PROVIDING SAMPLE SHAPE STATISTICS WITH FCA AND ISA APPROACHES

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ABSTRACT

Providing sample shape statistics for time warped signals or more generally for heterogeneous shape signals can be useful in many signal processing applications. In fact, the classical mean and the L^2 distance are inadequate in describing and quantifying the time or shape variability among such signals. Specific methods derived from the Curve Registration (CR) theory were developed to deal with this warping or shape problem. More recently, two approaches that seem similar in many points, Functional Convex Averaging (FCA) and Integral Shape Averaging (ISA), proposed an alternative to CR methods in averaging and comparing time warped signals. Both methods estimate a sample mean without requiring that observed signals are well structured and consider the warping as a stochastic process. The objective of the study is to compare ISA and FCA methods with respect to shape equality condition. For this purpose, shape equality is defined and the used methods are recalled with an emphasis on the similarities. After, a Corrected ISA (CISA) approach providing a mean and a distance is proposed and compared to the FCA approach. This comparison with simulations shows that the CISA approach is more suitable for quantifying shape variability than FCA one since it respects shape equality condition.

1. INTRODUCTION

The signal shape is a visual concept that has many definitions. It is an additive criterion that can be used for signal analysis and classification. Providing shape statistical tools should be helpful to these kinds of applications. Commonly, shape equality is assumed between two signals when one can be deduce from the other by amplitude and time affine transforms [1]. In fact, the major difficulty in designing such tool is that shape variability authorises both amplitude and time or warping variations on the signal. The conventional mean and the L^2 distance are, by definition, sensitive to time variations [2], [3], [4], [5]. Indeed, in shape analysis applications they are poor descriptors of the shape variability. Curve Registration (CR) theory was developed among the statistical community to address this warping problem [6]. Several methods were proposed to provide a sample mean estimated from realigned or registered signals [4], [7]. Distances measured on these registered signals and similarity criterions based on alignment cost or warping parameters were also proposed to quantify the warping dispersion [5], [7]. But these methods imposed the presence of a common structure and time localization among signals and, in fact, fixed strong limits on shape variability. Recently, two approaches, Functional Convex Averaging (FCA) [3] and Integral Shape Averaging (ISA) [2], [4] were proposed and applied to average warped signals. Both methods estimate a sample mean without requiring that observed signals are well structured and localised in time and consider the warping as a stochastic process. In addition, the FCA approach proposed a distance coupled to the FCA mean to describe the sample statistics of the warped signals. Our objective is to provide realistic shape statistical tools (respecting shape equality condition) applying ISA approach and compare them with FCA statistical tools. Our contribution can be summarised as follows. First, the shape equality and shape distance definitions are given. After, the FCA and ISA methods are recalled and the similarities and differences of the two approaches are clarified. Shape mean and distance candidates are proposed for a Corrected ISA (CISA) version. Some simulations are performed to show the performances of CISA and FCA methods in describing sample shape statistics. Finally, the results are discussed and a conclusion is given.

2. SHAPE EQUALITY

If we deal with the signal shape concept it is necessary to define shape equality. Following the Shape Invariant Model (SIM) definition of [1], the signals $x(t)$ and $y(t)$ are said *equals in shape* if:

$$y(t) = \alpha x((t-b)/a) + \beta, t \in [0,T] \quad (1)$$

or in a condensed form:

$$y = A_x \circ x \circ A_t, \{A_x, A_t\} \in \mathrm{A} \quad (2)$$

where '$\circ$' defines the composition operation (α, β) and (a, b) define the parameters of the affine amplitude and time transforms A_x and A_t respectively belonging to the affine group A.

