

# ADAPTIVE COHERENCE ANALYSIS OF NONSTATIONARY TIME SERIES USING THE ADAPTIVE LOMB SPECTRUM

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## ABSTRACT

This paper proposes an adaptive spectral coherence analysis of nonstationary time series using the adaptive Lomb spectrum. Unlike conventional coherence analysis using the Fourier transform, the proposed algorithm employs Lomb spectrum as the basic spectral analysis tool and the intersection of confidence intervals (ICI) rule for computing adaptively the window sizes to achieve a better time-frequency resolution of the coherence and the associated phase differences. Simulation results show that the proposed method can achieve a better time-frequency resolution than other conventional coherence measurements for nonstationary signals.

## 1. INTRODUCTION

Coherence analysis is a useful spectral analysis technique, which is used to analysis the correlation and the synchronization between different spectral components in two time series. The coherence between two time series can be computed from the Fourier transform or spectral of the two signals [1]. Let  $x(t)$  and  $y(t)$  be two multivariate random processes, then the coherence between  $x$  and  $y$  at frequency  $\omega$  can be calculated from the cross-spectral density  $S_{xy}(\omega)$  and the auto-spectral density  $S_{xx}(\omega)$ ,  $S_{yy}(\omega)$  as follows:

$$C_{xy}(\omega) = \frac{|S_{xy}(\omega)|}{\sqrt{S_{xx}(\omega) \cdot S_{yy}(\omega)}}, \quad (1)$$

where  $S_{xy}(\omega) = \int_{-\infty}^{+\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau$ , (2)

with  $R_{xy}(\tau) = E(x(t)y(t-\tau))$ , (3)

$$S_{xx}(\omega) = \int_{-\infty}^{+\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau, \quad (4)$$

with  $R_{xx}(\tau) = E(x(t)x(t-\tau))$ , (5)

$$S_{yy}(\omega) = \int_{-\infty}^{+\infty} R_{yy}(\tau) e^{-j\omega\tau} d\tau, \quad (6)$$

with  $R_{yy}(\tau) = E(y(t)y(t-\tau))$ . (7)

Based on the Cauchy-Schwarz inequality,  $S_{xx}(\omega)S_{yy}(\omega) \geq |S_{xy}(\omega)|^2$  and the coherence is always bounded between 0 and 1.

The coherence estimation above, which is based on Fourier transforms of the correlation sequences, is only suitable for stationary signals because the Fourier transform has an infinite support. For a nonstationary time series, the spectrum and hence its coherence with another time series will change considerably with time. Therefore, the conventional coherence function based on Fourier analysis cannot achieve the best bias-variance tradeoff and hence time-frequency resolution. To resolve the problem, we need a better time-frequency analysis technique such as the short time Fourier transform (STFT), which performs spectral analysis to the windowed input data centred at different time locations in order to achieve a better time resolution. This yields a time-frequency representation, which gives a spectral representation of a signal as a function of time. The selection of a proper window size is crucial to the time-frequency analysis. A very large window size will reduce the time resolution, while very small window sizes will reduce the frequency resolution. Ideally, the window size should vary at different time locations so that the best time-frequency resolution can be achieved.

In [2], a wavelet coherence function based on continuous wavelet transform (CWT) was proposed to address the time-frequency resolution problem of coherence function. Wavelet transform is a very popular time-frequency analysis tool and it has been studied and applied in many different areas. The basis functions of wavelet transform are obtained from the dilation and translation of a mother wavelet, usually with compact support. Wavelet transform gives a multi-resolution analysis (MRA) of the input signal. Typically, higher frequency resolution is given to low frequency components, while high frequency components are given a higher time resolution. It has been used to measure the synchronization as well as the information transfer between two channels of EEG signals [3]. However, the main drawback of CWT is that it can not vary its time-frequency resolution in the time-frequency plane according to the input signal, i.e. not signal adaptive. For example, in the high frequency band, the frequency resolution will be worse, while the time resolution of low frequency band is not as good as that of high frequency band.

