

# MAXIMUM LIKELIHOOD PARAMETER ESTIMATION OF SHORT-TIME MULTICOMPONENT SIGNALS WITH NONLINEAR AM/FM MODULATION

*M. Jabloun, N. Martin, M. Vieira*

LIS/ CNRS/ INPG,  
38402 Saint Martin d'Herès, France.  
meryem.jabloun@lis.inpg.fr

*F. Leonard*

Institut de recherches d'Hydro Quebec,  
Varennes, Canada.  
leonard.francois@ireq.ca

## ABSTRACT

Parameter estimation for closely spaced or crossing frequency trajectories is a difficult signal processing problem, especially in the presence of both nonlinear amplitude and frequency modulations. In this paper, polynomial models are assumed for the instantaneous frequencies and amplitudes (IF/IA). We suggest two different strategies to process multicomponent signals. In the first one, which is optimal, all model parameters are simultaneously estimated using a maximum likelihood procedure (ML), maximized via a stochastic technique called Simulated Annealing (SA). In the second strategy, which is suboptimal, the signal is iteratively reconstructed component by component. At each iteration, the IF and IA of one component are estimated using the ML procedure and the SA technique. To evaluate the accuracy of the proposed strategies, Monte Carlo simulations are presented and compared to the derived Cramer-Rao Bounds for closely spaced and crossing frequency trajectories. The results show the proposed algorithms perform well compared to existing techniques.

## 1. INTRODUCTION

The analysis of multicomponent signals has been investigated in the recent signal processing literature [1, 2, 3, 4, 5, 6, 7, 8]. These signals are encountered in applications such as radar, sonar, mobile communications and other engineering systems. Suboptimal approaches, based on the Higher Ambiguity Function (HAF) or Product HAF (PHAF) [1, 2, 4, 5, 7], provide performances close to the Maximum Likelihood estimators for high signal to noise ratio (SNR). However, the appearance of cross-terms due to the presence of multiple components deteriorates the estimation accuracy. Moreover the HAF and the PHAF must be used with caution at low SNR. In [1, 4, 5, 6], most of the proposed approaches are designed for linear amplitude and frequency modulations (AM/FM).

Here we are concerned with nonlinear AM and FM modulations. This paper is an extension of previous published

works [9, 10, 11], where only single component signals were considered. The instantaneous amplitude and frequency were modeled by nonlinear polynomial functions on contiguous short-time segments. A maximum likelihood procedure allowed the estimation of the model parameters. It was maximized via a Simulated Annealing technique. Encouraged by the estimation accuracy results, we now process multicomponent signals. We propose and compare two methods based on the same approach. In Section 2, we present our signal model. Section 3 describes the two proposed methods. The first one estimates all parameters at the same time, hence all components are reconstructed in an optimal way. The second method, which is suboptimal, operates component by component. As a result the instantaneous frequency and amplitude of one component are evaluated at each iteration. In Section 4, we derive the appropriate Cramer-Rao Bounds (CRB). In Section 5, Monte Carlo simulations show the algorithms perform well. Finally, we draw conclusions in Section 6.

## 2. AM/FM POLYNOMIAL MODELS

Let us consider a multicomponent signal  $s[n]$ , embedded in additive white Gaussian noise  $e[n]$  with zero mean and unknown variance  $\sigma^2$ ,

$$\begin{aligned} s[n] &= \sum_{i=1}^K A_i[n] \exp(j \Phi_i[n]) \\ y[n] &= s[n] + e[n], \text{ for } \frac{-N}{2} \leq n \leq \frac{N}{2}. \end{aligned} \quad (1)$$

$A_i[n]$  and  $\Phi_i[n]$  are the instantaneous amplitude and phase of the  $i^{th}$  component.  $F_i[n] = \frac{1}{2\pi} \frac{\partial \Phi_i[n]}{\partial n}$  is the instantaneous frequency. We assume positive amplitudes and non discontinuous phases.  $y[n]$  is the noisy signal.  $N + 1$  is the sample number assumed to be odd for simplicity.  $K$  is the number of the components.

