LOCALIZATION OF WIDEBAND SOURCES IN COLORED NOISE VIA GENERALIZED LEAST SQUARES (GLS)*

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ABSTRACT

In this paper we present a new method for localization of wideband sources in the presence of colored noise. The optimization criterion used is an extension of conventional least-squares method known as generalized least squares (GLS). The crux of the criterion is based on covariance matching and involves its transformation that yields a more efficient and consistent estimate of the direction of arrivals (DOAs) as compared to the LS and GLS methods that are only developed in the literature for narrowband sources.

1. INTRODUCTION

The problem of estimating the directions of multiple plane waves using an array of sensors has received considerable attention in the recent signal processing literature. A review of the most commonly used estimation techniques can be found in [3]. In sensor array applications, such as passive acoustic, passive sonar, and spread spectrum communications, there has been a growing interest in the analysis of wideband sources and data. One of the most important problems in wideband array processing is that of localization of sources whose data is received by an array of sensors. A common approach to wideband array processing is based on sampling the signals in the frequency domain that are measured at the sensors outputs. Each frequency component is then individually considered as corresponding to a narrowband signal. Incoherent signal-subspace based methods are concerned with processing the signal frequency components to estimate the Direction-of-Arrival (DOA) angles and then combining the results to localize the sources [4].

Majority of techniques developed in the array processing literature for performing direction of arrival (DOA) estimation make use of the assumption that the measurement noise is spatially white, or the covariance matrix of the sensor noise is known up to a single multiplicative factor. However, only a few techniques have been developed that address the DOA estimation problem in presence of colored noise, and these methods are working only for *narrowband* sources. In [5], an MDL-based criterion has been proposed to pose with this problem. In [6] and [9], a covariance-difference approach and an MAP-based approach, under some particular assumptions for each one, are considered to address the problem, respectively. The problem is specially solved for an ARMA-type noise of array in [10]. In [11], an approach based on modeling the noise covariance matrix by a parametric model and then fitting the the best modeled covariance to the sample covariance is employed. Most of the before-mentioned techniques suffer from a major drawback, which is an inconsistent estimate.

In this paper, motivated by the results in [1, 7], we present a reliable methodology to localize wideband sources in presence of colored noise based on what, in the statistical signal processing and linear algebra literature, is known as the generalized least squares (GLS) criterion. The obtained estimates turn out to be consistent and asymptotically efficient.

The paper is organized as follows. In Section 2 we will formulate and state the problem of interest and give a short description on GLS criterion. In Section 3 we will present the main result of the paper, which is mainly an optimization problem to be solved in order for localization of wideband sources of interest in an array of sensors in the presence of colored noise. In Section 4 we present simulation results to demonstrate the performance of the proposed methodology, and finally Section 5 concludes the paper.

The notation used throughout the paper is fairly standard. The Hermitian of a matrix A is denoted by $A^H = (A^*)^T$ and represents its conjugate transpose. $diag(\ldots)$ denotes a block diagonal matrix. $tr(A)$ stands for trace of matrix A. $vec(A)$ of an $m \times n$ matrix is defined by forming a new $mn \times 1$ vector by stacking the columns of A in the current order. Kronecker product, Frobenius norm of a ma-

* This work was supported in part by Natural Sciences and Engineering Research Council of Canada (NSERC) under Discovery Grant REPIN-42515.

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trix, and mathematical expectation are also denoted by $\otimes$, $\|.\|_F$, and $E(.)$, respectively.

2. PROBLEM FORMULATION AND GLS CRITERION

Assume that a single array with p sensors is impinged by q wideband sources. In addition, suppose that both $s_i(t)$ (source signal) and $v_i(t)$ (array noise) are mutually uncorrelated, second order stationary, zero-mean and wideband random processes with identical bandwidths BW. At each snapshot, performing a DFT on the time series, and sensor by sensor, results in the $p \times 1$ vectors of the transformed array output

$$\{X(f_j)\}_{j=1}^{J} = DFT\{x(nT_s)\}_{n=1}^{J} \tag{1}$$

In the frequency domain, the general model of an array for each frequency f_j can be written as

$$X(f_j) = A(f_j, \theta)S(f_j) + V(f_j) \tag{2}$$

where $j = 1, 2, \ldots, J$ with J denoting the number of frequency bins.

