TIME-FREQUENCY AND DUAL-FREQUENCY REPRESENTATION OF FRACTIONAL BROWNIAN MOTION

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ABSTRACT

Fractional Brownian motion (fBm) is a useful non-stationary model for certain fractal and long-range dependent processes of interest in telecommunications, physics, biology, and finance. Conventionally, the power spectrum of fBm is claimed to be a fractional power-law. However, fBm is not a wide-sense stationary process, so the precise meaning of this spectrum is unclear. In this paper, we model and analyze fBm in the context of harmonizable random processes. We derive and interpret exact expressions for novel useful complex valued second-order moment functions for fBm. These moment functions are time-frequency and dual-frequency correlation functions, connecting the random process to its infinitesimal random Fourier generator. In particular, we derive and discuss the time-frequency Rihaczek spectrum, and the dual-frequency Loève spectrum. Our main finding is that the dual-frequency spectrum of fBm has its spectral support confined to three discrete lines. This leads to the surprising conclusion that for fBm, the DC component of the infinitesimal Fourier generator is correlated with all other frequencies of the Fourier generator. We propose and apply multitaper based estimators for the moment functions, and numerical estimates based on synthetic fBm data and real world earthquake data confirm our theoretical results.

1. INTRODUCTION

Fractional Brownian motions provide useful models for a wide range of complex phenomena whose empirical spectra obey power laws of fractional order. In the literature the fBm is often claimed to have a spectrum $|\omega|^{-(2H+1)}$, where H is the Hurst parameter. This would be the spectrum of a stationary process. However, the fBm is a non-stationary random process, and therefore it cannot have such a spectrum. Many researchers have tried to develop the fBm spectrum, but no completely satisfactory result has been derived [7].

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In this paper, the fBm is treated as a non-stationary harmonizable random process, and we derive several new second order statistical characterizations of fBm. Work along these lines has been carried out in [1], where the time-frequency Wigner-Ville spectrum of fBm was derived and explored. The time dependent correlation function of fBm is well known, and this is a pure time-domain second order characterization of a non-stationary random process. In [4] it was shown that this time dependent quantity is just one of four ways of characterizing a harmonizable non-stationary random process. We follow their line of reasoning in order to derive the time-frequency Rihaczek spectrum, and the dual-frequency spectral correlation of fBm. These characterizations of fBm reveal new insight into the non-stationary behavior of fBm. Also, from these quantities, we suggest possible new estimators for the Hurst parameter.

2. FRACTIONAL BROWNIAN MOTION

The common way of formulating a fBm, $B_H(t)$, in the time-domain is [7]

$$B_H(t) - B_H(0) = \frac{1}{\Gamma(H+1/2)} \left[\int_{-\infty}^{0} \left((t-s)^{H-1/2} - (-s)^{H-1/2} \right) dB(s) + \int_{0}^{t} (t-s)^{H-1/2} dB(s) \right]. \tag{1}$$

Here, $B(t)$ denotes standard Brownian motion and $dB(t)$ is a differential increment of $B(t)$. The parameter $H \in (0,1)$ is the Hurst parameter. Eq. (1) generalizes ordinary Brownian motion, which is a special case of fBm with parameter $H = 1/2$. It can be shown that for $H \in (0, 1/2)$ the process $B_H(t)$ is obtained through fractional differentiation of $B(t)$, which causes $B_H(t)$ to fluctuate more than $B(t)$. For $H \in (1/2, 1)$ it can be shown that $B_H(t)$ is obtained by fractional integration of $B(t)$, and this has the effect of smoothing out $B(t)$. The fBm's possess numerous interesting properties. Among them, we can mention that its increments are stationary and self-similar.