As in [9, 10], we consider the signal on short-time segments. Actually, this helps us to locally track the frequency and amplitude modulations of highly nonstationary signals. So,  $N$  is about thirty samples. The polynomial model is motivated by Weierstrass' theorem and the shortness of

the segments. Let us consider  $(g_0[n], g_1[n], g_2[n])$ , a second order polynomial base defined on  $[\frac{-N}{2}, \frac{N}{2}]$ . We assume that the order is enough to model the nonstationarity which is actually true due to the shortness of the segment. The IA, IF and phase models of the  $i^{th}$  component are given in the following:

$$\begin{aligned} A_i[n] &= \sum_{m=0}^2 a_{i,m} g_m[n], \\ F_i[n] &= \sum_{m=0}^2 f_{i,m} g_m[n], \\ \Phi_i[n] &= \theta_{i,0} + 2\pi(\sum_{l=-\frac{N}{2}}^n F_i[l] - \sum_{l=-\frac{N}{2}}^0 F_i[l]). \end{aligned} \quad (2)$$

Initial phases are not necessarily set to zero. Consequently, we have to estimate seven parameters for each component: initial phase  $\theta_{i,0}$ , three amplitude parameters  $\{a_{i,0}, a_{i,1}, a_{i,2}\}$  and three frequency parameters  $\{f_{i,0}, f_{i,1}, f_{i,2}\}$ . In other words, we have to estimate  $\theta = \{\theta_1, \dots, \theta_K\}$  where  $\theta_i = \{a_{i,0}, a_{i,1}, a_{i,2}, \theta_{i,0}, f_{i,0}, f_{i,1}, f_{i,2}\}$ . We here use a ML procedure to carry out the parameter estimation. As the noise is Gaussian, the ML is equivalent to the Least Squares. So, the function to be minimized is given as follows:

$$\hat{\theta} = \underset{\theta \in \mathbb{R}^{7K}}{\operatorname{argmin}} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} |y[n] - \tilde{s}[n]|^2, \quad (3)$$

where  $\tilde{s}[n]$  is the signal model evaluated for each  $\theta$  by substituting (2) into (1). Since (3) is a nonlinear equation with  $7K$  unknown parameters, iterative optimization procedures such as quasi-Newton method converge to local minima unless a good initialization and high SNR. Instead, we propose to use stochastic optimization procedure in order to solve (3).

### 3. ALGORITHMS

For monocomponent signals, a comparison between some stochastic techniques in [9] shows the SA techniques is a good compromise in terms of bias and mean square errors (MSE) for the parameter estimates. So, we propose to develop methods including the SA technique in order to achieve multicomponent signal estimations. In this part, we describe two different strategies.

#### 3.1. Optimal Algorithm

The optimal algorithm is a modified version of SA whose main steps are in [10]. We initialize  $\theta$  the set of the  $7K$  parameters using a simple periodogram. Then, we run  $I$  iterations of the three first steps.  $I$  is a given iteration number, which is found experimentally in order to accelerate the convergence.  $I$  ranges between 1500 and 3000.

1. We generate new candidates  $\theta_C$  from a Gaussian probability law, centered on  $\theta$  and with variance  $\delta$ .  $\delta$  is

an agitation value, which avoids converging to local minima.

2. If  $\theta_C$  minimizes the Likelihood function (3), then we set  $\theta = \theta_C$ , otherwise  $\theta$  is not modified.
3. Then, we generate  $u$  from a binomial law with a probability belongs to  $[0, 0.5]$ . If  $u = 1$ , then  $\delta = 0.97x\delta$ . This step linearly reduces the agitation value in a random way in order to increase the convergence rate. We go to step 1 so far as  $I$  iterations are not achieved.
4. We remove the estimated signal from the noisy signal to generate a residue  $\tilde{e}[n]$ . We check if  $\tilde{e}[n]$  is a white process. If so, signal estimation is finished. If not, we restart  $I$  iterations of the estimation steps. In 98% of the cases, the convergence is guaranteed after the first  $I$  iterations and we don't need to restart the estimation steps.

