Analysis and Design of (non-)Isotropic Arrays for 3D Direction Finding

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***Abstract*— A simple Cramer-Rao bound on the estimation, using an array of sensors, of the azimuth and elevation angles of a radiating source has been developed recently. It is used here to develop a methodology to design ambiguity-free antenna arrays with a desired directive positioning performance. To reduce the number of the design parameters, A family of V-shaped arrays is investigated. Its gain and directivity (compared to the most commonly used uniform circular array) are defined and analytically computed as a function of the angle between the two branches of the V-shape. We obtain a class of antenna arrays whose performance and directivity can be adapted to the specified application.**

I. Introduction

We address direction of arrival (DoA) estimation using an array of sensors, a classic and largely investigated topic in signal processing [1]. However, limited results that deal with the impact of the array geometry on the estimation performance exist. The Cramer-Rao bound (CRB) is ideal as an algorithm-independent performance measure. Apart from some papers dedicated to some particular antenna geometries [2], the CRB was obtained only indirectly. In fact, its inverse, the Fisher matrix information matrix is expressed analytically in a simple manner only in the single source case [3], [4]. It is then matrix-valued for 3D sources. By examining off-diagonal entries, conditions were established on the sensors positions to ensure isotropic performance [5], [6], [7], [8]. This is the only existing result about the way the array geometry affects performance. Only lately, the azimuth and elevation CRBs of a single source were explicitly given as a function of the planar array geometry [9]. This is nothing but a cosine function, regardless of the array geometry. This allows to fully characterize the array performance and gives a strong tool to design arrays with desired isotropic or anisotropic behavior.

The bound from [9] serves as a starting point to our work. We show that the azimuth and elevation CRBs, and also the asymptotic normalized mean square angular error (ANMSAE) [7], are scaled, rotated or translated versions one of the other. They can be uniquely represented by a single scalar-valued cosine function that depends only on the source azimuth. The sensors locations affect only the maximum and minimum values of the cosine function which are always met at perpendicular directions. At the same time, we retrieve existing results on the conditions for isotropic performance [5], [6], [7], [8].

Anisotropic (directive) arrays can also be useful for a number of applications. The objective is to place sensors such that the estimation error is minimized within a given range of the source location. Such a problem is intractable in the general case because of the large number of parameters involved and because of the array ambiguity problem. We restrict our search within the family of V-shaped arrays. This is an extension of the L-shaped array, proved in [2] to outperform cross, rectangular, circular, triangular, $\ldots$, arrays. Applying [10] ensures (first-order) ambiguity-free arrays and completely determines the sensors positions (and, consequently, the array performance) as functions of the angle between the two branches of the V-shape. We give analytic expressions of the gain obtained w.r.t. the regular uniform circular array (UCA), as a function of this angle.

The paper is organized as follows. Results from [9] are recalled in Sec. II and used in Sec. III to propose an array performance measure. In Sec. IV, a class of V-shaped arrays is presented and their performance is analytically studied. A conclusion is given in Sec. V. We denote by $\Re(x)$ and $\Im(x)$ the real and imaginary parts of x, respectively.

II. Data Model and Previous Results

We consider an M-sized planar antenna array. To sensor m, located in the (x, y) plane, we associate the polar coordinates ρ_m and ϕ_m and the complex number $\gamma_m \hat{=} \rho_m e^{j\phi_m}$. The position of an emitting source located in the far field is characterized by its DoA angles : the azimuth Φ and the elevation Θ as depicted in Fig. 1. If the sensors are identical and omni-directional, the M-dimensional output of the antenna array can be expressed as phased replicas of the emitted signal, as follows

$$\begin{aligned}
\mathbf{x}(t) &\hat{=} [x_1(t) \cdots x_M(t)]^T \\
&= \begin{bmatrix} \exp\left(2j\pi\frac{\rho_1}{\lambda}\sin(\Theta)\cos(\Phi-\phi_1)\right) \\ \vdots \\ \exp\left(2j\pi\frac{\rho_M}{\lambda}\sin(\Theta)\cos(\Phi-\phi_M)\right) \end{bmatrix} s(t) + \mathbf{n}(t) \\
&\hat{=} \mathbf{a}(\Phi,\Theta)s(t) + \mathbf{n}(t),
\end{aligned}$$

