A PARTICLE FILTER TO MITIGATE JAMMING FOR GPS NAVIGATION

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ABSTRACT

This paper studies a deterministic Particle Filter for detecting and estimating jamming for GPS navigation. Jamming results in an sudden increase of the GPS measurement noise variance, hence a degradation of the positioning solution. The proposed methodology extends the state vector by including both the unknown noise variances and a latent process indicating variance jumps. A fixed lag smoothing algorithm makes the detection more robust to outliers. The sparseness of variance changes is advantageously taken into account in the proposed strategy.

1. INTRODUCTION

The Global Positioning system (GPS) is widely used either as a stand-alone navigation device or integrated with other sensors. However, satellite navigation is sensitive to external perturbations such as signal blocking, interferences or jamming. Jamming remains one of the most penalizing error sources. It can be classically modeled as white noise in the GPS useful bandwidth. This noise results in a sudden decrease of the signal to noise ratio (SNR). A far too abrupt jump can even prevent GPS measurements from being formed. Such scenarios are not considered in this paper wherein jamming is assumed to merely entail an increase of the GPS measurement noise. If left undetected, the navigation solution is bound to degrade.

The aim of this paper is to jointly estimate the solution to the navigation problem and the variance jumps caused by jamming from the noisy GPS measurements. Section 2 reviews the navigation model. The model parameters are assumed to depend on a latent process that represents the variance jumps (also referred to as indicator process). Thus, the state vector is composed not only of the navigation states (position, velocity...), but also of the variance of the measurement noise and the finite state process. Section 3 studies a deterministic fixed lag smoothing Particle Filter to solve the navigation problem in presence of jamming. The proposed algorithm is validated by simulation results presented in section 4. Conclusions and perspectives are reported in section 5.

2. PROBLEM FORMULATION

The joint detection/estimation of jamming can be formulated as a multiple model problem. Here, a model corresponds either to an absence of variance jump or to one of the possible combinations of simultaneous variance jumps on the GPS measurements. Such a problem is classically addressed by augmenting the state vector by an indicator vector that selects the appropriate model at each time instant. Equivalently, the navigation system subject to jamming can be modeled as a Jump Markov non linear system. The parameters of this system evolve with time according to a finite state Markov chain (the indicator vector) and the state vector (the unknown vehicle kinematics and the measurement variance noise) is non linearly related to the measurements. From now on, the following notations are used:

NOMENCLATURE

$\boldsymbol{x}_t$: tth time instance of the vector-valued process $\boldsymbol{x}$,
$\boldsymbol{x}_{p:q}$:$(\boldsymbol{x}_p, \ldots, \boldsymbol{x}_q)$,
$\boldsymbol{x}^T$: transpose vector $\boldsymbol{x}$,
$\boldsymbol{x}_{t,k}$:kth component of the vector $\boldsymbol{x}_t$,
$\boldsymbol{x}_t^{(i)}$: ith particle representing the process $\{\boldsymbol{x}_t\}_{t>0}$.

2.1. The state vector

The extended state vector is denoted by $\boldsymbol{X}_t = (\boldsymbol{x}_t, \boldsymbol{\phi}_t, \boldsymbol{\lambda}_t)$ where:

• $\boldsymbol{x}_t$ (of dimension n_x) is composed of the navigation states, i.e the parameters describing the vehicle motion (the position and its derivative) as well as GPS unknowns such as the receiver clock offset with respect to GPS time [1].

• $\boldsymbol{\phi}_t$ is the stacked vector of the variances associated to the GPS measurements at time t (not that the dimension of $\boldsymbol{\phi}_t$, denoted as n_y, is equal to the number of GPS measurements available at time instant t).

• $\boldsymbol{\lambda}_t$ (of dimension n_y) is the finite state latent Markov process. $\boldsymbol{\lambda}_t = (\boldsymbol{\lambda}_{t,1}, \ldots, \boldsymbol{\lambda}_{t,n_y})$, where $\boldsymbol{\lambda}_{t,j} = 1$ if there is a variance jump at time t on the ith measurement and $\boldsymbol{\lambda}_{t,j} = 0$ otherwise. Thus, this vector takes value in a

finite set of cardinal 2^{n_y} whose elements are denoted by $(\mathbf{\Lambda}_k)_{k=1,\dots,2^{n_y}}$.
The Jump Markov non linear system can be modeled as:

$$\begin{array}{rcl} \boldsymbol{x}_t &=& F_t\boldsymbol{x}_{t-1} + B_t\boldsymbol{v}_t, \\ \boldsymbol{\phi}_t &\sim& f_\phi(\boldsymbol{\phi}_t|\boldsymbol{\phi}_{t-1}, \boldsymbol{\lambda}_t), \\ \boldsymbol{\lambda}_t &\sim& f_\lambda(\boldsymbol{\lambda}_t|\boldsymbol{\lambda}_{t-1}), \\ \boldsymbol{y}_t &=& h_t(\boldsymbol{x}_t) + C(\boldsymbol{\phi}_t)\boldsymbol{w}_t, \end{array}$$

where $\boldsymbol{y}_t$ stands for the measurement vector (of dimension n_y) and $C(\boldsymbol{\phi}_t) = \mathrm{diag}(\boldsymbol{\phi}_t)$. The vectors $\boldsymbol{v}_t, \boldsymbol{w_t} \sim \mathcal{N}(\mathbf{0}, \mathrm{I})$ are independent white noise sequences ($\mathcal{N}(\boldsymbol{m}, P)$ denotes a Gaussian distribution of mean $\boldsymbol{m}$ and covariance matrix P). Note that the matrix F_t and the function h_t are defined in the next sections. The state vector mixes discrete valued with continuous valued parameters whose priors are non necessarily Gaussian. Particle filtering (PF) methods, also referred to as sequential Monte Carlo methods (SMC), have proved efficient to solve on-line such problems. They are based on a Bayesian approach whereby *prior* information is taken into account to estimate the unknown parameters. PF offers a unified framework to estimate as a whole the state vector, whatever the *prior* distribution of its parameters.

2.2. The state model

A hierarchical structure seems natural to describe the *a priori* behavior of the different components of the state vector. This paper considers the following model for the navigation states:

$$\boldsymbol{x}_t|\boldsymbol{x}_{t-1} \sim \mathcal{N}(F_t\boldsymbol{x}_{t-1}, B_tB_t^T).$$

The state matrices F_t and B_t depends on the vehicle dynamics. The reader is invited to consult [1] for possible motion models.
The variance of each measurement noise is classically well described by an Inverse Gamma distribution $\mathcal{IG}$ as in [2]. The parameters of this $\mathcal{IG}$ *prior* depend on the presence/absence of variance jumps. The following state equations are proposed for the variances:

• **Absence of variance jump**:

$$(\boldsymbol{\phi}_{t,i}|\boldsymbol{\phi}_{t-1,i}, \boldsymbol{\lambda}_{t,i} = 0) \sim \mathcal{IG}(\alpha^0_{t,i}, \beta^0_{t,i}).$$

The parameters $\alpha^0_{t,i}$ and $\beta^0_{t,i}$ are adjusted such that the mean of the resulting Inverse Gamma distribution is $\boldsymbol{\phi}_{t-1}$ and admits a small variance.

• **Presence of variance jump**:

$$(\boldsymbol{\phi}_{t,i}|\boldsymbol{\phi}_{t-1,i}, \boldsymbol{\lambda}_{t,i} = 1) \sim p\mathcal{IG}(\alpha^1_{t,i}, \beta^1_{t,i}) + (1-p)\mathcal{IG}(\alpha^2_{t,i}, \beta^2_{t,i}).$$

A mixture of Inverse Gamma distributions allows to model either the appearance or the disappearance of jamming.

i) Jamming appearance: $\alpha^1_{t,i}, \beta^1_{t,i}$ are chosen to favor higher values of the variance through a heavy tailed *prior*.
ii) Jamming disappearance of jamming: $\alpha^2_{t,i}, \beta^2_{t,i}$ correspond to a very informative *prior* that enforce the variance to the nominal GPS value.
The mixture probability p is chosen as follows. In the absence of jamming, the only possible transition is an increase of the variance and p is set to 1. In the presence of jamming, the noise level can increase or decrease as well and p can be chosen equal to $\frac{1}{2}$.
Each element of the indicator vector is assigned a Bernoulli distribution whose parameter γ ($0 \leq \gamma \geq 1$) allows to specify the frequency of the variance jumps. For the overall vector $\boldsymbol{\lambda}_t$:

$$p(\boldsymbol{\lambda}_t) = \gamma^{\sharp\boldsymbol{\lambda}_t}(1-\gamma)^{n_y - \sharp\boldsymbol{\lambda}_t},$$

where $\sharp\boldsymbol{\lambda}_t = \sum_{i=1}^{n_y} \boldsymbol{\lambda}_{t,i}$ is the number of GPS measurements simultaneously corrupted by jamming and $\gamma = P[\boldsymbol{\lambda}_{t,i} = 1]$.