Let us note :  $EQM = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} |y[n] - \tilde{s}[n]|^2$ .

For real parameter values,  $EQM$  is a chi-2 random variable with a degree of freedom equal to  $2(N+1)$ . So, its expectation is  $\sigma^2(N+1)$  and its variance is  $\sigma^4(N+1)$ . So we check the whiteness criterion by verifying that

$$\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} |\tilde{e}[n]|^2 \in D$$

$$D = \left[ \sigma^2(N+1) - \sigma^2\sqrt{N+1}, \sigma^2(N+1) + \sigma^2\sqrt{N+1} \right]$$

#### 3.2. Suboptimal Algorithm

We develop here an iterative algorithm: the estimation of  $A_i[n]$  and  $F_i[n]$  for the  $i^{th}$  component is carried out using the SA technique and equation (3). At each iteration, only one component is reconstructed. In this approach, we do not estimate  $7xK$  parameters simultaneously, so the computational time is reduced. The main steps of the algorithm are as follows:

1. Set  $i = 1$ ,
2. Initialize the parameter values of the  $i^{th}$  component. We determine the amplitude, the frequency and the phase of the FFT peak of the noisy signal  $y[n]$ . Then we set  $\theta_i = \{a_{FFT}, 0, 0, \theta_{FFT}, f_{FFT}, 0, 0\}$ , The other parameters are equal to 0.
3. Apply  $I$  iterations of the steps 1,2 and 3 described in paragraph (3.1) to estimate  $\theta_i$  only.
4. Once the frequency  $F_i[n]$  and the amplitude  $A_i[n]$  of the  $i^{th}$  component are evaluated using (2), we reconstruct the component  $s_i[n] = A_i[n].e^{j\Phi_i[n]}$ . We remove it from the noisy signal to generate a new noisy signal  $y[n]$ .

5. Check if the residue  $y[n]$  is a white process. If so, component estimation is finished. If not, set  $i = i + 1$  and restart step 2 in order to estimate the next component. The white process criterion is the same as in the paragraph (3.1).

The suboptimal algorithm gives an estimation of the component numbers with regard to the SNR level.

#### 4. CRAMER-RAO BOUNDS

According to [3], and with respect to our parameter definition, we derive the Fisher Information Matrix (FIM) for  $\theta$

$$FIM(\theta) = \frac{2}{\sigma^2} Re \left\{ \begin{bmatrix} \mathbf{A}_i^H \mathbf{A}_j & \mathbf{A}_i^H \phi_j \\ \phi_i^H \mathbf{A}_j & \phi_i^H \phi_j \end{bmatrix} \begin{matrix} 1 \leq i \leq K \\ 1 \leq j \leq K \end{matrix} \right\} \quad (4)$$

where

$$\begin{aligned} \mathbf{A}_i &= [g_0(\underline{n}) \bullet e^{j \Phi_i(\underline{n})}, g_1(\underline{n}) \bullet e^{j \Phi_i(\underline{n})}, g_2(\underline{n}) \bullet e^{j \Phi_i(\underline{n})}] \\ \phi_i &= j [\eta_{-1}(\underline{n}) \bullet s_i(\underline{n}), \eta_0(\underline{n}) \bullet s_i(\underline{n}), \eta_1(\underline{n}) \bullet s_i(\underline{n}), \\ &\quad \eta_2(\underline{n}) \bullet s_i(\underline{n})] \\ s_i(\underline{n}) &= A_i(\underline{n}) \bullet e^{(j \Phi_i(\underline{n}))} \end{aligned}$$

$\Phi_i(\underline{n})$  and  $s_i(\underline{n})$  are the phase and the signal time-vectors of the  $i^{th}$  component respectively.

Also, we note  $\underline{n} = [-\frac{N}{2}, -\frac{N}{2} + 1, \dots, \frac{N}{2}]$ ,  $\eta_{-1}[n] = 1$  and  $\eta_i[n] = 2\pi(\sum_{k=-\frac{N}{2}}^n g_i[k] - \sum_{k=-\frac{N}{2}}^0 g_i[k])$  for  $i = 0, 1, 2$  and  $n \in [-\frac{N}{2}, \frac{N}{2}]$ . ( $\bullet$ ) denotes element by element multiplication of vector entries.