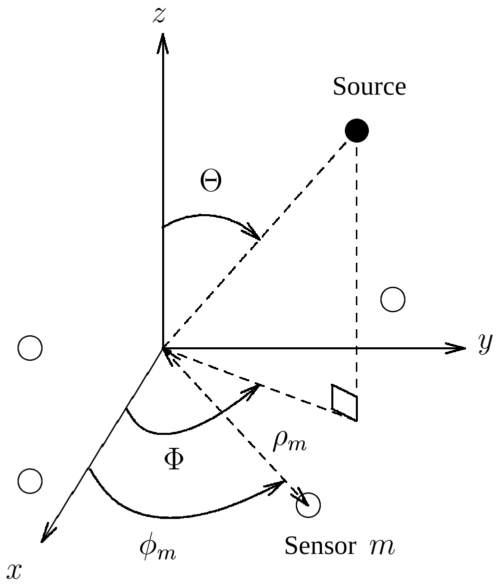


Fig. 1. Planar array and source DoAs.

where the m-th entry of $\mathbf{x}(t)$ (resp. $\mathbf{n}(t)$) is the signal (resp. noise) component collected at sensor m at time index t. They are assumed to be Gaussian-distributed, zero-mean, mutually-independent and spatially and temporally white, with variances σ_s^2 and σ_n^2, respectively. The antenna output is collected at time indexes $T_1, \cdots, T_N$ and snapshots $\mathbf{x}(T_1), \cdots, \mathbf{x}(T_N)$ are used to perform DOA estimation. The CRB expresses the lowest achievable estimation error. Usually attained by the maximum likelihood algorithm, it is, in this case, also achievable by the popular and less complex MUSIC algorithm [11]. The entries of the CRB matrix

$$\mathbf{C} \hat{=} \begin{bmatrix} \mathbf{C}_{\Phi\Phi} & \mathbf{C}_{\Phi\Theta} \\ \mathbf{C}_{\Theta\Phi} & \mathbf{C}_{\Theta\Theta} \end{bmatrix}$$

associated with the parameter vector $[\Phi, \Theta]^T$ have recently been proved [9] to be given by:

$$\mathbf{C}_{\Phi\Phi} = \frac{C_{\text{SNR}}}{N} \frac{1}{\sin^2(\Theta)} \mathcal{C}(\Phi) \tag{1}$$

$$\mathbf{C}_{\Theta\Theta} = \frac{C_{\text{SNR}}}{N} \frac{1}{\cos^2(\Theta)} \mathcal{C}\left(\Phi + \frac{\pi}{2}\right) \tag{2}$$

$$\mathbf{C}_{\Phi\Theta} = -\frac{C_{\text{SNR}}}{N} \frac{1}{\sin(2\Theta)} \frac{\Im\left[T_2 exp(-2j\Phi)\right]}{T_0^2 - |T_2|^2} \tag{3}$$

where

$$C_{\text{SNR}} \hat{=} \frac{1}{4\pi^2} \frac{\sigma_n^2}{\sigma_s^2} \left(1 + \frac{\sigma_n^2}{M\sigma_s^2}\right)$$

and

$$\mathcal{C}(\Phi) \hat{=} \frac{T_0 + \Re\left[T_2 exp(-2j\Phi)\right]}{T_0^2 - |T_2|^2} \tag{4}$$

is function of the array-dependent constants

$$T_0 \;\hat{=}\; \sum_{m=1}^{M} |\gamma_m|^2 - \frac{1}{M}\left|\sum_{m=1}^{M} \gamma_m\right|^2$$

$$T_2 \;\hat{=}\; \sum_{m=1}^{M} \gamma_m^2 - \frac{1}{M}\left(\sum_{m=1}^{M} \gamma_m\right)^2$$

III. Performance Measure

The developed CRBs are uniquely described by the azimuth-independent function $\mathcal{C}(\Phi)$. This is also the case of the (scalar-valued) ANMSAE proposed in [12] as an alternative performance measure. The ANMSAE is associated to the angle between the bearing vector pointing at the source and is equal to $N\left[\sin^2(\Theta)\mathbf{C}_{\Phi\Phi} + \mathbf{C}_{\Theta\Theta}\right]$. We, hence, have

$$\frac{\text{ANMSAE}}{C_{\text{SNR}}} = 2C_{\text{AV}} + \tan^2(\Theta)\mathcal{C}(\Phi + \frac{\pi}{2}).$$