In [4, 5], an adaptive time-frequency representation based on Lomb spectrum [6, 7] was proposed. This adaptive windowed Lomb periodogram (ALP) algorithm employs the intersection of confidence intervals (ICI) rule to determine the window size adaptively, so that it can approach the best possible time-frequency resolution for a particular input signal. Motivated by this novel time-frequency distribution (TFD), we propose in this paper a new coherence analysis algorithm based on ALP TFD. In the proposed method, windowed Lomb spectrums of the two signals are used to calculate a series of coherences using different temporal window sizes. The ICI rule is used to select the proper window for computing the adaptive Lomb coherence (ALC) of the two time series.

The paper is organized as follows: in Section 2, the proposed coherence analysis based on the windowed Lomb spectrum is proposed. The adaptive Lomb coherence and the phase difference estimation are introduced in Section 3. Simulation results and comparisons are described in Section 4. Finally, conclusions are drawn in Section 5.

## 2. COHERENCE BASED ON LOMB SPECTRUM

The Lomb spectrum is a very useful method for spectral analysis of non-uniformly sampled data. The basic idea of the Lomb spectrum is to estimate the amplitude of a given sinusoid with a certain angular frequency  $\omega$  by least squares fitting the sinusoidal to the observed non-uniform data points. The detailed computation can be summarized as follows: given a set of non-uniform discrete-time observations  $x_i = x(t_i); i=1,2,\dots,N$ , the mean  $\bar{x}$  and variance  $\sigma^2$  of the data are first computed. The Lomb spectrum at an angular frequency  $\omega$  is:

$$X(\omega) = \frac{1}{\sqrt{2}\sigma} \left\{ \frac{\sum_i (x_i - \bar{x}_i) \cdot \cos \omega t}{\sqrt{\sum_i \cos^2 \omega t}} - j \cdot \frac{\sum_i (x_i - \bar{x}_i) \cdot \sin \omega t}{\sqrt{\sum_i \sin^2 \omega t}} \right\}, \quad (8)$$

where  $\tau$  is given by  $\tan(2\omega\tau) = \frac{\sum_i \sin 2\omega t_i}{\sum_i \cos 2\omega t_i}$ .

In order to track time-varying signals and produce a time-frequency Lomb spectrum, a window  $w(t)$ , which is centered at time  $t$ , is introduced in [4, 5] so that the Lomb periodogram of the windowed data  $x_{t,w} = x_t w_t$  at a given time instant  $t$  is computed. It gives a windowed Lomb spectrum  $X(t, \omega)$  at different time instants  $t$ . This yields:

$$X(t, \omega) = \frac{1}{\sqrt{2}\sigma_t} \left\{ \frac{[\sum_w (x_{t,w} - \bar{x}_{t,w}) \cdot \cos \omega(t_w - \tau_t)]}{\sqrt{\sum_w \cos^2 \omega(t_w - \tau_t)}} - j \cdot \frac{[\sum_w (x_{t,w} - \bar{x}_{t,w}) \cdot \sin \omega(t_w - \tau_t)]}{\sqrt{\sum_w \sin^2 \omega(t_w - \tau_t)}} \right\}, \quad (9)$$

where  $\bar{x}_{t,w}$  and  $\sigma_t$  are respectively the mean value and the standard deviation of the data in the window  $w_t$ , and  $\tau_t$  is the offset in the window  $w_t$  at time  $t$ .

So, equation (9) allows us to compute the time-frequency distributions  $X(t, \omega)$  and  $Y(t, \omega)$  of the two given nonstationary signals  $x(t)$  and  $y(t)$ , respectively. In order to calculate the coherence, the auto- and cross-spectra  $S_{xx}$ ,  $S_{yy}$  and  $S_{xy}$  need to be estimated from time-limited observations according to the following equations:

$$S_{xx}(\omega) = E[X(\omega) \cdot X^*(\omega)], \quad (10)$$

$$S_{yy}(\omega) = E[Y(\omega) \cdot Y^*(\omega)], \quad (11)$$

$$S_{xy}(\omega) = E[X(\omega) \cdot Y^*(\omega)]. \quad (12)$$

The statistical expectation in (10)-(12) is practically realized by averaging in the time-frequency plane. In other words, smoothing is performed to the instantaneous estimate of various spectra. Here, for simplicity, we only consider smoothing across different time segments, that is, temporal smoothing. Since averaging spectral estimates across time segment requires that the time series to be stationary inside the window, for nonstationary time series, this temporal smoothing window  $w_s$ , which is different from the time window  $w$  in the calculation of windowed Lomb spectrum, should be small enough. This temporal smoothing is performed at every time instant  $t$  to produce  $S_{xx}(t, \omega)$ ,  $S_{yy}(t, \omega)$  and  $S_{xy}(t, \omega)$ , and the coherence can be obtained from:

$$C_{xy}(t, \omega) = \frac{|S_{xy}(t, \omega)|}{\sqrt{S_{xx}(t, \omega) \cdot S_{yy}(t, \omega)}}. \quad (13)$$

We can see from (13) that, if the auto-spectra  $S_{xx}$  and  $S_{yy}$  have very small values in some locations of the time-frequency plane, the coherence so calculated can assume a very large value due to the additive noise and finite size of the data samples. This is undesirable. To achieve a better performance and visualization of the coherence, a masking or thresholding operation on the coherence can be applied:

$$C_{xy}(t, \omega) = \begin{cases} \frac{|S_{xy}(t, \omega)|}{\sqrt{S_{xx}(t, \omega) \cdot S_{yy}(t, \omega)}} & S_{xx}(t, \omega) > \lambda, \\ & \text{and } S_{yy}(t, \omega) > \lambda. \\ 0 & \text{else} \end{cases} \quad (14)$$

where  $\lambda$  is a threshold to zero out those locations of  $S_{xx}$  or  $S_{yy}$  having very small amplitudes.

The selection of the window size is crucial to the time-frequency resolution of the Lomb spectra  $X(t, \omega)$ ,  $Y(t, \omega)$  and the Lomb coherence  $C_{xy}(t, \omega)$ . For a long window, the frequency resolution of the spectrum and coherence will improve, while the time resolution of fast varying components will be reduced. On the contrary, a small window will bring a better time resolution, but a worse frequency resolution. This bias variance tradeoff also exhibits in the Lomb spectrum, which is based on the least-squares fitting method [4, 5]. Motivated by the novel bandwidth selection rule in [8], we propose to use the ICI rule for determining the window length of  $w(t)$  at each time  $t$  in order to achieve a better bias-variance tradeoff and hence time-frequency resolution. The details will be given in the following section.

### 3. ADAPTIVE LOMB COHERENCE AND PHASE DIFFERENCE

Suppose that we are given a set of finite window sizes in ascending order of magnitude:  $H = \{h_k | h_1 < h_2 < \dots < h_K\}$ . For each window size  $h_k$ , we will obtain two Lomb spectra  $X(t, \omega; h_k)$  and  $Y(t, \omega; h_k)$ , and the coherence  $C_{xy}(t, \omega; h_k)$  can be estimated using equations (10)-(13). Consequently, we can get a series of coherence estimates from the given series of window sizes.

As discussed above, the variance,  $Var(\cdot)$ , and the bias,  $bias(\cdot)$ , of the coherence  $C_{xy}(t, \omega; h_k)$  based on Lomb spectrum are functions of the time window  $h_k$ . This also holds for the mean square error ( $MSE$ ), which is given by:

$$MSE(t, \omega; h_k) = Var[C_{xy}(t, \omega; h_k)] + bias^2[C_{xy}(t, \omega; h_k)]. \quad (15)$$

The variance and bias of  $MSE$  are respectively decreasing and increasing functions of the window size  $h_k$ . Therefore, there exists an optimal window such that the two terms is approximately equal so that the  $MSE(t, \omega; h_k)$  can be minimized. The ICI rule will examine a sequence of confidence intervals of the estimates  $C_{xy}(t, \omega; h_k)$  to determine the optimal window lengths. By using an adaptive window for each sinusoid and at each time instant  $t$ , better bias and variance trade-off of the coherence can be achieved. Hence, a better performance in time-frequency resolution is obtained. The optimal window size array of the whole time-frequency domain is  $h_{k^+}(t, \omega)$  and the optimal Lomb periodogram will be  $C_{xy}(t, \omega; h_{k^+})$ . More details of the ICI rule and the adaptive algorithm can be found in [4, 5 and 8].