Then, to characterise the shape statistics of a sample of signals respecting the shape equality condition, it is needed to have:

i) a distance d between signals which is in fact a distance between the signal shapes. Taking two signals $x_1(t)$ and $x_2(t)$:

$d\{x_1(t), x_2(t)\} = 0 \Leftrightarrow x_1(t)$ and $x_2(t)$ *are the same shape*

In fact, the group structure of A ensures our shape equality to be an equivalence among the signals. So equivalence classes are identified to shapes. It can be shown that the distance $d(x, y)$ between two signals is in fact a distance between their shape.

ii) a sample mean that is intrinsically invariant in shape to these affine transforms providing a realistic mean shape.

In the next sections, it is supposed that $\beta = 0$ for all signals or in other words, the offset is removed before averaging.

3. FCA METHOD

In the FCA approach, time-warped signals are considered to be realizations of a bivariate stochastic process [3]. Indeed, an observed process $(x, \tilde{Y}(x))$ in the warped space W (space of the warped signals) is generated from a latent bivariate process $(X(t), Y(t))$ in the synchronized space S trough the warping mapping ψ:

$$\psi : \{(X(t), Y(t)), t \in [0,1]\} \mapsto \{(x, \tilde{Y}(x)), x \in [0, T]\} \quad (3)$$

A family of monotone and invertible time-warping functions is defined as follows:

$$\varphi_{\tilde{Y}}(x) = \frac{\int_0^x |\tilde{Y}(s)| ds}{\int_0^T |\tilde{Y}(s)| ds}, x \in [0, T] \quad (4)$$

These warping functions correspond to area-under-the-curve synchronization. They can be also interpreted as mapping the warped time x of observed processes $\tilde{Y}$ to synchronized time t. We can rewrite eq.(3) in the following form:

$$\psi^{-1} : \{(x, \tilde{Y}(x)), x \in [0, T]\} \mapsto \{(\varphi_{\tilde{Y}}^{-1}(t), \tilde{Y}(\varphi_{\tilde{Y}}^{-1}(t))), t \in [0,1]\} \quad (5)$$

The functional convex mean or FCA mean in S is given by:

$$\{(\mu_X(t), \mu_Y(t)), t \in [0,1]\} \in S \quad (6)$$

where $\mu_X(t) = EX(t), \mu_Y(t) = EY(t)$ with $E(.)$ indicates the expectation operator. Here $\mu_x(.)$ corresponds to mean time transformation and $\mu_y(.)$ corresponds to mean amplitude function. In the warped space, we can define FCA mean as:

$$E_{FCA}\tilde{Y} = \mu_Y(\mu_X^{-1}(x)), x \in [0, T] \quad (7)$$

In this approach, a distance is also proposed to describe variability of time-warped signals. Its expression for two processes (or warped signals) $\tilde{Y}_1, \tilde{Y}_2 \in W$ is given by:

$$d_{FCA}(\tilde{Y}_1, \tilde{Y}_2) = \left[\int (X_1(t) - X_2(t))^2 dt + \int (Y_1(t) - Y_2(t))^2 dt\right]^{1/2} \quad (8)$$

We can remark from this equation that in the FCA approach shape variability is decomposed into amplitude and warping variability. For N observed processes $(x, \tilde{Y}_i(x))$, a consistent estimator of the FCA mean is given by the following expression [3]:

$$\bar{\bar{Y}} = \hat{\mu}_Y(\hat{\mu}_X^{-1}(x)), x \in [0, T] \quad (9)$$

with $\hat{\mu}_X(t) = \frac{1}{N}\sum_{i=1}^{N} \varphi_{\tilde{Y}_i}^{-1}(t), t \in [0, 1]$

and $\hat{\mu}_Y(t) = \frac{1}{N}\sum_{i=1}^{N} \tilde{Y}_i(\varphi_{\tilde{Y}_i}^{-1}(t)), t \in [0, 1]$

The FCA mean and distance respect also the following condition to be efficient statistical tools for sample signals:

$$\arg\min\left\{\sum_{i=1}^{N}\left[d_{FCA}^2\{\mu, \tilde{Y}_i\}\right]\right\} = \bar{\bar{Y}}, \mu \in W$$