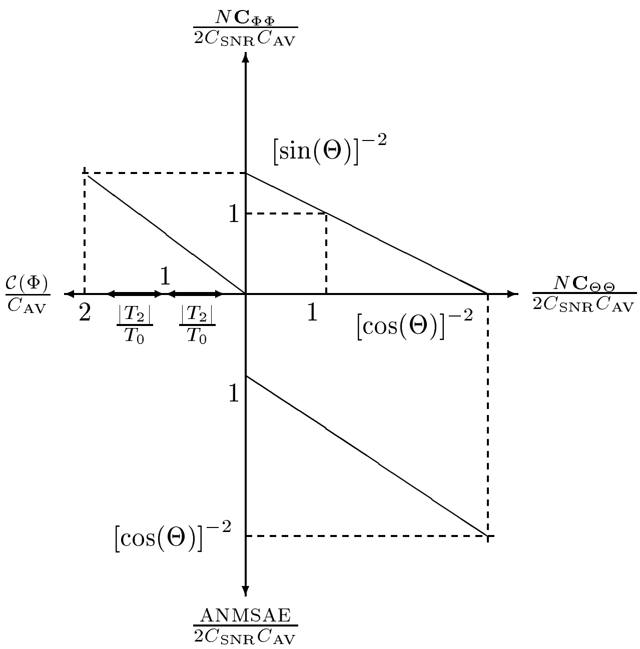


Fig. 2. Illustration of the linear dependence between azimuth and elevation CRBs, ANMSAE and $\mathcal{C}(\Phi)$ [equations (1), (6) and (7)]. The thick line in the horizontal left axis represents the interval of possible values.

Consequently, we consider the key $\mathcal{C}(\Phi)$ function as a unique performance measure that fully characterizes the way the antenna performance depends on the source position and the array geometry. It is π-periodical and fluctuates around the mean value

$$C_{\text{AV}} \hat{=} \frac{T_0}{T_0^2 - |T_2|^2}, \tag{5}$$

which, consequently, is a measure of the average (over all possible azimuth angles) estimation performance. The relationship

$$\mathcal{C}(\Phi) + \mathcal{C}\left(\Phi + \frac{\pi}{2}\right) = C_{\text{AV}}$$

allows us to highlight the linear dependence, illustrated in Fig. 2, between the azimuth and elevation CRBs and the ANMSAE since it enables us to write,

$$\frac{2}{N}C_{\text{SNR}}C_{\text{AV}} = \sin^2(\Theta)\mathbf{C}_{\Phi\Phi} + \cos^2(\Theta)\mathbf{C}_{\Theta\Theta} \quad (6)$$
$$\text{ANMSAE} = 2C_{\text{SNR}}C_{\text{AV}} + N\sin^2(\Theta)\mathbf{C}_{\Theta\Theta} \quad (7)$$

The first equation expresses the trade-off that needs to be made between azimuth and elevation estimation. An antenna array that optimizes both DoA estimates at the same look direction can not be found. Finally, we can easily show that $\mathcal{C}(\Phi)$ attains its maximum/minimum iff $\tan(2\Phi) = -\Im(T_2)/\Re(T_2)$ i.e., minimum and maximum values of $\mathcal{C}(\Phi)$ are met at perpendicular look directions.

To assess the DoA estimation performance of a given array, we refer to the UCA having the same number of sensors spaced by half the wavelength [13]. We introduce the following azimuth-dependent gain function

$$\mathcal{G}(\Phi) \;\hat{=}\; \frac{\mathcal{C}(\Phi)}{\mathcal{C}(\Phi)|_{UCA}} = \frac{M}{16\sin^2\left(\frac{\pi}{M}\right)}\mathcal{C}(\Phi)$$

as a measure of the antenna directivity and the estimation enhancement (w.r.t. to the UCA). If the source is such that $\mathcal{G}(\Phi) < 1$, the considered antenna array is able to more accurately estimate the source azimuth angle than the equivalent UCA. At the same time, however, the elevation estimate may be worse. Only when $\mathcal{G}(\Phi) < 1$ for all Φ, both DoA parameters can be more accurately estimated than using the UCA, regardless of the source location. The gain function $\mathcal{G}(\Phi)$ is better illustrated in polar representation, simultaneously with the unit radius circle, as in Fig. 4. The directional behavior of a given array can be more compactly represented by the following three parameters :