If we want to study how synchronized frequency components propagate between two time-series, their

phase differences are useful variables. In [2, 3], the wavelet phase difference can be defined as the phase of the cross wavelet spectrum:

$$\phi_{xy}(t, \omega) = \angle S_{xy}(t, \omega). \quad (16)$$

Similarly, we can define the phase difference of the ALC based on the Lomb spectrum:

$$\phi_{xy}(t, \omega) = \angle S_{xy}(t, \omega; h_{k^+}). \quad (17)$$

where  $S_{xy}(t, \omega; h_{k^+})$  is the adaptive cross-spectrum based on the adaptive Lomb spectrum.

## 4. SIMULATIONS

In our simulation, two nonstationary sinusoidal time series  $x(t)$  and  $y(t)$  with duration of 2 seconds were generated as follows:

$$x(t) = \begin{cases} \sin(20\pi t) + w_1(t) & \text{for } 0 \leq t < 0.5s \\ \sin(40\pi t) + w_1(t) & \text{for } 0.5 \leq t < 1.0s, \\ \sin(20\pi t) + w_1(t) & \text{for } 1.0 \leq t < 1.5s \\ \sin(40\pi t) + w_1(t) & \text{for } 1.5 \leq t < 2.0s \end{cases} \quad (18)$$

$$y(t) = \begin{cases} \sin(20\pi t) + w_2(t) & \text{for } 0 \leq t < 0.5s \\ \sin(40\pi t + \frac{\pi}{3}) + w_2(t) & \text{for } 0.5 \leq t < 1.0s \cdot \\ 2\sin(20\pi t + \frac{\pi}{3}) + w_2(t) & \text{for } 1.0 \leq t < 1.5s \\ 2\sin(40\pi t) + w_2(t) & \text{for } 1.5 \leq t < 2.0s \end{cases} \quad (19)$$

where  $w_1(t)$  and  $w_2(t)$  are two independent and identical distributed (i.i.d.) zero mean white Gaussian noise added to the two processes and the SNR is 20 dB. We can see that these two channels are correlated as follows: (1) in phase for the first quarter of time; (2) differ by a phase difference of  $\pi/3$  for the second quarter of time; (3) differ by a phase difference of  $\pi/3$  and a multiplicative factor of 2 for the third quarter of time; (4) differ by a multiplicative factor of 2 for the fourth quarter of time.

The windows lengths used to compute the Lomb spectrum and coherence are 16, 32, 64, and 128. The length of the temporal smoothing window  $w_s$  is 5. The threshold  $\lambda$  in (14) is given as 5% of the maximum value of  $S_{xx}$  and  $S_{yy}$ . The wavelet coherence used for comparison is computed from CWT using the complex Morlet wavelet basis. In Fig. 2, the coherence based on Lomb spectrum with a small window (fig.(a)) has a high time resolution but a bad frequency resolution, while the coherence with a large window (fig.(b)) has the opposite property. The coherence using adaptive window (fig.(c)) can achieve good time and frequency resolutions at the same time. Although the coherence based on the wavelet transform (fig.(d)) has a higher frequency resolution than that in fig.(a) and a higher time frequency resolution than that in fig.(b), its time-frequency resolution is much worse than that of the proposed adaptive Lomb coherence. The time-varying coherent structure of the two test signals is clearly identified in Fig. 2 (c) for the four segments: first

quarter at 10Hz, second quarter at 20Hz, third quarter at 10Hz, and fourth quarter at 20Hz. Fig. 3 shows the phase differences between the two channels at 10 and 20Hz along the time axis. We can see that the phase difference based on Lomb spectrum is smoother than that using the wavelet transform.