3. SECOND ORDER ANALYSIS OF FBM

According to [9] the fBm belongs to the class of *harmonizable* random processes and therefore has the following Cramér-Loève spectral representation,

$$B_H(t) = \int_{-\pi}^{\pi} e^{jt\omega} dZ(\omega), \tag{2}$$

where $dZ(\omega)$ is the complex valued non-orthogonal increment process (or the generalized Fourier transform) with the following incremental spectral covariance

$$(2\pi)^2 \operatorname{E}\{dZ(\omega_1)dZ^*(\omega_2)\} = S_L(\omega_1, \omega_2)d\omega_1 d\omega_2. \tag{3}$$

The process $dZ(\omega)$ is a random set function that assigns random variables to frequency intervals [9]. The spectral correlation function $S_L(\omega_1, \omega_2)$ is often called the Loève spectrum [3, 4]. This equation describes the essential feature of non-stationary random processes, namely that there is correlation between the different frequency components of $dZ(\omega)$ [3, 4]. Conversely, a stationary random process would have $S_L(\omega_1, \omega_2) = S_0(\omega_1)\delta(\omega_1 - \omega_2)$, for some spectrum $S_0(\omega_1)$. The function $S_0(\omega_1)$ is the conventional power spectrum. According to Eq. (2), we may think of $B_H(t)$ as a superposition of complex oscillations $\exp(j\omega t)$ with complex infinitesimal random amplitudes $dZ(\omega)$ that in general are correlated for different frequencies [3, 4].

The bivariate inverse Fourier transform of $S_L(\omega_1, \omega_2)$ is the bivariate temporal correlation function for the time series

$$\begin{aligned} R_L(\tau_1, \tau_2) &= \operatorname{E}\{B_H(\tau_1)B_H(\tau_2)\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_L(\omega_1, \omega_2)e^{j(\omega_1\tau_1 - \omega_2\tau_2)}\frac{d\omega_1 d\omega_2}{(2\pi)^2} \end{aligned} \tag{4}$$

From [7] we have that the dual-time temporal correlation function for fBm is given by

$$R_L(\tau_1, \tau_2) = \frac{V_H}{2}\left(|\tau_1|^{2H} + |\tau_2|^{2H} - |\tau_1 - \tau_2|^{2H}\right), \tag{5}$$

where

$$V_H = \Gamma(1 - 2H)\frac{\cos(\pi H)}{\pi H}, \tag{6}$$

and $\Gamma(\cdot)$ is the ordinary Gamma function. This bivariate temporal correlation function $R_L(\tau_1, \tau_2)$ reveals the non-stationary structure of the fBm.

In order to gain more insight into the global and local non-stationarity of the random process, we will transform the time and frequency coordinates as follows [4]: $\tau_1 = t$, $\tau_2 = t - \tau$, $\omega_1 = \omega + \nu$, and $\omega_2 = \omega$. The time t and frequency ω are *global* variables. The time τ and frequency ν are *local* variables. With these transformations of variables, we may rewrite the Fourier transform pair of Eq. (4) as the temporal correlation $R_{B_H}(t, \tau)$ and the spectral correlation $S(\nu, \omega)$. We may rewrite Eq. (5) as

$$R_{B_H}(t, \tau) = \frac{V_H}{2}\left(|t|^{2H} + |t - \tau|^{2H} - |\tau|^{2H}\right), \tag{7}$$

in terms of global time t and local time τ.

3.1. The time-frequency Rihaczek spectrum

The Rihaczek spectrum $P_{B_H}(t, \omega)$ is a global time, global frequency representation of the second order statistical properties of fBm. This spectrum is derived by means of the Fourier transform of $R_{B_H}(t, \tau)$ with respect to the local time variable τ [4]. However, the Fourier transform of Eq. (5) does not exist in the ordinary sense, since the integral $\int_{-\infty}^{\infty} e^{j\omega t}|\tau|^{\alpha}d\tau$ is formally divergent. Hadamard's finite part is a technique for extracting a finite part from such divergent integrals. By using the formula on page 364 in [2] to integrate $\int_{-\infty}^{\infty} e^{j\omega t}|\tau|^{\alpha}d\tau$, we obtain the following Rihaczek spectrum for fBm

$$P_{B_H}(t, \omega) = \pi V_H|t|^{2H}\delta(\omega) + 2j\sin\frac{\omega t}{2}e^{-j\omega t/2}|\omega|^{-(2H+1)}, \tag{8}$$

where $\delta(\cdot)$ is the Dirac delta function.