- $\min_\Phi \mathcal{G}(\Phi)$: it represents the maximal enhancement w.r.t. the azimuth estimation.
- $\max_\Phi \mathcal{G}(\Phi)$: if $\Phi_{\text{MIN}} \hat{=} \text{argmin}_\Phi \mathcal{G}(\Phi)$, then $\mathcal{G}(\Phi_{\text{MIN}} + \pi/2) = \max_\Phi \mathcal{G}(\Phi)$ i.e. $\mathbf{C}_{\Theta\Theta} = [\max_\Phi \mathcal{G}(\Phi)] \times (\mathbf{C}_{\Theta\Theta}|_{UCA})$ for every source with azimuth Φ_{MIN}. Hence, $\max_\Phi \mathcal{G}(\Phi)$ expresses eventually a degradation of the elevation estimation.
- I_{MIN} : the width of the sector over which the studied array outperforms the UCA, i.e. the interval within which $\mathcal{G}(\Phi) \le 1$.

From (4), if $T_2 = 0$, azimuth and elevation estimates are uncorrelated ($\mathbf{C}_{\Phi\Theta}$ equals zero) and the respective CRBs are constant, in concordance with [5], [6], [7]. This, in fact, is the only known result about the array geometry impact on DoA estimation. For a variety of application, however, isotropy is not a desired property. For air-borne emergency positioning systems, for example, some a priori knowledge is available about the target location. At the same time, the number of sensors that can be afforded may be very limited. Hence, directive antenna arrays can be of a practical interest. The proposed gain function is relevant to design and assess performance of directive arrays. A pragmatic approach is presented in the next section.

IV. Antenna Array Design Methodology

Antenna design must take into account the ambiguity problem. Only its simplest form, the so-called first-order ambiguity, is considered here. Higher-order ambiguities are mainly analytically intractable. First-order ambiguity occurs when two steering vectors associated with two distinct look directions are identical. Only sufficient conditions for (first-order) ambiguity-free antenna are known. For instance, an antenna is ambiguity-free if at least one set of 3 sensors fulfills some spacings conditions [10].

To further simplify the design problem, we impose a particular structure to the antenna array. V-shaped arrays are a trivial choice for planar array. They include the efficient L-shaped antenna array [2]. As depicted in Fig. 3, the two branches of the V shape start at the origin. The coordinates of sensors 1, 2 and 3 are fixed in order to satisfy [10]. The other sensors are placed such that each branch forms a uniform linear array (ULA) with half-the-wavelength spacing. Apart from ensuring a (first-order) ambiguity-free antenna, the application of [10] affects the performance of the antenna array only marginally, if the antenna size is large enough. This fact can be verified analytically.

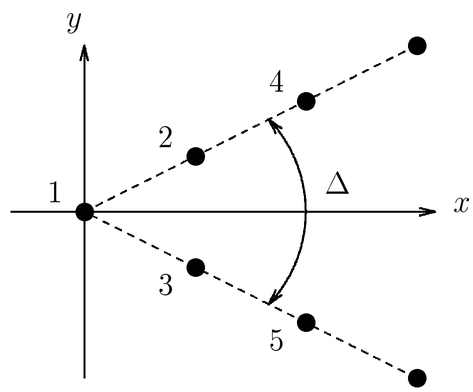


Fig. 3. The V-shaped antenna.

The sensors positions (and hence, the array performance) are functions of the angle Δ between the two branches. Examples of the gain function are represented in Fig. 4. The V-shaped geometry shows to be suitable as directive arrays. The V-shaped array always outperforms the UCA (w.r.t. azimuth estimation) over some (azimuth) sector. The larger the enhancement, the narrower the sector. The directivity can be managed by varying the angle Δ. The gain function $\mathcal{G}(\Phi)$ can be computed analytically. When M tends to infinity, it is given by

$$\mathcal{G}(\Phi) = \frac{3}{\pi^2 \sin^2(\Delta)} \times \left\{ 4 - 3\cos^2\left(\frac{\Delta}{2}\right) + \left[5\cos^2\left(\frac{\Delta}{2}\right) - 4\right]\cos(2\Phi) \right\}.$$