The CRB for  $\theta$  is the inverse of the FIM given by (4). In [9], for a single component signal, the estimation of amplitude and frequency parameters is decoupled. Moreover, when we use an orthonormal base, the FIM for the amplitude parameters is diagonal and there where no correlation between estimated parameters. Here the correlation between all the parameters is very high in the presence of multicomponent signals. The FIM (4) is a badly conditioned matrix especially when crossing frequency trajectories occur. More specifically it tends to a singular matrix for closely spaced frequencies. Therefore estimating such cases is a real challenge.

#### 5. RESULTS

In this section, we give some numerical examples demonstrating the performances of the two proposed algorithms. We also evaluate the CRB given in Section 4. All the considered signals are of 33 samples. The sampling frequency is 1 Hz. The SNR is defined as the ratio of the energy of a

constant amplitude signal, whose energy equals that of the time-varying signal, to noise variance. We estimate two-component of quadratic AM/FM signals, which means cubic polynomial phase signals, embedded in Gaussian noise. The experimental plots are based on 50 independent noise realizations. Two cases of quadratic amplitude and frequency modulations are discussed.

- Case I: The Frequency trajectories are separated,
- Case II: The Frequency trajectories are crossing one another.

##### 5.1. IF/IA Reconstruction

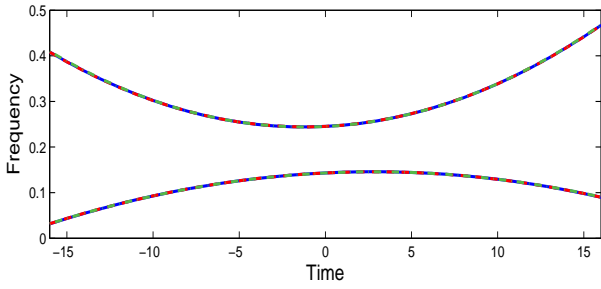
Fig.1(a) and Fig.2(a) show the reconstruction of the frequency and the amplitude in case I, (using the optimal algorithm compared to the suboptimal one) for 20 dB and 10 dB respectively. The estimations of the corresponding signal are reported in Fig. 3. Fig.1(b) and Fig.2 (b) display the IF and IA estimates versus the original ones in case II for SNR equal to 20 dB and 10 dB respectively. All estimated curves are the mean of 50 Monte Carlo simulations.

Comparing the two different proposed procedures, we deduce that the suboptimal method gives the better accuracy when we estimate the most energetic component. However since the estimation of the next components strongly depends on the first one, the performance gradually decreases as the component number increases. This means that the estimation error is more important for the last estimated component than for the first one. Nevertheless this procedure is a good compromise between computation time and estimation accuracy and gives an estimation of the component number conditionally to the SNR level.

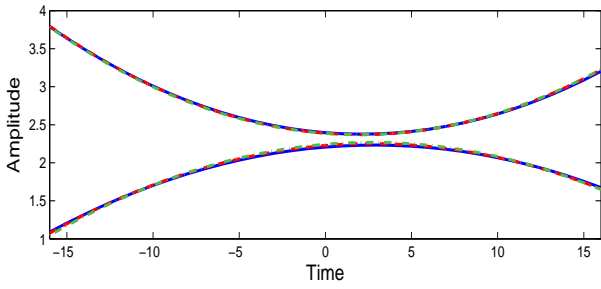
On the other hand, a zoom at the window center (see Fig. 4), shows the curves estimated by the optimal algorithm are the closest to the original ones. Actually, the estimation error is reduced in the window center with this procedure. We note that the local SNR is smaller at the window center than at the window extremities. However the error estimation is spread out on all components in the opposite of the suboptimal method. The estimation of one parameter is conditioned by the estimation of the others.