Consider the $pJ \times 1$ "generalized vector" $\bar{X}$ formed by concatenating the $X(f_j)$'s as

$$\bar{X} = [X^*(f_1), \ldots, X^*(f_J)]^* \tag{3}$$

and the resulting covariance matrix of the array given by

$$R = E[\bar{X}\bar{X}^*] \tag{4}$$

Since $X(f_i)$ and $X(f_j)$, for $i \neq j$, are uncorrelated, R will have a block diagonal form

$$R(\theta) = diag\left(P(f_1), \quad \ldots \quad, P(f_J)\right) \tag{5}$$

where $P(f_j) = A(f_j, \theta)R_s(f_j)A^H(f_j, \theta) + R_v(f_j)$ is the covariance matrix of the array output at the jth frequency. Assuming that corresponding to each frequency point f_j there are N statistically independent samples of data, we define

$$\begin{aligned} X(f_j) &= [X_1(f_j), \ldots, X_N(f_J)] \\ \hat{P}(f_j) &= \frac{1}{N} X(f_j)X^*(f_j) \end{aligned} \tag{6}$$

2.1. The GLS Estimator

The basic idea for the approach in [1] associated to a single frequency is to select the DOA estimates $\hat{\theta}$ such that the "best fit" between the sample-covariance $\hat{R}$ and the array data covariance is obtained. A reasonable goodness-of-fit criterion that may be chosen is the sum of squares of the entries of the difference matrix $R - \hat{R}$. A least square (LS) criterion is then given by the following cost function

$$CF_{LS}(\theta) = \frac{N}{2}\|\Delta\|_F^2; \quad \Delta = R - \hat{R}$$

However, the drawback with the LS is that the resulting estimated DOAs are not asymptotically efficient. A better criterion that can be used to remedy this problem is by taking the summation over squares of transformed representation of the above difference, that is, $\tilde{\Delta} = \hat{R}^{-1/2}\Delta\hat{R}^{-1/2}$, where $\hat{R}$ is assumed to be full rank. The resulting estimates can be shown to be asymptotically efficient. Therefore, the modified cost function may be expressed as

$$\begin{aligned} CF_{GLS}(\theta) &= \frac{N}{2}\|\hat{R}^{-1/2}\Delta\hat{R}^{-1/2}\|_F^2 \\ &= \frac{N}{2}\|I - \hat{R}^{-1/2}R\hat{R}^{-1/2}\|_F^2 \end{aligned} \tag{7}$$

The estimator obtained by minimizing the cost function (7) is referred to as the generalized least squares (GLS) estimator, namely

$$\hat{\theta}_{GLS} = \arg\min_{\theta}[CF_{GLS}(\theta)] \tag{8}$$

2.2. Modification of the GLS criterion for Wideband Case

Let us now define the respective cost function for the wideband case. Based on the above formulation, the proposed error criterion is given by

$$L(\theta) = \|R(\theta) - \hat{R}(\theta)\|_F^2 = K + \sum_{j=1}^{J}\|P(f_j) - \hat{P}(f_j)\|_F^2 \tag{9}$$

where $R(\theta)$ is given by (5), and K is a constant value, independent of θ, that can be obtained from the non-block diagonal elements of $\hat{R}$.

The above cost function can be modified as a GLS cost function alternatively,

$$L(\theta) = \|I - \hat{R}^{-1/2}R\hat{R}^{-1/2}\|_F^2 \tag{10}$$

If we define the square-root of the data covariance matrix of the array as

$$\hat{R}^{-1/2} = \begin{bmatrix} \hat{R}_{11} & & \hat{R}_{1J} \\ & \ddots & \\ \hat{R}_{J1} & & \hat{R}_{JJ} \end{bmatrix}$$

the cost function in (10) may be rewritten as follows

$$L(\theta) = \sum_{j=1}^{J}\|I - \hat{R}_{jj}P(f_j)\hat{R}_{jj}\|_F^2 \tag{11}$$

Consequently, only $\hat{R}_{jj}$'s are the relevant blocks used in our formulation.

3. LOCALIZATION OF WIDEBAND SOURCES BASED ON GLS CRITERION

The main attributes of the proposed DOA estimates algorithm are stated below, which is expressed as solving an optimization problem.

Theorem 1 *The DOA estimates of the wideband sources are obtained by solving the following nonlinear optimization problem:*

$$\hat{\theta} = \arg\max_{\theta} \|\eta_0 \eta_0^{\dagger} \bar{I}\|_F^2 \tag{12}$$

where

$$\begin{aligned} \bar{I} &= vec[I_p, \ldots, I_p] \\ \eta_0 &= [\eta^*(f_1), \ldots, \eta^*(f_J)]^* \\ \eta(f_i) &= [\hat{A}(f_i, \theta), \hat{\phi}(f_i)] \\ \hat{A}(f_i, \theta) &= \tilde{A}^*(f_i, \theta) \otimes \tilde{A}(f_i, \theta) \\ \hat{\phi}(f_i) &= \phi^T(f_i) \otimes I_p \\ \tilde{A}(f_i, \theta) &= \hat{R}_{ii} A(f_i, \theta) \end{aligned} \tag{13}$$

and also $\phi(f_i)$'s are known matrices in the factorization $\hat{R}_{ii} R_v(f_i) \hat{R}_{ii} = \Gamma \phi(f_i)$. Note that Γ is unknown and the same for all frequency bins and can be estimated from the optimization problem above.