The associated parameters $\min_\Phi \mathcal{G}(\Phi)$, $\max_\Phi \mathcal{G}(\Phi)$ and I_{MIN} are reported as function of Δ, in Fig. 5(a), Fig. 5(b) and Fig. 6, respectively. For a varying angle Δ, Fig. 5 shows that the estimation errors on the azimuth and elevation angles evolve in opposite manners. Interestingly, for a given range of Δ, they are both lower than those obtained using the UCA. This range is the one for which I_{MIN} in Fig. 6 equals 180 [DEG]. We also notice that a given gain $\min_\Phi \mathcal{G}(\Phi)$ may be reached by two V-shaped arrays. In this case, the one with the larger Δ is preferred because the associated interval width I_{MIN} is larger.

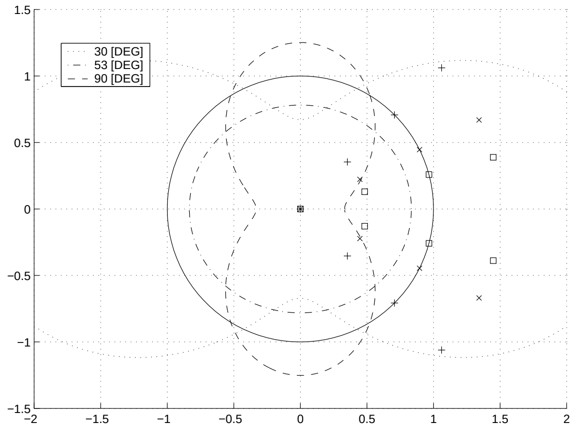
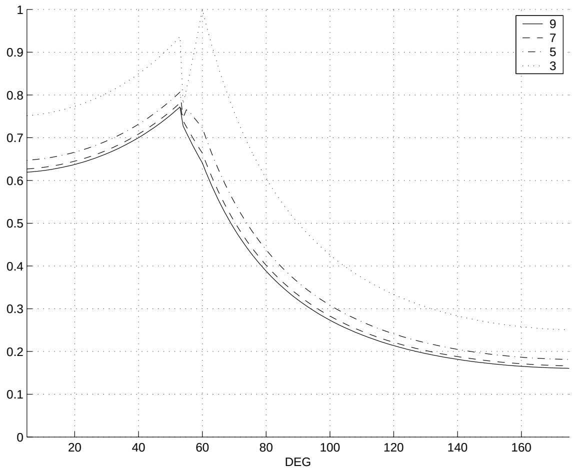


Fig. 4. Polar representation of $\mathcal{G}(\Phi)$ for V-shaped arrays with 7 sensors. The legend shows the angle Δ. The solid line represents the unit radius circle.

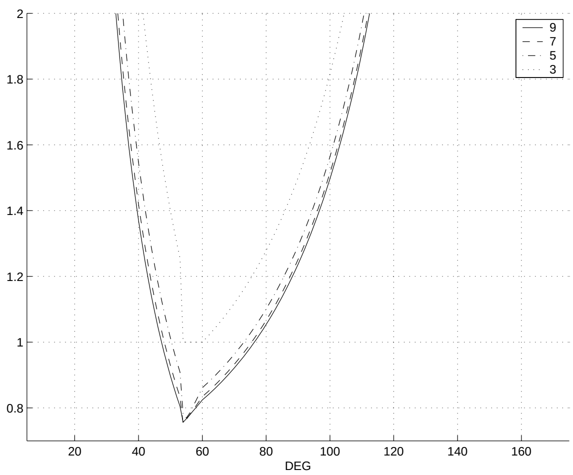
The angle Δ can be chosen to obtain an isotropic behavior as well. When M tends to infinity, such angle Δ is given by $2\arctan(1/2)$ and the (constant) gain $\mathcal{G}(\Phi)$ of the isotropic V-shaped array tends to $15/2\pi^2$ i.e. 24% enhancement w.r.t. the UCA.