From equations (18) and (19), we know that these two processes should synchronize in four different periods at 10Hz and 20 Hz and other areas in the time-frequency domain should be zero. These phase differences and coherence between the two nonstationary signals are quantified effectively by the proposed method and it has a better localization in the time-frequency plane than that from wavelet transform.

### 5. CONCLUSION

This paper proposed an adaptive coherence analysis algorithm called adaptive Lomb coherence, which is based on the Lomb spectrum. The ICI rule is employed to adaptively select the window length to achieve a better time-frequency resolution for computing the coherence and phase difference from the adaptive Lomb spectrum. Our simulation results show that a better time-frequency resolution over the wavelet transform based coherence can be obtained. The method can be applied to wide range of nonstationary signals such as EEG signals, and the details will be left for future work.

### 6. REFERENCE

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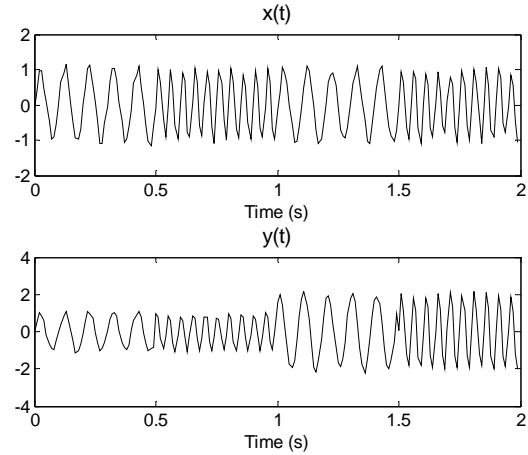


Fig. 1: simulated signals which are mutually coherent at different frequencies within four consecutive periods.

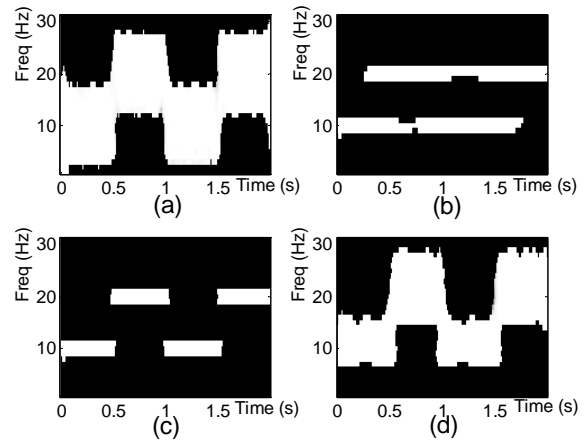


Fig. 2: Coherence based on: (a) Lomb spectrum with window length 16, (b) Lomb spectrum with window length 128, (c) Lomb spectrum with adaptive window lengths, (d) continuous wavelet transform.

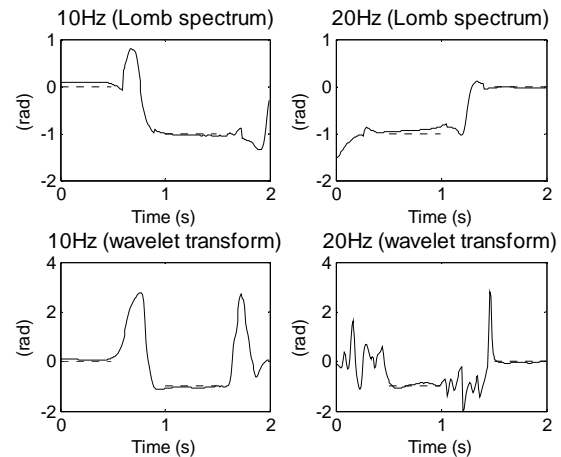


Fig. 3: Wavelet phase difference between the two simulated signals at 10Hz and 20Hz.