The MSE of the estimation of the IF and IA are plotted in Fig.5 versus the CRB (derived from [3] with respect to our base) at 20 and 10 dB. We run 50 independent realizations. For the optimal algorithm, the estimation accuracy of the IF and IA waveform is more accurate than the suboptimal for low SNR. The MSE are symmetric too. This is mainly due to the procedure optimality. But since the computational time is increased compared to the suboptimal method, we prefer to use the last one in further optimization problems.

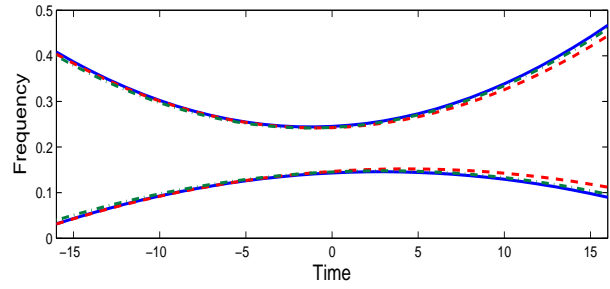
Finally, with regard to the low sample number, the nonlinear FM (ie the cubic phase) and especially the nonlinear



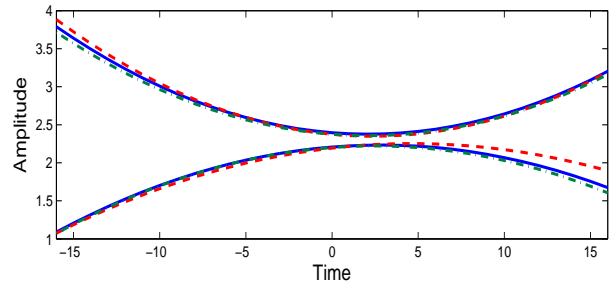
(Figure 1. a)



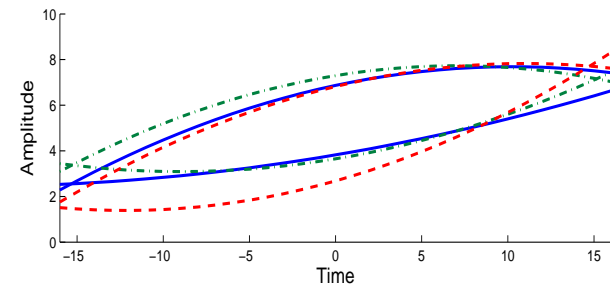
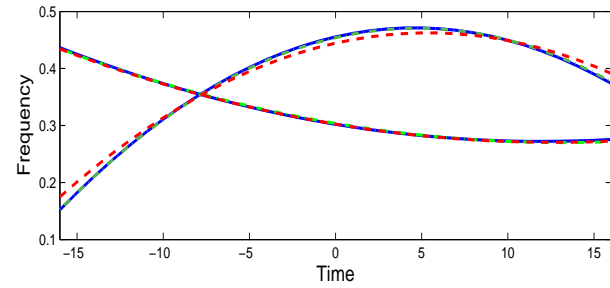
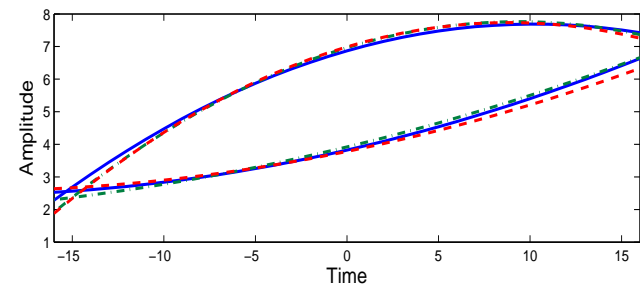
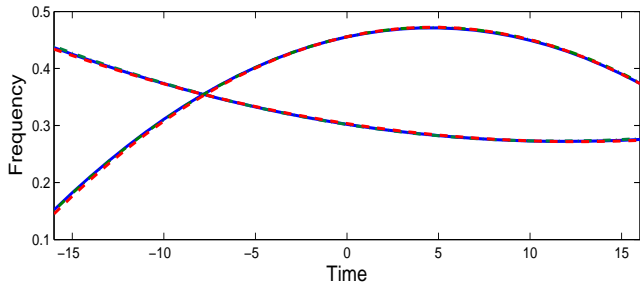
(Figure 1. b)