V. CONCLUSION

A CRB-based scalar-valued criterion is proposed to evaluates the estimation performance of a planar antenna array used to locate a 3D source in the far-field. The criterion is a cosine function of the source azimuth angle that reaches its maximum and minimum points at perpendicular look directions. Hence, regardless of their geometry, antenna arrays are directive in nature. The (*amount* of) directivity depends, in a complex manner, on the sensors positions. Ambiguity consideration further complicates the design problem and motivates the proposition of a pragmatic approach. The performance of the antenna, now required to have a V shape, can be fully characterized and analytically computed and compared to the equivalent UCA. The results can be used to choose the (V-shaped) array that have the desired directivity.



(a)



(b)

Fig. 5. Minimum (a) and maximum (b) values of the gain function $\mathcal{G}(\Phi)$ for a varying angle Δ (reported on the horizontal axis). The legends show the number of sensors.

REFERENCES

[1] H. Krim and M. Viberg, “Two decades of array signal processing research,” IEEE Signal Processing Mag., pp. 67-94, July 1996.

[2] Y. Hua, T. K. Sarkar and D. D. Weiner, “An L-Shaped array for estimating 2-D directions of wave arrival,” IEEE Trans. Antennas Propagat., vol. 44, pp. 889-895, June 1996.

[3] B. Porat and B. Friedlander, “Analysis of the asymptotic relative efficiency of the MUSIC algorithm,” IEEE Trans. Acoust., Speech, Signal Processing, vol. 36, pp. 532-544, Apr. 1988.

[4] Y. Hua and T. K. Sarkar, “A note on the Cramer-Rao bound for 2-D direction finding based on 2-D array,” IEEE Trans. Signal Processing, vol. 39, pp. 1215-1218, May 1991.

[5] A. Mirkin and L. H. Sibul, “Cramér-Rao bounds on angle estimation with a two-dimensional array,” IEEE Trans. Signal Processing, vol. 39, pp. 515-517, Feb. 1991.

[6] R. O. Nielsen “Azimuth and elevation angle estimation with a three-dimensional array,” IEEE J. Oceanic Eng., vol. 19, pp. 84-86, Jan. 1994.

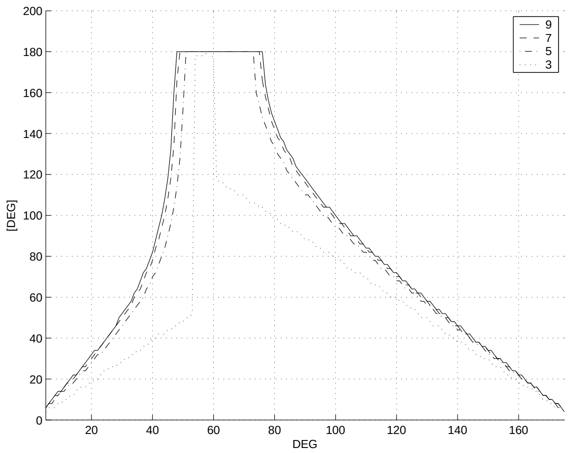


Fig. 6. Parameter I_{MIN} as function of the angle Δ (reported on the horizontal axis). The legend shows the number of sensors.

[7] M. Hawkes and A. Nehorai "Effects of sensor placement on acoustic vector-sensor array performance," IEEE J. Oceanic Eng., vol. 24, pp. 33-40, Jan. 1999.

[8] Ü. Baysal and R. L. Moses, "On the geometry of isotropic arrays," IEEE Trans. Signal Processing, pp. 1469-1478, June 2003.

[9] H. Gazzah and S. Marcos, "Antenna Arrays for Enhanced Estimation of Azimuth and Elevation," IEEE ICASSP Conf., April 2003.

[10] L. C. Godara and A. Cantoni, "Uniqueness and linear independence of steering vectors in array space," J. Acoust. Soc. Am. 70(2), pp. 467-475, Aug. 1981.

[11] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," IEEE Trans. Antennas Propagat., vol. AP-34, pp. 276-280, Mar. 1986.

[12] A. Nehorai and E. Paldi, "Vector-sensor array processing for electromagnetic source localization," IEEE Trans. Acoust., Speech, Signal Processing, vol. 42, pp. 376-398, Feb 1994.

[13] C. P. Mathews and M. D. Zoltowski, "Eigenstructure techniques for 2-D angle estimation with uniform circular arrays," IEEE Trans. Signal Processing, vol. 42, pp. 2395-2407, Sep. 1994.