2.3. The measurement model

Navigation with GPS is based on triangulation. A vehicle is equipped with a receiver that measures propagation delays of signals emitted by the GPS satellites (or equivalently vehicle to satellite distances). The GPS measurement equation $\boldsymbol{y}_t = h_t(\boldsymbol{x}_t) + C(\boldsymbol{\phi}_t)\boldsymbol{w}_t$ has been explicitly presented in [1]. The nonlinear function $h_t(\boldsymbol{x}_t)$ expresses the satellite to vehicle rangings, biased by the receiver clock and satellite offsets. There are as many measurements as satellites in line-of-sight of the vehicle.

$$h_t(\boldsymbol{x}_t) = \begin{cases} \sqrt{(x_t - x_t^1)^2 + (y_t - y_t^1)^2 + (z_t - z_t^1)^2} + b_t, \\ \vdots \\ \sqrt{(x_t - x_t^i)^2 + (y_t - y_t^i)^2 + (z_t - z_t^i)^2} + b_t, \\ \vdots \\ \sqrt{(x_t - x_t^{n_y})^2 + (y_t - y_t^{n_y})^2 + (z_t - z_t^{n_y})^2} + b_t, \end{cases}$$

where (x_t, y_t, z_t) and (x_t^i, y_t^i, z_t^i) refer respectively to the position of the vehicle and the ith satellite in the ECEF (earth-centered earth-fixed) frame of coordinate. The scalar b_t denotes the GPS receiver clock offset.

3. THE FIXED LAG DETERMINISTIC PF

This section first briefly reviews particle filtering methods. The algorithm solving the joint detection/estimation problem is then thoroughly described.

3.1. Estimation objectives

In a Bayesian context, all inference about the unknown states is obtained from the *posterior* probability density function (pdf) $p(\boldsymbol{X}_t|\boldsymbol{y}_{1:t})$. For instance, the minimum mean square error (MMSE) estimates of the continuous valued vectors $\boldsymbol{x}_t$ and $\boldsymbol{\phi}_t$ are defined as $\widehat{\boldsymbol{x}}_t = \mathrm{E}(\boldsymbol{x}_t|\boldsymbol{y}_{1:t})$ and $\widehat{\boldsymbol{\phi}}_t = \mathrm{E}(\boldsymbol{\phi}_t|\boldsymbol{y}_{1:t})$. These *posterior* expectations are approximated by sample averages due to the PF approach. Jamming detection is based on the change probability, defined as:

$$P^c_{t,k} = P(\boldsymbol{\lambda}_{t,k} = 1|\boldsymbol{y}_{1:t}).$$

This term represents the *a posteriori* probability that the kth GPS measurement incurs a variance jump. A high value of $P^c_{t,k}$ coincides with jamming appearance or disappearance. Equivalently, the smoothed MMSE estimates are $\widehat{\boldsymbol{x}}_t = \mathrm{E}(\boldsymbol{x}_t|\boldsymbol{y}_{1:t+L})$, $\widehat{\boldsymbol{\phi}}_t = \mathrm{E}(\boldsymbol{\phi}_t|\boldsymbol{y}_{1:t+L})$ and the smoothed *posterior* probability $P^c_{t,k} = P(\boldsymbol{\lambda}_{t,k} = 1|\boldsymbol{y}_{1:t+L})$, where L is a positive integer.