(Figure 2. a)

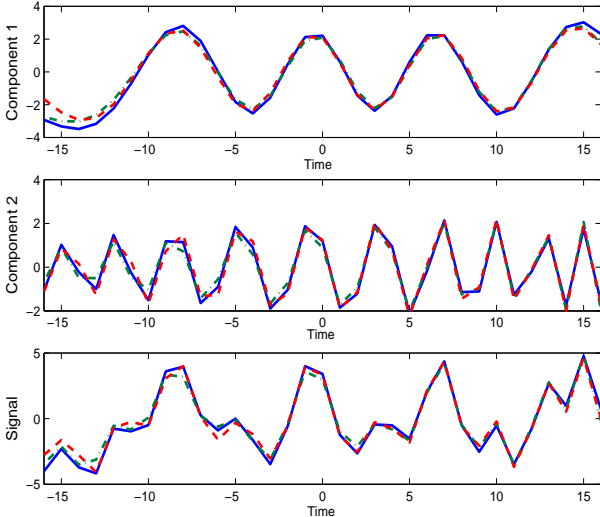


(Figure 2. b)



**Fig. 1.** (a) Example of closely spaced frequency trajectories,(b) Example of crossing frequency trajectories: The reconstructed curves of IA and IF via the optimal algorithm (—) and the sub-optimal one (-) versus the original (-) IA and IF curves for an SNR equal to 20 dB .

**Fig. 2.** (a) Example of closely spaced frequency trajectories,(b) Example of crossing frequency trajectories: The reconstructed curves of IA and IF via the optimal algorithm (—) and the sub-optimal one (-) versus the original (-) IA and IF curves for an SNR equal to 10 dB .



**Fig. 3.** The component Reconstruction in the case I: The signal component estimates via the optimal algorithm (---) and the suboptimal one (· · ·) versus the original (-) curves for an SNR equal to 10 dB .

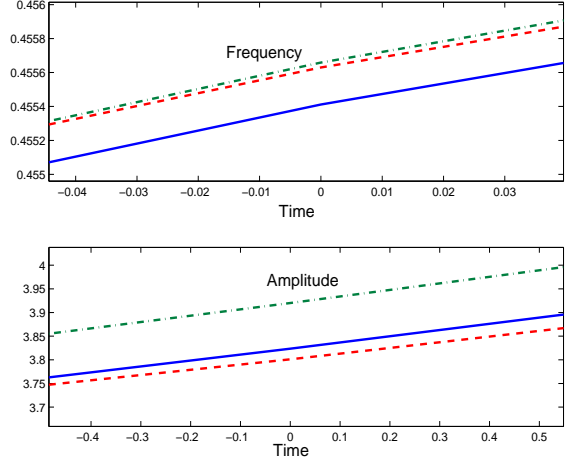
AM, the accuracy of the IF estimation is sufficiently high and estimated curves are close to the original ones. The IA estimation is also acceptable. The two proposed algorithms are able to estimate crossing or close frequency trajectories which was a challenge.