4. ISA METHOD

In the ISA approach [2], [4] N strictly positive signals $s_i \in \mathbb{R}^+$ generated by a stochastic process, normalised by their integral or area, are assimilated to probability densities of time r.v's T_i. So each probability law P_{T_i} is expressed by the density p_i:

$$p_i : t_i \in [0, \mathrm{T}] \mapsto s_i(t_i) = \frac{s^*_i(t_i)}{\int_0^{\mathrm{T}} s^*_i(\tau) d\tau} \in \mathbb{R}^+ \quad (10)$$

where s^*_i are the signals without area normalisation.

The associated distribution function $F_i \in \mathrm{F}$ is defined by:

$$F_i : t_i \in [0,\mathrm{T}] \mapsto S_i(t_i) = \frac{\int_0^{t_i} s^*_i(\tau)ds}{\int_0^{\mathrm{T}} s^*_i(\tau)ds} \in [0,1] \quad (11)$$

Then the mean of the inverse distribution functions is calculated as follows:

$$\tilde{S}^{-1}(y) = \frac{1}{N}\sum_{i=1}^{N} S_i^{-1}(y), y \in [0,1] \quad (12)$$

where $(.)^{-1}$ denotes the inverse operation.

Finally, the expression of the ISA mean in the signal space is given by:

$$\tilde{s} = \left(\left(\tilde{S}^{-1}\right)^{-1}\right)' \quad (13)$$

where $(.)'$ denotes the derivation operation.

From eq.(4), (9) and (12), we can easily remark that for positive signals:

$$\mu_X \equiv \tilde{S}^{-1} \quad \text{and} \quad \varphi_{\tilde{Y}}^{-1} \equiv S^{-1} \quad (14)$$

As for FCA approach, a warping mapping Φ linking the observed sample distribution functions S_i and $\tilde{S}$ can be defined as follow:

$$\Phi : \{t_i, S_i(t_i), t_i \in [0,\mathrm{T}]\} \mapsto \{\overline{t}, \tilde{S}(\overline{t}), \overline{t} \in [0,\mathrm{T}]\} \quad (15)$$

where the stochastic warping functions ψ_i are by definition strictly increasing and respect $\overline{t} = \psi_i(t_i)$ [4]. From the ISA definition we can observe that the method, prior to FCA approach, uses the surface-under-the-curve mapping to link between the observed time warped signals and the ISA mean.

4.1 CISA mean and distance

A similarity criterion with the ISA mean had been already used in shape classification applications [8]. But the obtained results were difficult to interpret in term of quantifying shape variability. Although, the proposed criterion respected the shape equality condition but it was not a distance. In this communication, taking as a starting point the ISA approach, we propose to develop a shape distance as defined in Section 2. First let us assume that:

$$S_i = \tilde{S} \circ \varphi_i, \varphi_i = \psi_i^{-1} \quad (16)$$

we suppose that the time transformation φ_i can be decomposed on two parts, a stochastic affine time transform $A_{\overline{t},i}$ (as defined in Section 2.) that represents possible jitter and scale effects and a stochastic non linear time transform ω_i that represents the shape variation, on a constant time support, of S_i related to $\tilde{S}$. We can write:

$$\varphi_i = A_{\overline{t},i} \circ \omega_i \quad (17)$$

where $A_{\overline{t},i}(\overline{t}) = (\overline{t} - b_i)/a_i$ and a_i and b_i are realisations of the r.v's A and B with E[A]=1 and E[B]=0 respectively. Replacing eq.(17) in (16) we obtain:

$$S_i = \tilde{S} \circ A_{\overline{t},i} \circ \omega_i \text{ or } S_i^{-1} = \omega_i^{-1} \circ A_{\overline{t},i}^{-1} \circ \tilde{S}^{-1} \quad (18)$$

or in another form:

$$S_i^{-1}(y) = \omega_i^{-1}\left(a_i \tilde{S}^{-1}(y) + b_i\right), y \in [0,1] \quad (19)$$