3.2. Sequential Monte Carlo methods

Particle filters provide online an empirical estimate of the *posterior* pdf of interest:

$$\widehat{p}(\boldsymbol{X}_{0:t}|\boldsymbol{y}_{1:t}) = \sum_{i=1}^{N} w_t^{(i)} \delta(\boldsymbol{X}_{0:t} - \boldsymbol{X}_{0:t}^{(i)}).$$

Contrary to grid-based methods, the support points $\boldsymbol{X}_{0:t}^{(i)}$, also called particles, are not chosen arbitrary. They evolve randomly according to an appropriate simulation-based rule. Ideally, the particles should be generated sequentially from the distribution of interest and therefore equally weighted. However, this distribution is not standard and only known up to a normalizing constant. As a result, an alternative sampling technique called sequential importance sampling is used. At each time instant, the new particles are generated according to a proposal distribution $\pi(\boldsymbol{X}_t|\boldsymbol{X}_{0:t-1}, \boldsymbol{y}_{1:t})$. They are then assigned importance weights w_t to correct for the discrepancy between the pdf of interest p and π. These importance weights are defined as:

$$w_t^{(i)} \propto w_{t-1}^{(i)} \frac{p(\boldsymbol{X}_t^{(i)}|\boldsymbol{X}_{0:t-1}^{(i)}, \boldsymbol{y}_{1:t}) p(\boldsymbol{y}_t|\boldsymbol{X}_{0:t-1}^{(i)}, \boldsymbol{y}_{1:t-1})}{\pi(\boldsymbol{X}_t^{(i)}|\boldsymbol{X}_{0:t-1}^{(i)}, \boldsymbol{y}_{1:t})}.$$

The optimal proposal distribution minimizing the variance of the importance weights has been proposed in [3]:

$$\pi(\boldsymbol{X}_t|\boldsymbol{X}_{0:t-1}, \boldsymbol{y}_{1:t}) = p(\boldsymbol{X}_t|\boldsymbol{X}_{0:t-1}, \boldsymbol{y}_{1:t}).$$

However, sampling from this distribution is most of the time impossible. In this study, the proposal distribution is a Gaussian approximation of the optimal one obtained by a local linearization of the measurement equation. For proposals of this form, the variance of the importance weights is well-known to increase until all but one particle have negligible weight. An additional resampling step from the estimated pdf is introduced to prevent the particle degeneracy. The state of art of SMC is detailed in [4].

3.3. Algorithm description

The generic particle filter is modified to meet the estimation objectives and make the most of the analytical structure of the model. The state vector mixes finite (the indicator vectors) with continuous (the navigation states and the measurement variance) state Markov processes. Therefore, the sampling procedure can be improved by fully exploring the indicator vector state space while the continuous states are still drawn randomly. This approach, known as deterministic particle filter [5], allows to improve the variance jump detection. Even if the variance jumps occur relatively infrequently (i.e γ is very small), the "jump" hypothesis is represented by many particles at each time instant. The resulting approximation of $p(\boldsymbol{X}_{0:t}|\boldsymbol{y}_{1:t})$ can be written:

$$\widehat{p}(\boldsymbol{X}_{0:t}|\boldsymbol{y}_{1:t}) = \sum_{i=1}^{N} \sum_{j=1}^{2^{n_y}} w_t^{(i,j)} \delta(\boldsymbol{X}_{0:t} - \boldsymbol{X}_{0:t}^{(i,j)}).$$

In the above formula, the following notation is used:

$$\boldsymbol{X}_{0:t}^{(i,j)} = \left(\boldsymbol{X}_{0:t-1}^{(i)}, \boldsymbol{\lambda}_t^{(i,j)} = \boldsymbol{\Lambda}_j, \boldsymbol{x}_t^{(i,j)}, \boldsymbol{\phi}_t^{(i,j)}\right),$$

where $\boldsymbol{\phi}_t^{(i,j)}, \boldsymbol{x}_t^{(i,j)} \sim \pi(\boldsymbol{\phi}_t, \boldsymbol{x}_t|\boldsymbol{\phi}_{t-1}^{(i)}, \boldsymbol{x}_{t-1}^{(i)}, \boldsymbol{\lambda}_t^{(i,j)}, \boldsymbol{y}_t)$. More precisely, the particles are drawn in two steps:

- $\boldsymbol{\phi}_t^{(i,j)} \sim \pi(\boldsymbol{\phi}_t|\boldsymbol{\phi}_{t-1}^{(i)}, \boldsymbol{x}_{t-1}^{(i)}, \boldsymbol{\lambda}_t^{(i,j)}, \boldsymbol{y}_t)$,
- $\boldsymbol{x}_t^{(i,j)} \sim \pi(\boldsymbol{x}_t|\boldsymbol{\phi}_t^{(i,j)}, \boldsymbol{x}_{t-1}^{(i)}, \boldsymbol{\lambda}_t^{(i,j)}, \boldsymbol{y}_t)$,

where π is the approximated optimal proposal distribution. The main difficulty lies in the navigation vector $\boldsymbol{x}_t$ that cannot be marginalized out to yield $\pi(\boldsymbol{\phi}_t|\boldsymbol{\phi}_{t-1}^{(i)}, \boldsymbol{x}_{t-1}^{(i)}, \boldsymbol{\lambda}_t^{(i,j)}, \boldsymbol{y}_t)$. The variance particles are thus instead sampled from $\pi(\boldsymbol{\phi}_t|\boldsymbol{\phi}_{t-1}^{(i)}, \boldsymbol{x}_t = F_t\boldsymbol{x}_{t-1}^{(i)}, \boldsymbol{\lambda}_t^{(i,j)}, \boldsymbol{y}_t)$. The corresponding importance weights are expressed as:

$$w_t^{(i,j)} \propto w_{t-1}^{(i)} p(\boldsymbol{y}_t|\boldsymbol{x}_{t-1}^{(i)}, \boldsymbol{\phi}_{t-1}^{(i)}, \boldsymbol{\lambda}_t^{(i,j)}) p(\boldsymbol{\lambda}_t^{(i,j)} = \boldsymbol{\Lambda}_j).$$

The resampling step then selects N particles among the $N \times 2^{n_y}$ candidates to keep the computational complexity constant. In this study, a more efficient algorithm is proposed. Detecting a variance jump is quite a challenging problem depending on the magnitude of the change. If the application allows a fixed delay L before the estimation, a fixed-lag smoothing approach can make the detection more robust. The parameter L is adjusted to keep the balance between the PF performance and the real time requirement. Once more, all the possible future paths (from time t to $t + L$) of the finite state particles can be explored to estimate the pdf of interest. However, such a procedure is computationally demanding and the calculatory cost is due to increase with the lag L. This limitation can be overcome by taking advantage of the sparseness of the variance jumps. If the stake is to detect a jump at time t, it can be assumed that no transition between time $t+1$ and $t+L$ is the far more likely hypothesis. The explored indicator paths then take the form: $\boldsymbol{\lambda}_{t:t+L}^{(j)} = (\boldsymbol{\Lambda}_j, \mathbf{0}, \ldots, \mathbf{0})$. The approximated fixed lag smoothing pdf can be written:

$$\widehat{p}(\boldsymbol{X}_{0:t}|\boldsymbol{y}_{1:t+L}) = \sum_{i=1}^{N}\sum_{j=1}^{2^{n_y}} w_{t+L}^{(i,j)} \delta(\boldsymbol{X}_{0:t+L} - \boldsymbol{X}_{0:t+L}^{(i,j)}).$$

The particles are expressed as:

$$\boldsymbol{X}_{0:t+L}^{(i,j)} = \left(\boldsymbol{X}_{0:t-1}^{(i)}, \boldsymbol{\lambda}_{t:t+L}^{(i,j)} = \boldsymbol{\lambda}_{t:t+L}^{(j)}, \boldsymbol{x}_{0:t+L}^{(i,j)}, \boldsymbol{\phi}_{0:t+L}^{(i,j)}\right),$$

where the continuous state particles are drawn iteratively as described above. Provided the lag is not too large, no resampling step is performed between time t and $t+L-1$. At time $t+L$, the fixed-lag weights $w_{t+L}^{(i,j)}$ are used to compute the fixed-lag smoothed estimates. The continuous state estimates are obtained as:

$$\begin{aligned} \widehat{\boldsymbol{x}}_t &= \sum_{i=1}^{N}\sum_{j=1}^{2^{n_y}} w_{t+L}^{(i,j)} \boldsymbol{x}_t^{(i,j)}, \\ \widehat{\boldsymbol{\phi}}_t &= \sum_{i=1}^{N}\sum_{j=1}^{2^{n_y}} w_{t+L}^{(i,j)} \boldsymbol{\phi}_t^{(i,j)}. \end{aligned}$$

The estimated change probability for the kth measurement is given by:

$$\widehat{P}_{t,k}^c = \sum_{i=1}^{N}\sum_{j=1}^{2^{n_y}} w_{t+L}^{(i,j)} \delta(\boldsymbol{\lambda}_{t,k}^{(i,j)} - 1).$$