### 5.2. Comparison with the CRB

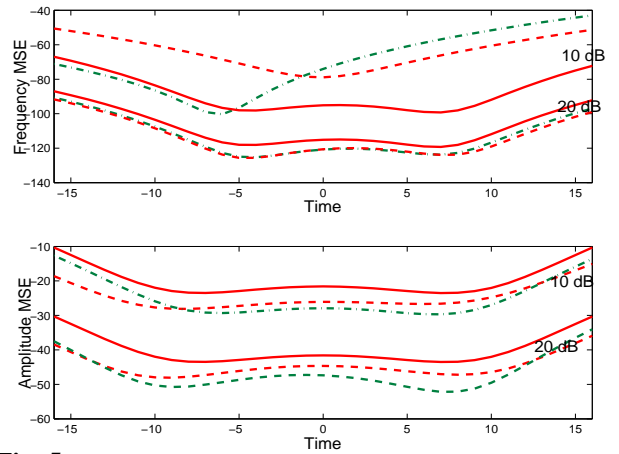
In the following, we consider case I for a statistical parameter study. The performance estimation of the frequency and the amplitude parameters  $\{a_{1,0}, a_{1,2}, a_{2,1}, f_{1,0}, f_{2,0}, \theta_{1,0}\}$  are reported in Fig.6. In Fig.7, the bias of the parameter estimates are plotted. The solid line denotes the CRB, which are given by the main diagonal of the matrix  $FIM^{-1}$ . As the proposed estimators are biased, the direct comparison is not evident. We can see the MSE of the amplitude parameters very close to the corresponding CRB. Similar results are obtained for the other parameters. This highlights the performance of the two proposed methods in a noisy environment. Works are in progress for evaluating the influence of the correlation between estimates.

## 6. CONCLUSION

In this paper, two methods based on the maximum likelihood procedure and Simulated Annealing technique, have been proposed in order to estimate short-time multicomponent signals with nonlinear amplitude and frequency modulation. By considering all component parameters at the same time in the estimation process, the first method allows us to keep optimality. In the second method, the use of the iterative technique as a tool to estimate multicomponent signals loses the optimality but provides a simple way to estimate the number of components with a reduced com-



**Fig. 4.** The MSE of the IF and IA estimates presented for only one component of case I using the optimal algorithm (---) and the suboptimal one (· · ·) versus the appropriate CRB.

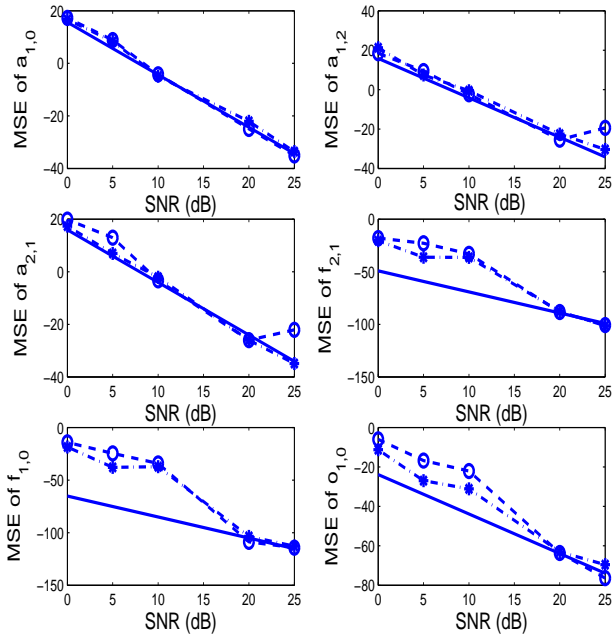


**Fig. 5.** The MSE of the IF and IA estimates presented for only one component of the case I using the optimal algorithm (---) and the suboptimal one (· · ·) versus the appropriate CRB.

putation time. It is shown that closely spaced or crossed frequency trajectories on short time duration are well estimated in the presence of nonlinear amplitude modulation and multicomponent signals. This was a great challenge. In future, we intend to study highly non stationary signals on long duration. We will estimate signals over contiguous short-time segments. Then we will merge all estimated signal parts to reconstruct the entire signal as we did for one component [9, 11].