In Fig. 1a, 1b, and 1c we show the modulus, the real part, and the imaginary part of $P_{B_H}(t, \omega)$, respectively, for $H = 0.1$. The periodic harmonic effects shown for ω close to zero disappears when $H \to 1$ and the Rihaczek spectrum is reduced to one single line at $\omega = 0$. From Eq. (8) we see that, for $\omega = 0$ on log-log scale, we get a line with slope $2H$. In Fig. 1d we have plotted the line $P_{B_H}(t, 0)$ on log-log scale, and we see that it is a perfect line with slope $2H$. Thus, it is clear that if we have good estimators for the Rihaczek spectrum, we will be able to estimate the Hurst parameter by estimating the slope of this line.

3.2. The dual-frequency spectral correlation

Since the $P_{B_H}(t, \omega)$ and $S_{B_H}(\nu, \omega)$ is a Fourier pair in the $t - \nu$ variables [4], we Fourier transform the Rihaczek spectrum with respect to the global time variable t to produce the dual-frequency spectrum $S_{B_H}(\nu, \omega)$

$$\begin{aligned} S_{B_H}(\nu, \omega) = 2\pi \Big[&|\omega|^{-(2H+1)}\delta(\nu) \\ &-|\omega|^{-(2H+1)}\delta(\nu + \omega) - |\nu|^{-(2H+1)}\delta(\omega)\Big]. \end{aligned} \tag{9}$$

This result bears comment. We see that the frequency correlation is real, and it is represented as a line spectrum with spectral support on three discrete lines. For $\nu = 0$ we have the stationary manifold, so the spectrum $S_{B_H}(0, \omega)d\omega = \operatorname{E}\left\{|dZ(\omega)|^2\right\} = 2\pi|\omega|^{-(2H+1)}d\omega$ is the conventional stationary power spectrum. Stationary random processes would

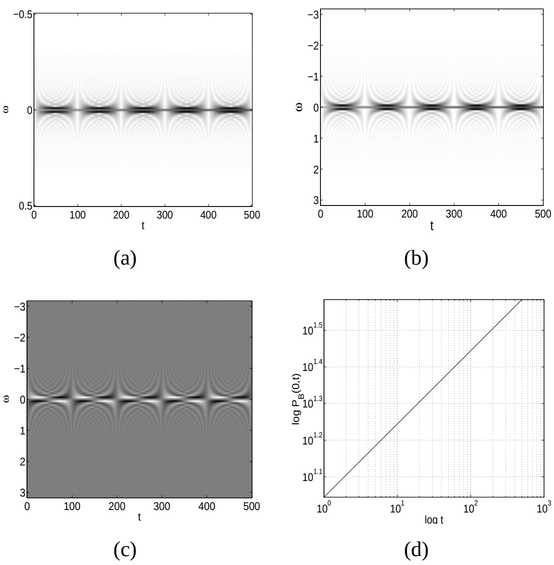


Fig. 1. (a) The modulus of the Rihaczek spectrum, $|P_{B_H}(t,\omega)|$, (b) the real part of the Rihaczek spectrum, $\mathrm{Re}\, P_{B_H}(t,\omega)$, (c) the imaginary part of the Rihaczek spectrum, $\mathrm{Im}\, P_{B_H}(t,\omega)$, and (d) the DC line $P_{B_H}(t,0)$. Here, $H = 0.1$.

have a contriobution only on $\nu = 0$. However, if the process is non-stationary, the dual-frequency spectrum $S(\nu,\omega)$ will have non-zero contributions also outside the stationary manifold. For $\omega = 0$, we have that $S_{B_H}(\nu,0)d\omega = \mathrm{E}\{dZ(0)dZ^*(\nu)\} = 2\pi|\nu|^{-(2H+1)}d\omega$ is the correlation of the DC component of the increment process with all other frequency components of the increment process. For $\nu = -\omega \neq 0$, $S_{B_H}(-\omega,\omega) = \mathrm{E}\{dZ(\omega)dZ^*(0)\} = 2\pi|\omega|^{-(2H+1)}$ is also a manifestation of the correlation between the DC component of the increment process, and the increment process for all other frequencies ω. The two lines that represent the correlation between the DC component of the increment process and the increment process for all other frequencies, are clearly outside the stationary manifold, and they give rise to the non-stationary behavior of fBm.