Two sets of weights are available with this method: the filtering weights $w_t^{(i,j)}$ and the smoothing weights $w_t^{(i,j)}$. Therefore, a more elaborated resampling strategy can be applied. Indeed, the resampled particles should represent the filtering distribution, but relevant particles are characterized by high smoothing weights. The dilemma is solved by resorting to an importance sampling scheme. The particles are resampled according to the smoothing pdf and they are then assigned new filtering weights computed as:

$$w_t^{(i,j)} \propto \frac{w_t^{(i,j)}}{w_{t+L}^{(i,j)}}.$$

The whole procedure can then be reiterated at the next time step.
Finally, the algorithm can be improved by introducing a constraint about the time interval between two variance jumps. The approach consists of enforcing a minimal duration τ_m for the nearly constant variance periods. For any particle, as long as the delay from the last variance jump is less than τ_m, only the offsprings denying a transition are selected.

A decision step can also be introduced once the transition probability has been estimated. If $\widehat{P}_{t,k}^c$ is greater than a fixed threshold, a jump variance is decided and only the particles indicating the jump are propagated. The resulting variance estimate is more accurate. The whole algorithm proceeds as in table (1).
Remark: the "no jump" assumption may seem restrictive. If a transition occurs at time t, this hypothesis only entails an artificical increase of the estimated change probability from time $t-L$ to t. However, the probability peak still coincide with the jump instant.

4. SIMULATION RESULTS

A classical GPS scenario is used to validate the proposed method. Several Monte Carlo runs corresponding to different realizations of the GPS measurement noise are carried out. A constant velocity trajectory is considered (the acceleration is simulated as a white noise sequence). The GPS measurements are computed all along this trajectory according to the measurement equation from real GPS satellite trajectories. Random variance jumps are added to the nominal GPS variance of the measurement noise. The number of measurements that are affected at the same time by jamming is limited to ensure the system observability. The simulations presented in this section have been obtained from $n_y = 6$ line-of-sight satellites. Since there are 4 unknown navigation components, a maximum of 2 channels can be jammed at the same time.
The results depicted on figures 1, 2, 3 and 4 are obtained from 100 Monte Carlo simulations, 2000 particles and a lag $L = 8$. Figure 1 and 3 shows that the peaks of the estimated change probability coincide with the variance jump instants (indicated by vertical lines). Besides, the measurement noise variance is properly estimated as illustrated on figure 2 and 4. These results are confirmed by table (2) which shows the mean and standard deviations of the estimated changepoints. Finally, the estimated *posterior* pdf of the unknown variance after a jump (time $t = 134$) is shown on figure 5. The vertical lines indicate respectively the true parameter value and the estimated one. A good matching is observed altough a small delay is necessary before the exact variance is recovered.

5. CONCLUSION

This study has addressed the problem of detecting and estimating jamming in GPS. A fixed lag deterministic Particle Filter has been proposed to detect variance jumps due to jamming while estimating the navigation states. The main originality of this work is to include variance jump indicators in the state vector to perform the detection. The sparseness of these jumps has allowed to reduce the computational complexity of the algorithm.

A fixed lag smoothing PF iteration (time t)

- For $i = 1, \ldots, N$,
  - for $k = 1, \ldots, n_y$,
    compute $\boldsymbol{\tau}_{t,k}^{(i)} = t - \max(p < t, \boldsymbol{\lambda}_{p,k}^{(i)} = 1)$,
  - for $j = 1, \ldots, 2^{n_y}$,
    i) set $\boldsymbol{\lambda}_{t:t+L}^{(i,j)} = (\boldsymbol{\Lambda}_j, \mathbf{0}, \ldots, \mathbf{0})$,
    ii) for $k = 1, \ldots, n_y$,
    if $\boldsymbol{\tau}_{t,k}^{(i)} < \tau_m$ whereas $\lambda_{t,k}^{(i,j)} = 1$, discard $\boldsymbol{\lambda}_{t:t+L}^{(i,j)}$.
  - for $p = 0, \ldots, L$,
    i) generate $(\boldsymbol{x}_{t+p}^{(i,j)}, \boldsymbol{\phi}_{t+p}^{(i,j)})$ according to $\pi(\boldsymbol{x}_{t+p}, \boldsymbol{\phi}_{t+p} | \boldsymbol{x}_{0:t+p-1}^{(i,j)}, \boldsymbol{\phi}_{0:t+p-1}^{(i,j)}, \boldsymbol{\lambda}_{0:t+p}^{(i,j)}, \boldsymbol{y}_{0:t+p})$,
    ii) compute $w_{t+p}^{i,j}$.
- Compute the smoothed estimates,
- Resample according to the smoothing weights $w_{t+L}^{i,j}$,
- Compute the modified filtering weights $w_t^{i,j} \propto \frac{w_t^{i,j}}{w_{t+L}^{i,j}}$.