## 7. REFERENCES

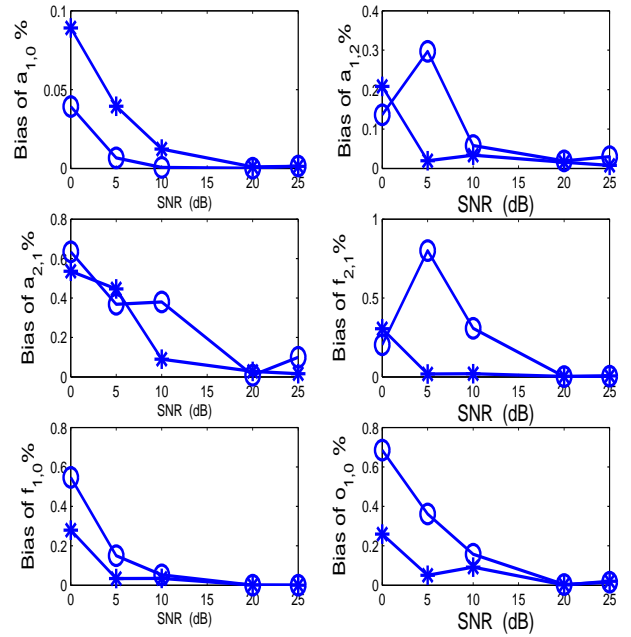
- [1] S. Barbarossa A. Scaglione and G. B Giannakis Nikias, “Product higher order function for multicomponent polynomial phase signal modeling,” *IEEE Transactions On Signal Processing*, vol. 49, pp. 691–708, Mar 1998.
- [2] A. Francos and M. Porat, “Analysis and synthesis of



**Fig. 6.** The MSE of the parameter estimates  $\{a_{1,0}, a_{1,2}, a_{2,1}, f_{1,0}, f_{2,0}, \theta_{1,0}\}$  using the optimal algorithm (-o), the suboptimal algorithm (-\*) versus the SNR(dB) and compared to the CRB (-).

multicomponent signals using positive time-frequency distributions,” *IEEE Transactions On Signal Processing*, vol. 47, pp. 493–504, Feb 1999.

- [3] A. Francos and M. Porat, “Estimation of amplitude and phase parameters of multicomponent signals,” *IEEE Transactions On Signal Processing*, vol. 43, pp. 917–926, Apr 1995.
- [4] M. Z. Ikram and G. T Zhou, “Estimation of multicomponent polynomial signals of mixed orders,” *IEEE Transactions On Signal Processing*, vol. 81, pp. 2293–2308, Nov 2001.
- [5] C. Ioana C. Cornu and A. Quinquis, “Polynomial phase signal processing via warped higher-order ambiguity function,” in *Proc. of EUSIPCO*, Vienna, Austria, Sept 2004, pp. 1159–1162.
- [6] R. M. Liang and K. S. Arun, “Parameter estimation for superimposed chirp signals,” in *Conf. on Acoustics, Speech, and Signal Processing*, San Francisco, California, Mar 1992, pp. 273–276.
- [7] S. Peleg B. Porat and B. Friedlander, “The achievable accuracy in estimating the instantaneous phase and frequency of a constant amplitude,” *IEEE Transactions On Signal Processing*, vol. 71, pp. 2216–2224, Jun 1993.
- [8] S. Saha and S. M. Kay, “Maximum likelihood parameter estimation of superimposed chirps using monte



**Fig. 7.** The bias of the parameter estimates  $\{a_{1,0}, a_{1,2}, a_{2,1}, f_{1,0}, f_{2,0}, \theta_{1,0}\}$  using the optimal algorithm (-o), the suboptimal algorithm (-\*) versus the SNR(dB).

carlo importance sampling,” *IEEE Transactions On Signal Processing*, vol. 50, pp. 224–230, Feb 1995.

- [9] M. Jabloun M. Vieira F. Leonard and N. Martin, “Local orthonormal decomposition for both instantaneous amplitude and frequency of highly nonstationary discrete signals,” in *IMA: The Institute of Mathematics and Its Applications*, Cirencester, United Kingdom, Dec 2004, pp. 107–110.
- [10] M. Vieira F. Leonard M. Jabloun and N. Martin, “Short time single polynomial phase using legendre functions,” in *Proc. of EUSIPCO*, Vienna, Austria, Sept 2004, pp. 793–796.
- [11] M. Jabloun M. Vieira F. Leonard and N. Martin, “A am/fm single component signal reconstruction using a nonsequential time segmentation and polynomial modelling,” in *NSIP: IEEE workshop on Nonlinear Signal and Image Processing*, Sapporo, Japan, May 2005.