Proof: Considering single terms of cost function (11) individually results in

$$L_j(\theta) = \|I - \hat{R}_{jj} P(f_j) \hat{R}_{jj}\|_F^2 \tag{14}$$

Substituting $P(f_j)$, as defined before, into (14) and assuming $\hat{R}_{jj} R_v(f_j) \hat{R}_{jj} = \tilde{R}_v(f_j)$, $\hat{R}_{jj} A(f_j, \theta) = \tilde{A}(f_j, \theta)$, and $\tilde{R}_v(f_j) = \Gamma \phi(f_j)$, where Γ is an unknown weight matrix that is the same for all frequency bins and $\phi(f_j)$ is assumed to be known for various frequencies, gives us

$$L_j(\theta) = \|I - \tilde{A}(f_j, \theta) R_s(f_j) \tilde{A}^H(f_j, \theta) - \Gamma \phi(f_j)\|_F^2 \tag{15}$$

Noting that

$$\begin{aligned} \|X\|_F^2 &= \|vec(X)\|_F^2 \\ vec(X + Y) &= vec(X) + vec(Y) \\ vec(ABC) &= (C^T \otimes A) vec(B) \end{aligned} \tag{16}$$

for matrices of appropriate dimensions [8], (15) can be rewritten as

$$L_j(\theta) = \|vec(I) - \begin{bmatrix} \hat{A}(f_j, \theta) & \hat{\phi}(f_j) \end{bmatrix} \begin{bmatrix} vec(R_s(f_j)) \\ vec(\Gamma) \end{bmatrix} \|_F^2 \tag{17}$$

where

$$\begin{aligned} \hat{A}(f_j, \theta) &= \tilde{A}^*(f_j, \theta) \otimes \tilde{A}(f_j, \theta) \\ \hat{\phi}(f_j) &= \phi(f_j)^T \otimes I \end{aligned}$$

The last identity in (16) has been used in order to separate variables in the nonlinear cost function and converting it to a linear optimization problem. Now, assume that

$$\zeta = \begin{bmatrix} vec(R_s(f_j)) \\ vec(\Gamma) \end{bmatrix}$$

and that $R_s(f_j)$'s are the same for all frequency components. In this case, ζ will represent a unique variable vector to be estimated from the obtained optimization problem. Now, we have

$$\begin{aligned} L(\theta) &= \sum_{j=1}^{J} \|vec(I) - \eta(f_j) \zeta\|_F^2 \\ &= \|\bar{I} - \eta_0 \zeta\|_F^2 \end{aligned} \tag{18}$$

where

$$\begin{aligned} \bar{I} &= vec(I) \\ \eta(f_j) &= \left[\hat{A}(f_j, \theta), \hat{\phi}(f_j)\right] \\ \eta_0 &= [\eta^*(f_1), \ldots, \eta^*(f_J)]^* \end{aligned}$$

Minimizer to the problem of optimizing $L(\theta)$ is now given by $\hat{\zeta} = \eta_0^{\dagger} \bar{I}$ where $\eta_0^{\dagger} = (\eta_0^H \eta_0)^{-1} \eta_0^H$. substituting the obtained minimizer back into the cost function (18) gives the maximization problem (12) where available variables are given by (13), and this proves the theorem.

Remark 1 *The only restrictive point in our formulation above is that we have assumed that signal spectral density matrices $R_s(f_j)$ are the same for all frequency bins.*

Remark 2 *Our proposed methodology is even capable of estimating the noise spectral density matrix at each frequency bin. The reason behind this is that $R_v(f_j)$'s can be determined from $\hat{R}_{jj} R_v(f_j) \hat{R}_{jj} = \Gamma \phi(f_j)$, which is a result of our assumptions in Theorem 1., since an estimate of Γ is obtained from the minimizer of (18) and rest of the matrices are known a priori.*

4. SIMULATION RESULTS

In this section, we will give illustrative examples in order to demonstrate the theoretical results as well as the relative capabilities of the proposed scheme as compared to Cramer-Rao bound (CRB).