Table 1. Algorithm description

Changepoint	Mean estimated changepoint	Standard deviation
$t = 50$	50	2.7
$t = 67$	67	3.3
$t = 100$	99	2.8
$t = 134$	135	2.9
$t = 150$	151	3

Table 2. Estimated changepoints statistics.

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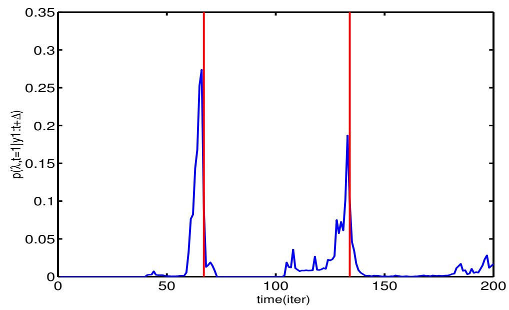

Fig. 1. Detection probability, 1st measurement.

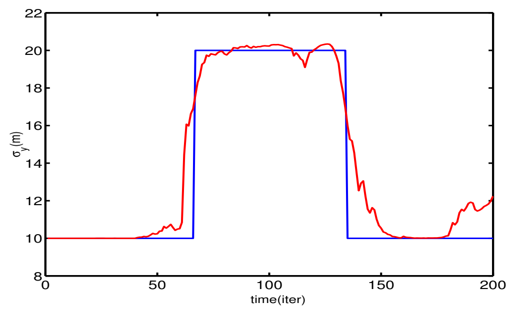

Fig. 2. $\sqrt{\widehat{\phi}_{t,1}}$ versus time.

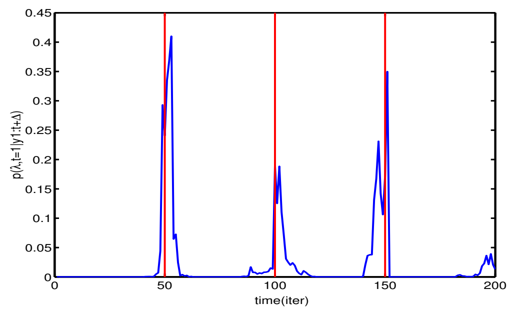

Fig. 3. Detection probability, 2nd measurement.

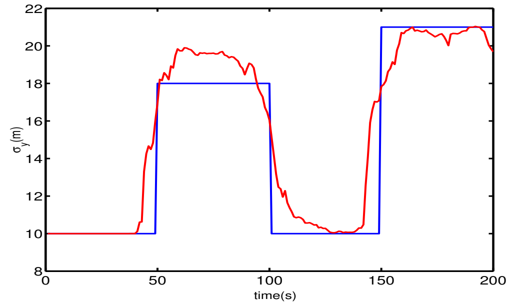

Fig. 4. $\sqrt{\widehat{\phi}_{t,2}}$ versus time.

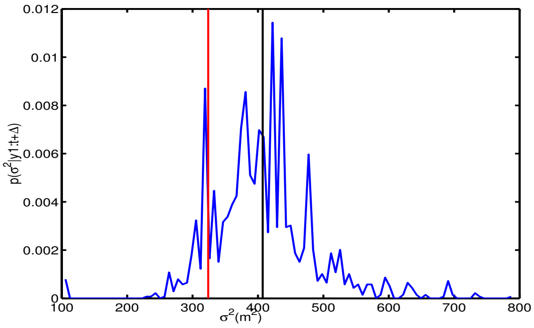

Fig. 5. $\widehat{p}\left(\boldsymbol{\phi}_t|\boldsymbol{y}_{1:t+L}\right)$ after a jump ($t = 134$).