GEOMETRIC ANALYSIS OF FILTERED-X LMS ALGORITHMS

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ABSTRACT

In this paper, we present a geometric based analysis of the "filtered" LMS algorithms such as the filtered-x LMS algorithm and the adjoint LMS algorithm. Using this method, the maximum step size, the effects of the secondary path estimation error and the relationship between the adjoint, secondary path equalization and FXLMS LMS algorithms can be easily obtained. Furthermore, our introduced method yields these conditions with some geometrical meaning and intuitive explanation. Based on our introduced method, we also discuss the convergence speed of the filtered-x LMS algorithm for the first time.

1. INTRODUCTION

The "filtered" LMS algorithm is defined as an adaptive filtering system where the input or the output of the adaptive filter or the feed-back error signal are linearly filtered prior to applying the LMS method to update the adaptive filter coefficients [1]. Some good examples of "filtered" LMS algorithms are the filtered-x LMS algorithm [2], the adjoint LMS algorithm [3], and the secondary path equalization algorithm [2]. These algorithms are widely applied in active noise control (ANC) systems. Our analysis in this paper is focused on the filtered-x LMS algorithm, but it can be directly extended to all "filtered" LMS algorithms.

The filtered-x LMS algorithm could be applied to both feed-forward ANC and feed-back ANC (see Figure 1 from reference [2]). $P(z)$, $S(z)$ and $\hat{S}(z)$ represent the main path, secondary path and the estimated secondary path, respectively; $W(z)$ is the adaptive filter; $x(n)$ is the reference signal, and $v(n)$ is an additive zero-mean noise that is uncorrelated with $x(n)$. Using the reference signal vector $X(n)$ as $[x(n)\ x(n-1)\ \ldots\ x(n-M)]$ where M is the order of the adaptive filter $W(n)$, then the adaptive filter coefficients are updated by

$$W(n) = W(n-1) + \mu e(n) X_f^*(n) \qquad (1)$$

where $X_f(n)$ is the reference signal vector $X(n)$ filtered by the estimated secondary path $\hat{S}(z)$. The positive, real number μ is the step size that controls the convergence speed and stability. To date, there are many results (see [4],[5],[6] for example) available concerning the convergence analysis of the filtered-x LMS algorithm. However, our introduced method examines these conditions from a geometrical viewpoint that yields some elegant convergence results for the filtered-x LMS algorithm.

We first introduce the basic concepts of the geometric analysis method, and based on this method, we derive the step size upper bound for the FXLMS algorithm in Section 2. In Section 3, using our proposed method, we analyze the effects of the secondary path estimation error on the stability of the FXLMS algorithm. Section 4 provides the relationship between the different filtered-error LMS algorithms. Section 5 provides the convergence rate analysis of the FXLMS algorithm based on the geometric analysis method. Simulation results are provided in Section 6, and we draw our conclusions in Section 7.

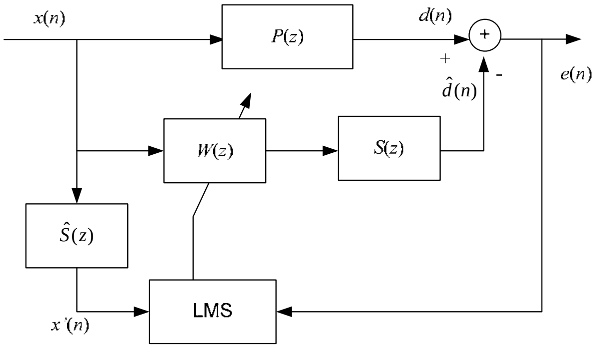

Figure 1. Block diagram of filtered-x LMS

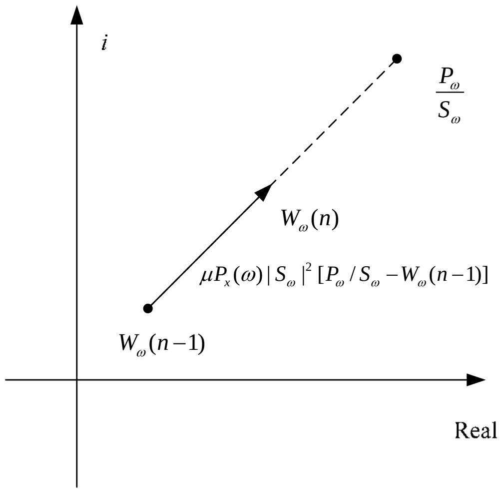

Figure 2. The complex plane expression of Eq. (3)