In Fig. 2a we show the dual-frequency spectrum $S_{B_H}(\nu,\omega)$. The vertical line is the stationary manifold, and the horizontal and the diagonal lines are the zero frequency component correlated with all the other frequencies. From Eq. (9) we see that any of those three lines of support contains a line with the slope $-(2H+1)$. In Fig. 2b we show the line $S_{B_H}(\nu,0)$ for $\nu \in (0,1/2)$ on log-log scale. Hence, we obtain yet another estimator for the Hurst parameter. If we have available a good estimator for the dual-frequency spectrum, we can obtain the Hurst parameter by the use of linear regression to estimate the slope of these lines. In fact, as opposed to the Rihaczek spectrum, we now have three lines available. We can estimate the Hurst parameter from any of these lines, and by averaging over these estimates we may form a final composite estimate of H. This is a way to reduce the variance of the estimate of H.

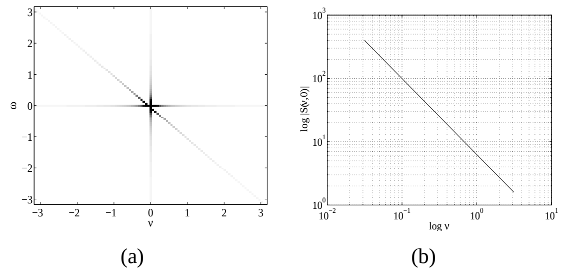


Fig. 2. (a) The spectral correlations $|S_{B_H}(\nu,\omega)|$ of fBm, and (b) the cut $|S_{B_H}(\nu,0)|$ for $\nu \in (0,1/2)$ on a log-log scale. Here, $H = 0.1$.

3.3. Estimation of the spectral correlation

The dual frequency spectral correlation is a mathematical expectation. As a practical matter, it will have to be estimated. In [8] the multitaper approach to spectral estimation was introduced. The method is based on finding the data tapers that maximize the energy concentration in the frequency band $(-B,B)$, and averaging over a set of spectral estimates. We can derive an estimator for the spectral correlation in the same lines as in [5] by starting with the *baseband complex demodulate*

$$\widehat{x}_\alpha(n,f) = \sum_\alpha \tilde{x}_\alpha(f) v_\alpha(n) \tag{10}$$

where

$$\tilde{x}_\alpha(f) = \sum_n x[n] v_\alpha[n] e^{-j2\pi nf}. \tag{11}$$

Here, $\alpha = 0,\ldots\Delta-1$ denotes the data taper order, where Δ is the number of data tapers, and $v_\alpha[n]$ is data taper of order α. The real valued sequences

$$v_\alpha[n] = \int_{-1/2}^{1/2} V_\alpha(f) e^{j2\pi nf} df \tag{12}$$

are the *discrete prolate spheroidal sequences* (DPSS), and $V_\alpha(f)$ are the *discrete prolate spheroidal wave functions* (DPWF) [8]. The DPWF's are a sequence of orthogonal eigenfunctions that maximizes the energy in the band $(-B,B)$. Eq. (11) is a windowed *discrete-time Fourier transform* of the data with the kth DPSS acting as a data taper. Using the $\Delta = \lfloor 2NB \rfloor - 2$, lowest order DPSSs as data windows is a common choice, which ensures an acceptable trade-off between variance reduction and spectral leakage.