Since our shape distance proposes to measure the inverse shape fluctuation functions ω_i^{-1} independently from the affine time transforms, we propose to cancel the effect of $A_{\overline{t},i}^{-1}$ before calculating our distance by estimating it from the precedent equation. For this purpose, since $A_{\overline{t},i}^{-1}$ is not directly observable we can suppose that:

$$S_i^{-1}(y) = a_i(\tilde{S}^{-1}(y)) + b_i + \varepsilon_i(y), y \in [0,1] \quad (20)$$

where ε_i is a noisy term that depends on the affine time transform $A^{-1}_{\overline{t},i}$, $\tilde{S}^{-1}$ and the non linearity of ω_i^{-1}. The principal advantage of working with eq.(20) is that affine parameters become linear with $\tilde{S}_i^{-1}$. Then, we can easily estimate an affine transform $\hat{A}_{\overline{t},i}$ by linear regression between S_i^{-1} and $\tilde{S}^{-1}$ and calculate for each signal a corrected signal $S_{\hat{A},i}^{-1}$ with the following expression using eq.(18):

$$S_{\hat{A},i}^{-1} = \hat{\omega}_i^{-1} \circ \tilde{S}^{-1} \quad (21)$$

with:

$$\hat{\omega}_i^{-1} = \hat{A}_{\overline{t},i} \circ \omega_i^{-1} \circ A_{\overline{t},i}^{-1}$$

Where $\hat{\omega}_i^{-1}$ is an estimation of ω_i^{-1} obtained following the proposed method. It can be observed that the estimation of $\hat{A}_{\overline{t},i}$ using the linear regression is suboptimal. Indeed, the non linearity represented by ε_i for each signal can introduce an error in the estimation. As a result, $\tilde{S}^{-1}$ may be not the mean of $S_{\hat{A},i}^{-1}$. To take into account this error, we propose to recalculate a new mean from the new samples $S_{\hat{A},i}^{-1}$:

$$\tilde{S}_{\hat{A}}^{-1}(y) = \frac{1}{N}\sum_{i=1}^{N} S_{\hat{A},i}^{-1}(y), y \in [0,1] \quad (22)$$

or in another form:

$$\tilde{S}_{\hat{A}}^{-1}(y) = \frac{1}{N}\sum_{i=1}^{N} \hat{\omega}_i^{-1}\left(S^{-1}(y)\right), y \in [0,1] \quad (23)$$

and in the signal domain:

$$\tilde{s}_{\hat{A}} = \left(\left(\hat{S}_{\hat{A}} \right)^{-1} \right)' \quad (24)$$

We propose to name the obtained mean as Corrected Integral Shape Averaging (CISA) mean. Finally, the following CISA distance in $L^2[0,1]$ between samples s_i and s_j is defined as follows:

$$d_{CISA}(s_i, s_j) = \left[\int \left(S_{\hat{A},i}^{-1}(y) - S_{\hat{A},j}^{-1}(y) \right)^2 dy \right]^{1/2} \quad (25)$$

by replacing $S_{\hat{A},i}^{-1}$ and $S_{\hat{A},j}^{-1}$ by their value, we can rewrite eq.(25) in the following form:

$$d_{CISA}(s_i, s_j) = \left[\int \left(\left(\hat{\omega}_i^{-1} - \hat{\omega}_j^{-1} \right) \left(\tilde{S}^{-1}(y) \right) \right)^2 dy \right]^{1/2} \quad (26)$$