2. STEP SIZE UPPER BOUND CONDITION FOR FILTERED-X LMS

If we assume the input is a pure sinusoid with a frequency ω, then each of the filters $P(z)$, $W(z)$, $S(z)$, and $\hat{S}(z)$ at this frequency can be represented by the complex numbers P_ω, $W_\omega(n)$, S_ω and $\hat{S}_\omega$, respectively, which represent gains and phase differences. So for a single frequency input, Eq. (1) is

$$\begin{aligned} W_\omega(n) &= W_\omega(n-1) + \mu x_\omega^*(n)\hat{S}_\omega^*[x_\omega(n)P_\omega - x_\omega(n)W_\omega(n-1)S_\omega] \\ &= W_\omega(n-1) + \mu x_\omega^*(n)x_\omega(n)\hat{S}_\omega^* S_\omega[P_\omega/S_\omega - W_\omega(n-1)] \\ &\approx W_\omega(n-1) + \mu P_x(\omega)\hat{S}_\omega S_\omega^*[P_\omega/S_\omega - W_\omega(n-1)] \end{aligned} \tag{2}$$

$P_x(\omega)$ represents the power spectral amplitude of the reference signal at frequency ω. Note that here we do not include the additive noise *v(n)* due to its zero mean and the fact that it is uncorrelated with the reference signal *x(n)*. When the adaptive filter converges, $W_\omega(n) = W_\omega(n-1)$. Consequently we have $W_\omega(\infty) = P_\omega / S_\omega$. If the estimated secondary path $\hat{S}(z)$ has no error, i.e. $\hat{S}(z) = S(z)$, Eq. (2) can be written as

$$W_\omega(n) \approx W_\omega(n-1) + \mu P_x(\omega)|S_\omega|^2[P_\omega/S_\omega - W_\omega(n-1)] \tag{3}$$

We can plot $W_\omega(n)$, $W_\omega(n-1)$ and P_ω/S_ω in the complex plane. The physical meaning of Eq. (3) is that $W_\omega(n)$ goes in the point-to-point direction from $W_\omega(n-1)$ toward P_ω/S_ω with a length of $\mu P_x(\omega)|S_\omega|^2\left|P_\omega/S_\omega - W_\omega(n-1)\right|$ as shown in Figure 2. Based on the complex plane plot, we find that in this case that we need

$$\mu P_x(\omega)|S_\omega|^2 < 2 \tag{4}$$

to ensure the convergence of the adaptive filter. Though this analysis is based on single frequency inputs, the result can be extended to broad-band input signals due to the orthogonality of different frequency signals. However, the step size μ should take the smallest value for the entire frequency range, so we find that

$$\mu < \frac{2}{\max\left(P_x(\omega)\left|S_\omega\right|^2\right)}. \tag{5}$$

This result is the same as the one developed in [1].

3. THE SECONDARY PATH ESTIMATION ERROR

However, in practice, there is always some estimation error. At the frequency ω, the estimated secondary path $\hat{S}(z)$ can be expressed as:

$$\hat{S}_\omega = c_\omega S_\omega e^{j\theta_\omega} \tag{6}$$

where c_ω is a real constant representing the amplitude estimation error, and θ_ω represents the phase estimation error. Combining Eqs. (6) and (2), we have

$$W_\omega(n) \approx W_\omega(n-1) + \mu P_x(\omega) \,|\, S_\omega \,|^2\, c_\omega [P_\omega / S_\omega - W_\omega(n-1)] e^{-j\theta_\omega} \quad (7)$$

Consequently, we see that $W_\omega(n)$ doesn't go in the point-to-point direction from $W_\omega(n-1)$ toward P_ω / S_ω; instead there is an angle difference θ_ω, which is shown in Figure 3. If $|\theta_\omega| < 90^o$ and $\mu c_\omega P_x(\omega) \,|\, S_\omega \,|^2 < 2\cos(\theta_\omega)$, then the distance from $W_\omega(n)$ to P_ω / S_ω will be less than $W_\omega(n-1)$ to P_ω / S_ω, which means that $W_\omega(n)$ is closer to the optimal solution than is $W_\omega(n-1)$. Thus, the adaptive filter will eventually converge in this case. On the other hand, if $|\theta_\omega| \geq 90^o$, the adaptive filter will never converge, no matter how small the step size is. Thus we find that we need to meet the bound

$$\mu < \min \frac{2\cos(\theta_\omega)}{c_\omega P_x(\omega) \,|\, S_\omega \,|^2} \text{ for all } \omega \quad (8)$$

And have phase error less than $90°$. This analysis gives us a clear and intuitive view about the $\pm 90^o$ stability bound of the filtered-x LMS algorithm, and the amplitude estimation error of $\hat{S}(z)$ will only affect the bound of the step size μ. It will not cause the adaptive filter to diverge for an appropriate choice of μ, which has been observed by many researchers [1,5-7]. However our analysis uses a geometric argument that provides an intuitive explanation that we use to develop a new, computationally efficient, FXLMS algorithm.