Finally, to estimate the dual frequency spectral correlation we compute the sample covariance of the baseband complex demodulate at two different frequencies

$$\widehat{S}_{B_H}(\nu, f) = \frac{1}{\Delta} \widehat{x}_\alpha(n,\nu) \widehat{x}_\alpha^H(n,f). \tag{13}$$

By inserting Eq. (10) into Eq. (13), we obtain

$$\widehat{S}_{B_H}(\nu, f) = \frac{1}{\Delta} \tilde{x}_\alpha(\nu) \tilde{x}_\alpha(f), \tag{14}$$

where $\tilde{x}_\alpha(f)$ is defined in Eq. (11).

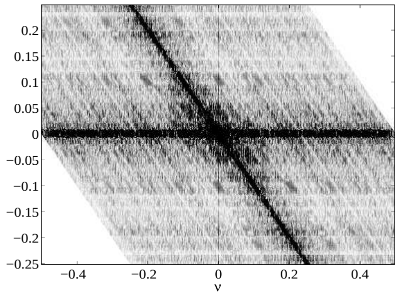
4. CASE STUDIES

In this Section we show examples of analysis utilizing both real and synthetic data. For the synthetic part we generate fBm time series and estimate the dual-frequency spectrum. The same analysis is conducted on a real world time series containing the number of earthquakes with magnitude greater than 7.0 on Richter's scale for every year since 1900.

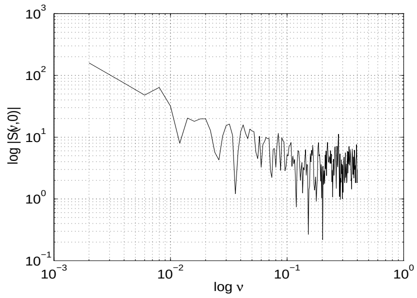
4.1. Spectral analysis of synthetic fBm

As mentioned in Sec. 3.2 the spectral correlation of fBm has spectral support on three discrete lines of support. To verify this experimentally, we conducted a numerical example where we generated $N = 512$ samples of fBm with Hurst parameter $H = 0.1$. The samples were generated by using the method in [6]. In Fig. 3a we show the estimated spectral correlation $\widehat{S}_{B_H}(\nu, \omega)$. The time-bandwidth product was chosen to be $NB = 3.5$. This implies that 5 DPSS-tapers were applied when constructing the estimator $\widehat{S}_{B_H}(\nu, \omega)$. We observe that the predicted horizontal and diagonal lines are clearly present. The vertical line is also present, but it is only one pixel wide, and therefor difficult to spot. The reason that the vertical line is so thin is that the estimator only smooths along the ω-axis. The two lines off $\nu = 0$, are however, outside the stationary manifold. As mentioned, these are the lines that manifest the non-stationarity of fBm.

In Fig. 3b we show the horizontal line of $|\widehat{S}_{B_H}(\nu, 0)|$ for $\nu \in (0, 1/2)$. The figure is shown on a log-log scale, and the linear dependency is obvious. The same occurs for the other two lines. From the figure we see that the line is linear for a certain range of ω. Beyond a certain ω the spectrum is flat. When estimating the slope we used only the linear part of the spectrum. We estimated the slope of (1) The line $|\widehat{S}_{B_H}(\nu, 0)|$ for $\nu \in (0, 1/2)$, (2) the line $|\widehat{S}_{B_H}(0, \omega)|$ for $\omega \in (0, 1/2)$ and (3) the line $|\widehat{S}_{B_H}(-\omega, \omega)|$ for $\omega \in (0, 1/2)$. The estimated Hurst parameters were, $\widehat{H}_1 = 0.1100$, $\widehat{H}_2 = 0.0972$, and $\widehat{H}_3 = 0.0921$, respectively. After averaging these estimates we obtain $\widehat{H} = 0.0997$, which is very close to the true Hurst parameter, $H = 0.1$. These results were obtained from a simple linear regression. By the use of more sophisticated methods for estimating the Hurst parameter from the spectrum, one might obtain better results.



(a)



(b)

Fig. 3. For synthetic fBm: (a) Modulus of the estimated spectral correlation $|\widehat{S}_{B_H}(\nu, \omega)|$, and (b) The line $|\widehat{S}_{B_H}(\nu, 0)|$ for $\nu \in (0, 1/2)$ shown on a log-log scale. Here, $H = 0.1$.