The maximization problem (12) is computationally intensive over q-dimensional space. To optimize the modified GLS-based cost function, we use the Alternating Projection (AP) method of [2]. The method reduces (12) to a few one-dimensional search problems independent of the number of sources. An important point we come up with in our simulations is that care should be taken to choose the initial

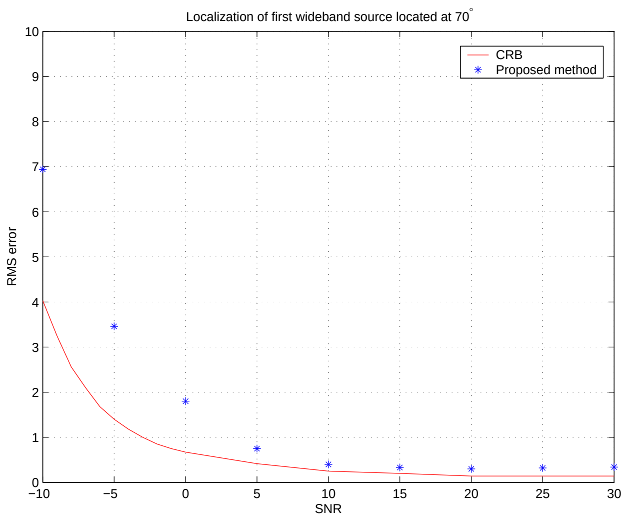

Fig. 1. RMS error of DOA estimate of first source versus SNR in the presence of colored noise.

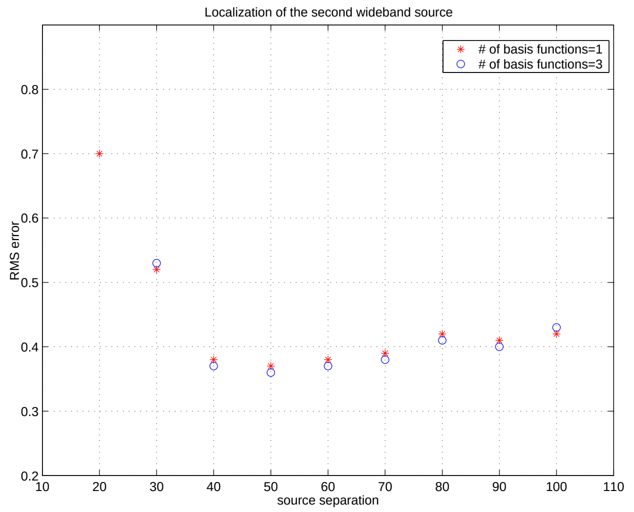

Fig. 2. RMS error of estimate of second source for different values of the number of basis functions.

condition for the AP method in the above optimization procedure.

As the example to verify qualifications of the proposed methodology to localize the wideband sources in the presence of colored noise, consider a uniform linear array of eight equispaced sensors impinged by two wideband sources located at $70°$ and $105°$. The spacing between the two adjacent sensors is assumed to be half a wavelength at center frequency of the wideband sources. The array noise is assumed to be colored generated by an AR model. In our experiments, 50 snapshots are used and for each snapshot 17 frequency components of the array output are taken into account. We performed 50-trial Monte-Carlo simulations in order to compare the root-mean-square (RMS) error of the localized sources as a function of SNR with Cramer-Rao (CR) bound. Figure 1 shows the RMS error for localization of the first wideband source versus different values of SNR. As can be observed, the localization error obtained by making use of the proposed method is not considerable when compared to CRB.

In the first simulation, we considered the number of basis functions (number of rows of matrices $\phi(f_i)$) to be 1. Now assume that position of first source is fixed and δ is the distance between two sources (in degree), and that the number of basis functions is 3. We might expect to obtain results not better than first case since we are trying to estimate more parameters, as appeared in Γ, from the available data which has been gathered of the signals reaching from two sources. But simulation results show that this is only the case for sources separation less than $30°$ and for separation size over $30°$, higher number of basis functions does not lead to degradation of DOA estimates. Figure 2 illustrates the RMS error for localization of the second source in two cases number of basis functions 1 and 3 versus δ. For the part of simulation, we assumed $SNR = 20\ db$ for both sources. Note that for the case of $\delta = 20°$, considering more number of basis functions than the number of sources leads to localization failure.

5. CONCLUSION

In this paper, we presented the required framework for developing wideband DOA estimation in the presence of colored sensor noise by utilizing an asymptotically-efficient method of estimation called generalized least squares (GLS). A few number of techniques have been reported in the narrowband array processing literature in order to deal with spatially colored measurement noise, most of which end up with non-asymptotically efficient estimates of direction of arrival. The present work, however, takes advantage of the benefits of GLS to give consistent and efficient DOA estimates of wideband sources. Simulation results based on generated synthetic data were presented to demonstrate the advantages and capabilities of our proposed method subject to existence of colored noise in the array model and/or low SNR values of source signals.

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