From this last definition, we can observe that this distance is directly related to difference between two estimated inverse shape fluctuation functions $\hat{\omega}_i^{-1}$ and $\hat{\omega}_j^{-1}$ defined from the reference $\tilde{S}^{-1}$. In addition, CISA distance is invariant to an affine time transforms on the signals since it is the same transforms as on their integral [2], [4]. However, we can easily deduce that the CISA distance is invariant to the amplitude transform $A_{x,i}$ (defined in Section.2) since the signals are normalised by their area. We can summarise the CISA mean and distance properties by writing:

i) $d_{CISA}\{s(t), A_x(s(A_t(t)))\} = 0$

ii) $\arg\min \left\{ \sum_i^N d_{CISA}^2 \{\mu, s_i\} \right\} = \tilde{s}_{\hat{A}}, \mu \in \mathbb{R}^+$

From eq.(20), (21) and (22), we can observe that in the case of equal shape signals $(\omega_i^{-1} = Id, \forall i \in [1, N])$, ISA and CISA means are the same.

5. SIMULATION STUDY

5.1 Signals with the same shape

In this simulation, we generate $N = 15$ Gaussian signals with the following equation:

$$x_i(t) = \alpha_i G\left((t - b_i) / a_i \right), i = 1 : N \quad (27)$$

where $G(\sigma = 0.11, m = 0.5)$ and α_i, b_i and a_i are realisations of the Gaussian r.v's $\alpha(\sigma = 0.34, m = 1)$, $b(\sigma = 0.05, m = 0)$ and $a(\sigma = 0.09, m = 1)$ respectively. From this set of signals, CISA and FCA means are computed and a distance histogram (related to the mean of each approach) is proposed. The obtained results are shown on Fig.1. and Fig.2. The simulated signals are linearly warped in time. We can observe that CISA and FCA mean are superimposed. In all simulations, CISA mean amplitude is restored by area rectification [4]. Studying the distance histograms, we can note a large histogram for the FCA approach due to amplitude, scale and jitter variations. For the CISA distance histogram, a very narrower histogram closer to 0 value is observed. This last result is theoretically justified since the signals are the same shape.

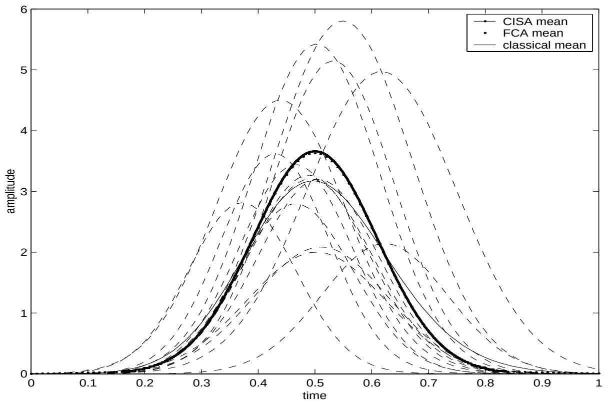

Fig.1. : Signals, FCA, CISA and classical means.

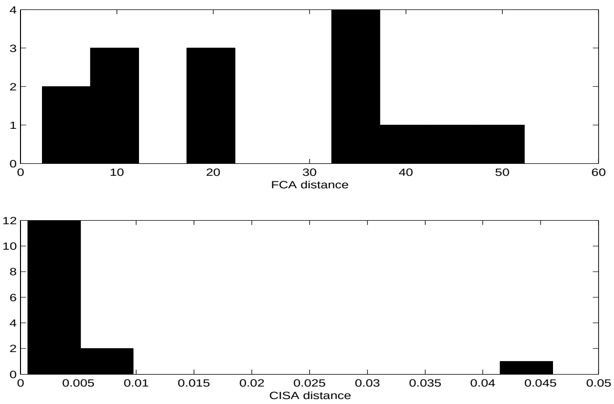

Fig.2. : FCA distance and CISA distance histograms.