4.2. Spectral analysis of earthquake data set

In this section we analyse a real world time series containing the number of earthquakes with magnitude greater than 7.0 on Richter's scale. The time series contains data for every year since 1900. In Fig. 4 we show the time series, and the data can be downloaded from the Earthquake Data Base System of the U.S. Geological Survey, National Earthquake Information Center, Golden CO, `http://neic.usgs.gov/neis/eqlists/7up.html`.

From Fig. 5a we see the estimated dual-frequency spectrum. Three characteristic lines of spectral support are clearly visible, so this earthquake data set has the spectral correlation structure of fBm. In Fig. 5b we show the cut of $\widehat{S}(\nu, 0)$ for $\nu \in (0, 1/2)$ on a log-log scale. For the linear part of this curve we estimate the Hurst parameter to be $\widehat{H} = 0.29$.

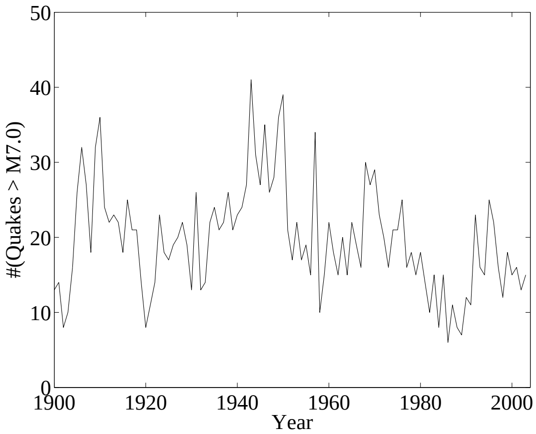


Fig. 4. The number of earthquakes with magnitude greater than 7.0 on Richter's scale for every year from year 1900 to 2004.

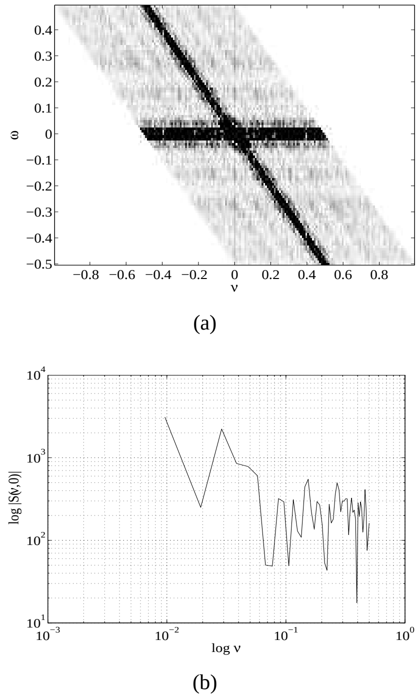


Fig. 5. For earthquake data: (a) Modulus of the estimated spectral correlation $|\widehat{S}_{B_H}(\nu, \omega)|$ for the quake data set, and (b) The line $|\widehat{S}_{B_H}(\nu, 0)|$ for $\nu \in (0, 1/2)$ shown on a log-log scale.

5. CONCLUSION

In this paper we proposed new tools for representing and analyzing fBm. These methods focus on the underlying nonstationary character of fBm. The well known temporal correlation function $R_{B_H}(t, \tau)$ is a pure time-domain second order characterization of fBm. To obtain more insight of the second-order statistical properties of fBm, we derived the time-frequency Rihaczek spectrum $P_{B_H}(t, \omega)$, and the dual-frequency spectral correlation function $S_{B_H}(\nu, \omega)$.

Our main result was that the dual-frequency spectral correlation moment function of fBm has its spectral support on three discrete lines. These lines show that the nonstationarity of fBm is due to the correlation between the DC frequency component of the increment process and all other frequencies of the increment process. We reviewed a method for estimating the dual-frequency spectral correlation, and demonstrated the theoretical predictions on synthetic fBm data and on real world earthquake data.

6. REFERENCES

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