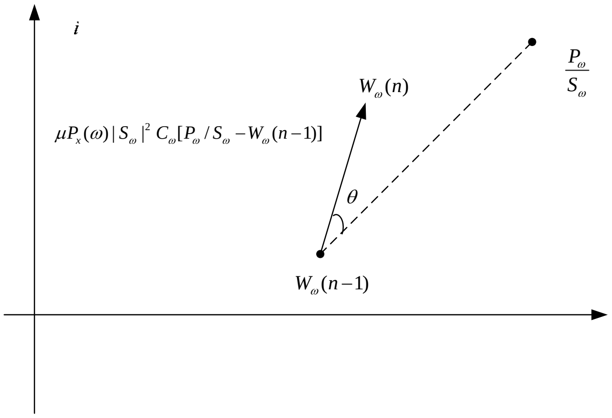

Figure 3. The complex plane expression of Eq. (7)

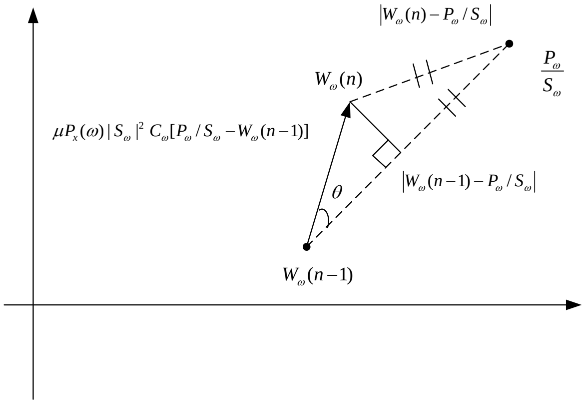

Figure 4. Convergence analysis

4. RELATIONSHIP BETWEEN DIFFERENT FILTERED LMS ALGORITHMS

The adjoint LMS algorithm and the secondary path equalization method are alternatives to the filtered-x LMS algorithm. Using our proposed geometric analytical method, these two algorithms can be easily developed. To ensure that the adaptive filter coefficients $W_\omega(n)$ go in the point-to-point direction from $W_\omega(n-1)$ toward P_ω / S_ω, we could either filter the reference signal by the estimated secondary path (as in FXLMS) or filter the error signal by the adjoint of the estimated secondary path (as in the Adjoint LMS algorithm) or by filtering the error signal by the inverse of the estimated secondary path (as in the Secondary Path Equalization Method). We find that each of these filtered LMS algorithms cancel out the secondary path phase response, and so the adaptive filter coefficients $W_\omega(n)$ go in the point-to-point direction from $W_\omega(n-1)$ toward P_ω / S_ω. Alternatively, one may view these algorithms as having aligned the reference and the error signals.

5. CONVERGENCE RATE ANALYSIS

If we subtract $P(j\omega)/S(j\omega)$ (the expected value of $W_\omega(n)$) from both sides of Eq. (7), then as $n \to \infty$

$$\left|W_\omega(n) - P_\omega / S_\omega\right| \approx \left|1 - \mu c_\omega P_x(\omega) \left|S_\omega\right|^2 \cos(\theta)\right| \left|W_\omega(n-1) - P_\omega / S_\omega\right| \quad (9)$$

The physical meaning of Eq. (9) is that at the frequency ω, the adaptive filter distance from its optimal value, i.e. $W_\omega(n)-P(j\omega)/S(j\omega)$, decays by a factor of $\left|1-\mu c_\omega P_x(\omega)|S_\omega|^2\cos(\theta)\right|$ per iteration, as shown in Figure 4. Using this decay factor, we can define a time constant for the adaptive filter at frequency ω as:

$$\tau_\omega \approx \frac{1}{\mu c_\omega P_x(\omega)|S_\omega|^2\cos(\theta_\omega)} \tag{10}$$

