) &= E\left[B_H(\boldsymbol{x}), B_H(\boldsymbol{y})\right] \\ &= \frac{1}{2}V'_H\left(\|\boldsymbol{y}\|^{2H} + \|\boldsymbol{x}\|^{2H} - \|\boldsymbol{y} - \boldsymbol{x}\|^{2H}\right) \end{aligned} \quad (30)$$

where V'_H is defined as above.

Theorem 4[11]*: The power spectrum of an n-dimensional fBm, for* $0 < H < 1$ *is given as*

$$\phi_{B_H}(\boldsymbol{k}) = k^{-n-2H} \quad (31)$$

This the similar to result proved in (27)

Furthermore, the power spectrum of the fractional Gaussian noise obtained from the fractional Brownian motion has a Hurst parameter $H' = H + 1$ and hence $P_1(\omega) \to \|\omega\|^{-2H'}$.

4. SIMULATION RESULTS

In this section we present synthesis examples of 2D self-similar fields generated by filters given in [14]. The input to the filters was the white Gaussian random field with zero mean and unit variance and the filter size was 128 by 128. With different self-similarity parameters we can obtain roughness of various degree. Synthesis examples of 2-D self-similar random fields for isotropic and directional are shown in Fig.1. for *r = -1.4* and *r = -0.6.* Another synthesis example is the directional SS random field $x_2(n)$ with the power spectrum

$$P_2(\omega) = \frac{|f(\omega_1)|^r}{|f'(\omega_1)|^r} g(\omega_2) \quad (32)$$

where $f(\omega_1)$ is a 1-D warping transform function, and $g(\omega_2)$ is an arbitrary 1-D power spectrum. Then $x_2(n)$ is SS with $H = -r - 1$ with respect to the class of matrices as in (7)[14].

Theorem 5[14] : *Let* $P_x(\omega)$ *be a power spectrum of a DS SS random field* $x(n)$ *with respect to the class of matrices A according to* (24)*. Then, a stationary random field* $y(n)$ *with the power spectrum* $P_y(\omega) = \left|\det\frac{d\omega}{d\Lambda_C(\omega)}\right| P_x\left[\Lambda_C(\omega)\right]$ *is SS with respect to the class of matrices* $B = CAC^{-1}$ *and* $y(n)$ *has the same H as* $x(n)$.

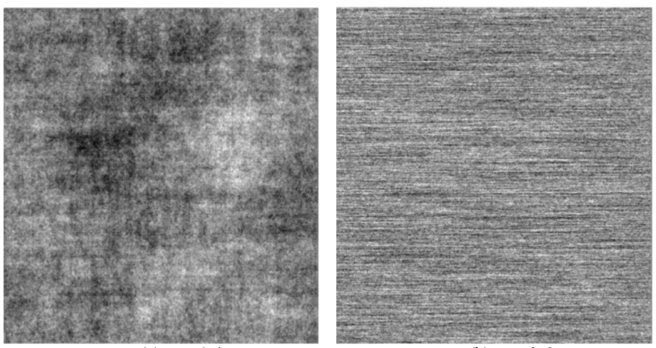
Proof: See [14].

A random field $h_2(t_1, t_2) = f_2(t_1)g_2(t_2)$, which is the product of two statistically independent 1-D SS random processes with same degree *H*, is SS with respect to the class of matrices given in (6) with the parameter $H_2 = 2H$. For example, a random field $x_4(n)$ with the power spectrum

$$P_4(\omega) = \frac{|f(\omega_1)|^r |f(\omega_2)|^r}{|f'(\omega_1)|^r |f'(\omega_2)|^r} \quad (33)$$

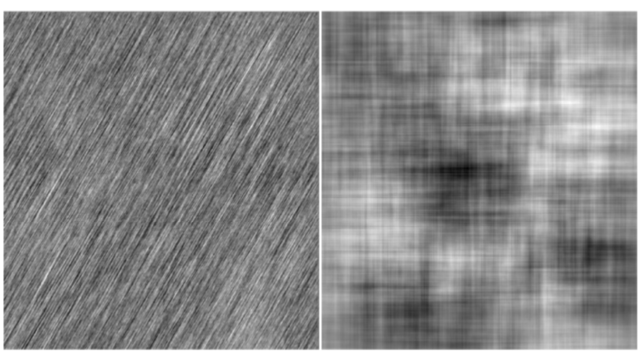
where $f(\omega)$ is a 1-D warping transform function and is SS with respect to a class of matrices given in (6) and

$H_4 = -r - 1$. Directional self-similar random fields synthesized from the power spectrum $P_2(\omega)$ with BLT and $g(\omega) = 0.5(1+\cos\omega)$ are shown in Fig.2(a). The power spectrum $P_2(\omega)$ is transformed to $P_3(\omega)$ with $\theta = \pi/3$ using Theorem 5 which results in directional self-similar random fields oriented by an angle $\theta = \pi/3$. Self-similar random fields in Fig. 2(b) are generated by using separable 1-D filters. In a similar fashion as mentioned in discrete-space SS random processes in higher dimensions can be synthesized by passing an *n-dimensional* uncorrelated white noise through an *n-dimensional* fractal filter.



(a) r = -1.4 (b) r = -0.6

Fig. 1. Synthesized discrete SS random fields, (a) Isotropic, (b) Directional.



(a) r = -0.6, θ=π/3 (b) r = -0.6

Fig. 2. Synthesized discrete SS random fields, (a) Oriented, (b) Seperable.

5. CONCLUSION

A *n-dimensional* definition of DS self-similarity is proposed using a matrix scaling operation. A wider class of self-similarity could be characterized through the new definition and the proposed model has an additional parameter that decides the class of self-similarity. Synthesis examples for discrete-space SS random fields are shown for *n=2*. As part of the future work we propose to generalize the model to multifractal applications and also investigate the applicability of the proposed modeling approach to simulation of print defects.

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