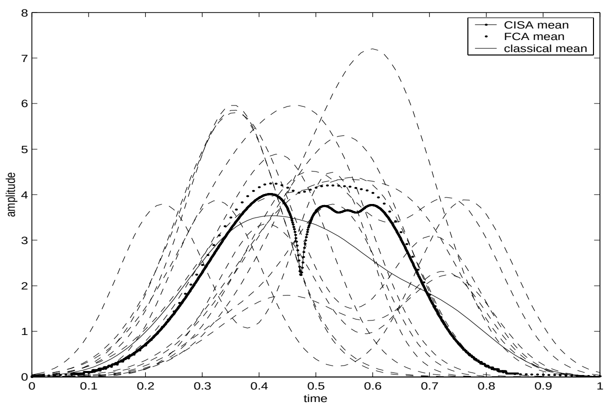

Fig.3. : Signals, FCA, CISA and classical means.

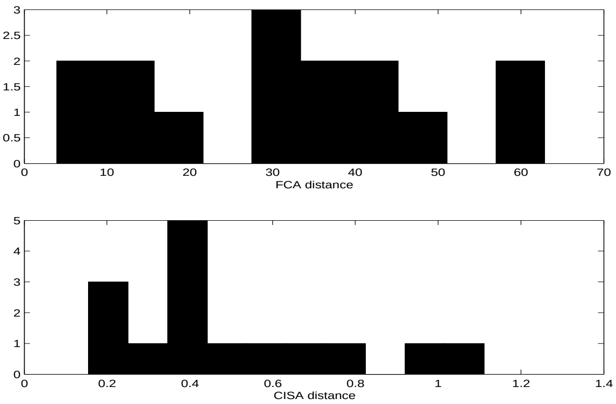

Fig.4. : FCA distance and CISA distance histograms.

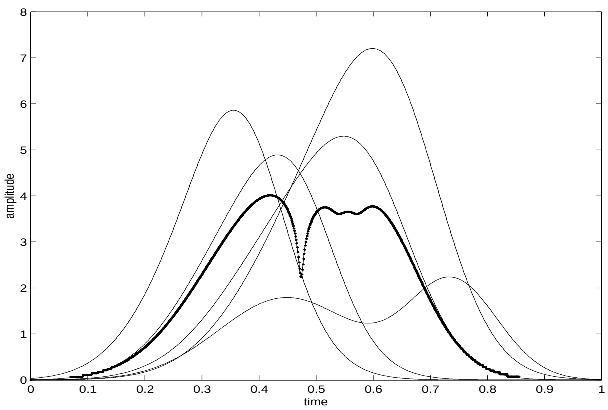

Fig.5. : Selected signals with $d_{CISA} \in [0.35, 0.45]$ and the CISA mean.

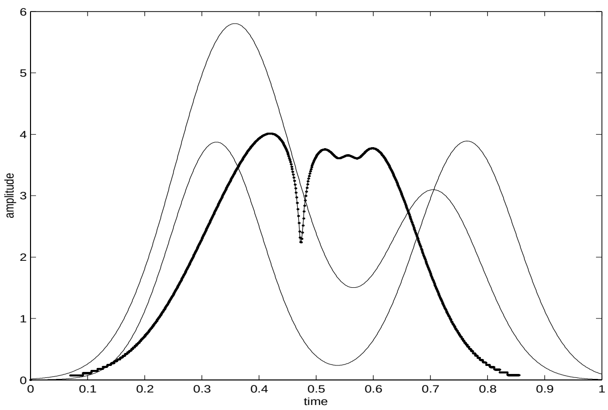

Fig.6. : Selected signals with $d_{CISA} \in [0.9, 1.2]$ and the CISA mean.