An adaptive filter convergence speed is decided by the slowest convergence component, which has the maximum time constant. Similar to the analysis of the LMS adaptive filter in [8], we define this maximum time constant as the overall time constant:

$$\tau=\max\{\tau_\omega\}\approx\frac{1}{\min\left[\mu c_\omega P_x(\omega)|S_\omega|^2\cos(\theta_\omega)\right]}. \tag{11}$$

If our step size μ takes the upper bound in Eq. (8), then the overall time constant is

$$\tau\approx\frac{\max\left[c_\omega P_x(\omega)|S_\omega|^2\right]}{\min\left[c_\omega P_x(\omega)|S_\omega|^2\cos^2(\theta_\omega)\right]} \tag{12}$$

So, this overall time constant determines our filtered-x LMS filter convergence speed. We want the overall time constant to be as small as possible. If there is no error in the estimation of the secondary path, then Eq. (12) reduces to Eq. (3.4.12) in [8]. From this result, we can see that the phase error and amplitude estimation error will reduce the filtered-x LMS algorithm convergence speed. Note that the derivation of the overall time constant τ requires many approximations, so that the overall time constant τ does not represent the actual minimum number of iterations required for the adaptive filter to converge. However, it does reflect the relative convergence speed of the adaptive filter as indicated in the simulation results.

6. SIMULATION RESULTS

We simulate the FXLMS algorithm as shown in Figure 1. The main path is modeled as an FIR filter with coefficients [0 0 1.0000 0.7083 0.1861]. To simplify the simulation, the secondary path has the transfer function [1 α], with $\alpha\in(0,1)$. FIR adaptive filters with order 30 are implemented. We assume the secondary path estimation has no error. So the overall time constant τ can be calculated from (12) as:

$$\tau=\frac{1+\alpha}{1-\alpha} \tag{13}$$

For different values of α, the stepsize μ is chosen by the trial and error method to ensure that the adaptive filters converge at their fastest rate. We denote I_α as the minimum number of iterations required for the adaptive filter to converge when the secondary path is modeled as [1 α], so I_0 represents the convergence speed for the LMS based adaptive filter. We define the normalized number of iterations $\bar{I}_\alpha$ as

$$\bar{I}_\alpha=\frac{I_\alpha}{I_0} \tag{14}$$

Figure 5 shows the relationship of the time constant τ to the parameter α, and the relationship between $\bar{I}_\alpha$ and α. From this figure, we find that the proposed overall time constant is linearly related to the actual number of iterations.

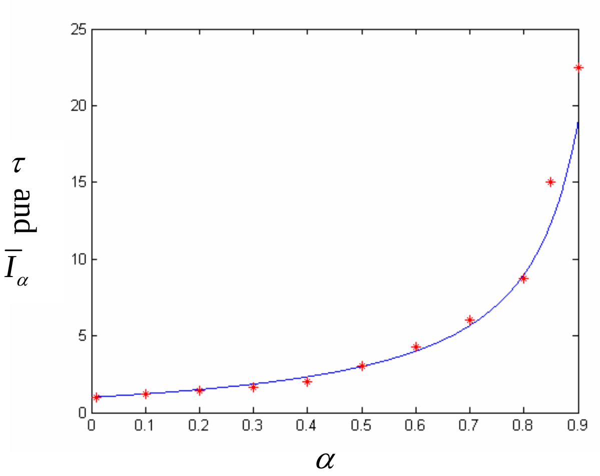

Figure 5. Relationship between the time constant τ and parameter α (blue line), and the relationship between $\bar{I}_\alpha$ and α (red star curve).

7. CONCLUSION

In this paper, we developed a geometric method to analyze the filtered-x LMS algorithm. Based on our analysis, we derived the upper bound for the allowable step size, discussing the secondary path estimation error on the convergence property of the filtered-x LMS algorithm and the relationship between the different filtered LMS algorithms. Although this analysis results has been previously reported, our method is more direct and provides some geometrical meaning and intuitive explanations. Furthermore, using this tool, we ***quantatively*** analyzed the effect of the secondary path estimation error on the adaptive filter convergence rate. ***To the best of our knowledge, this result is provided for the first time.*** Our result shows that both the amplitude estimation error and the phase estimation error will reduce the convergence speed of the adaptive filter. Our analysis result can be extended to any filtered LMS algorithm.

In our future work, we want to develop a fast converging adaptive filter based on our proposed analysis results.

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