5.2 Signals with different shapes

In this part, $N = 15$ signals with different shapes are generated following the model with the equation:

$$x_i(t) = \alpha_{1,i} G_1\left(\left(A_{t,i}(t) - b_{1,i}\right)/a_{1,i}\right) + \alpha_{2,i} k G_2\left(\left(A_{t,i}(t) - b_{2,i}\right)/a_{2,i}\right)$$

$$\text{with } A_{i,t}(t) = \left(t - b_{A,i}\right)/a_{A,i}, i = 1 : N \quad (28)$$

where $G_1(\sigma = 0.11, m = 0.4)$, $G_2(\sigma = 0.08, m = 0.6)$, $k = 0.6$ and $\alpha_{1,i}, \alpha_{2,i}, b_{A,i}, b_{1,i}, b_{2,i}, a_{A,i}, a_{1,i}$ and $a_{2,i}$ are realisations of the following Gaussian r.v's respectively:
$\alpha_1(\sigma = 0.31, m = 1)$, $\alpha_2(\sigma = 0.25, m = 1)$, $b_A(\sigma = 0.04, m = 0)$, $b_1(\sigma = 0.04, m = 0)$, $b_2(\sigma = 0.09, m = 0)$, $a_A(\sigma = 0.04, m = 1)$, $a_1(\sigma = 0.15, m = 1)$, and $a_2(\sigma = 0.07, m = 1)$.
From Fig.3, we can observe that the signals are not well structured and localised in time. In addition, we can note shape similarities and differences between FCA and CISA means. The FCA mean is smoother than the CISA one. However, the CISA mean shape contains, in a condensed form, features present among signals like double bumping, flat peak etc.. On Fig.4, we can observe an FCA distance histogram that is not readily interpretable in term of shape as CISA one. Indeed, the affine transforms effects are mixed with the shape difference in FCA distance calculation. From the figure, we can see that the CISA distance histogram is directly related to the shape variability independently from affine amplitude and time variations defined in the model in eq.(28). In fact, a signal set corresponding to $d_{CISA} \in [0.35, 0.45]$ (the histogram peak) is extracted and shown on Fig.5. In this figure, some of the extracted signals are visually similar and one is different in shape. But all the signals are at the same CISA distance from the CISA mean. Signals far from the mean shape in term of CISA distance ($d_{CISA} \in [0.9, 1.2]$) are shown on Fig. 6. It can be noted that their shape are visually very different from the CISA mean shape. We can remark that the "pronounced two peaks" shape feature is included in the CISA shape (the narrowed valley).

6. DISCUSSIONS AND CONCLUSION

In this work, two recent approaches FCA and ISA providing statistical tools to describe warped signals were studied. Theoretically, both methods consider the warping as a stochastic process and no a priori are made about shape and warping dispersion as for other CR approaches. It is important to keep in mind that the main objective of both methods is to provide us with adequate statistical tools for warped signals or more generally signals with different shapes. It can be noted that both methods gave acceptable results in classical CR applications [3], [4]. For FCA approach, shape variability is the result of a latent bivariate process representing stochastic amplitude and time variability. For this study, we propose a new approach derived from the ISA approach and name it Corrected ISA (CISA). For ISA and CISA approaches, shape variability results only of stochastic time warping fluctuations on the distribution functions of the signals. Both FCA and CISA approaches use the area-under-the-curve mapping but differ in mean and distance computing. We showed by simulation that FCA and CISA means are superimposed if the signals to average are the same shape. They are different in shape if the signals are different in shape too. In addition, we observed that the CISA mean shape contains principal shape features present in the signals.

In term of shape quantification, we showed that despite the fact that FCA statistical tools can describe time and amplitude variability (and implicitly shape variability) among time-warped signals in a more general sense that classical CR methods, they are not suited for quantifying shape variability. This is due to the non respect of the FCA approach to shape equality condition as defined in the SI model. On the other hand, we proposed a distance for CISA approach and proved by theory and simulation that CISA mean and distance are suited for measuring shape variability among time-warped signals since shape equality condition is respected. Indeed, the proposed CISA distance histogram is directly related to the shape variability present among signals. It can be considered as a shape histogram and used for further applications. Finally, an interesting application could be signal shape clustering since CISA method provides us with a real shape class centre and a shape distance.

7. REFERENCES

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