AN APPROXIMATION OF EIGENVALUE DISTRIBUTION IN I.I.D MIMO CHANNELS UNDER RAYLEIGH FADING

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ABSTRACT

In multiple input multiple output (MIMO) communication systems, eigenvalues of channel correlation matrices play an essential role for the performance analysis, and particularly, the investigation about their behavior under time-variant environment ruled by a certain statistics is an important problem. This paper presents an approximation formula of the marginal distribution of all the eigenvalues of MIMO correlation matrices under i.i.d. (independent and identically distributed) Rayleigh fading environment. We will show that the theory of SIMO space diversity using maximal ratio combining (MRC) is applicable to the approximation of statistical distributions of all eigenvalues in MIMO systems with same number of diversity branches. The derived approximation has a monomial form suitable for the calculation of various performance indeces used in MIMO systems. Through computer simulations, the effectiveness of the proposed method is demonstrated.

1. INTRODUCTION

Recently, multiple input multiple output (MIMO) systems adopting array antennas in transmitter and receiver are collecting attentions as a reliable communication scheme which achieves high data rate transmission [1]. It is well known that MIMO channels are expressed by using singular value decomposition (SVD) [2], and eigenvalues of MIMO channel correlation matrices have an essential meaning in the performance analysis (For example, the capacity of MIMO channel is expressed by using eigenvalues [3]). In fading environment, a time-varying channel matrix is considered to be a stochastic process, and the analysis of statistical distribution of eigenvalues is very important to know the total performance of the system. The literature [4] describes the joint distribution of eigenvalues for i.i.d. (independent and identically distributed) Rayleigh fading channel, and the marginal distribution has been derived for unordered eigenvalues [3]. As for the ordered case, the explicit expressions for the marginal distribution of eigenvalues are not found in literatures except the case of the largest eigenvalue. The closed form expression for the largest eigenvalue is known, but the actual formula described in [5, 6] has a so much complicated form not easy to use in various calculations. Under this situation, reference [7] has given an approximation of the marginal distribution of the largest eigenvalue using gamma distribution, by connecting MIMO MRC (Maximal Ratio Combining) transmission problem with that of single input multiple output (SIMO) space diversity. But the straight extension of the result of [7] to the case of other eigenvlues is impossible, since the SIMO system has only one eigenvalue. Hence a novel idea is required to improve the theory in [7] to be applicable to eigenvalues smaller than the largest one.

In this paper, we give a solution to this problem, and present an approximation of the marginal distribution of all the eigenvalues of MIMO correlation matrices under i.i.d. Rayleigh fading environment. We will show that the SIMO space diversity is again applicable after modification to the estimation of the statistical distribution of eigenvalues in MIMO communication systems.

The remainder of this paper is organized as follows: Section 2. briefly describes the model of MIMO communication system, and its statistical eigenanalysis is given in section 3. In section 4, the approximation formula of the marginal distribution of all the eigenvalues is proposed based on the theory of space diversity in SIMO systems. In section 5, computer simulations are carried out to verify the effectiveness of the proposed method. Finally, in section 6, conclusions and future works are described.

2. MIMO SYSTEM

In this section, the model of MIMO communication system used through this paper is briefly described.

Figure 1 shows an $N_t \times N_r$ MIMO system considered in this study. Assuming frequency flat fading environment, the MIMO channel is expressed by a matrix $H \in \mathbb{C}^{N_r \times N_t}$

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having a n_r-th column and n_t-th raw element H_{n_r,n_t} which shows the channel response between n_t-th and n_r-th antenna elements in the transmitter and the receiver. The channel matrix H is equivalently expressed using SVD as follows:

$$
\begin{aligned}
H &= U\Lambda V^H \\
\Lambda &= diag(\lambda_0^{1/2}, \cdots, \lambda_{R-1}^{1/2}) \\
&\lambda_0 \geq \cdots \geq \lambda_{R-1} \geq 0 \\
R &= \min\{N_t, N_r\}, \quad S = \max\{N_t, N_r\} \\
&U \in \mathbb{C}^{N_r \times N_r}, \quad V \in \mathbb{C}^{N_t \times N_t}
\end{aligned}
$$

In above equations, λ_r denotes the r-th eigenvalue of channel correlation matrix $H^H H$ or HH^H, and the column vectors of matrices U and V are the eigenvectors of $H^H H$ and HH^H which belong to the eigenvalue λ_r. Figure 2 shows that MIMO channels are decomposed into R independent virtual eigenpaths and each eigenvalue expresses the power gain of each path.

To describe the role of eigenvalues in MIMO systems, let us consider an example of the capacity inherent in the MIMO channel. The capacity of MIMO channel is formulated using those eigenvalues as follows:

$$
C = \sum_{r=0}^{R-1} \log_2 (1 + \gamma_r \lambda_r) \tag{1}
$$

where γ_r denotes the signal to noise ratio (SNR) of r-th eigenpath when the transmission power is allocated based on the water filling (WF) theorem [3]. As shown here, the eigenvalues in MIMO systems are important parameters, and the statistical distribution of them has a significant role to evaluate the total performance of the system under time-variant fading.

3. EIGENVALUE DISTRIBUTION

This section deals with the statistical eigenanalysis of MIMO communications system.

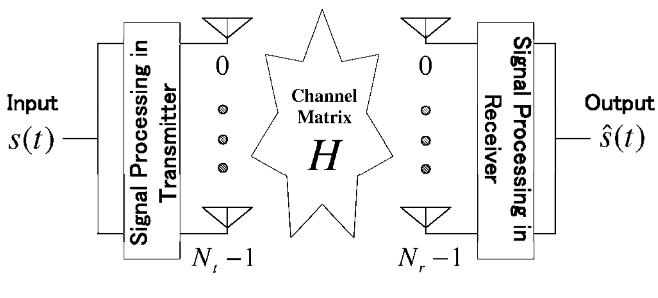


Fig. 1. $N_t \times N_r$ MIMO system.

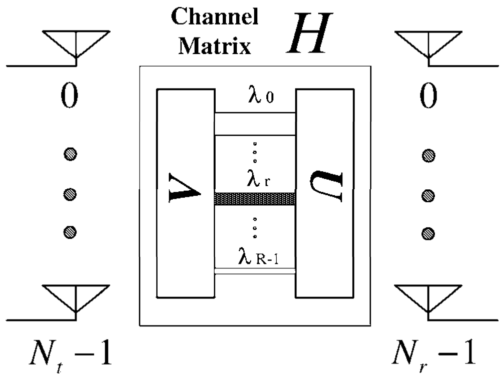


Fig. 2. SVD-based representation of $N_t \times N_r$ MIMO channel.

In case of i.i.d. Rayleigh fading, the channel matrix H is a Gaussian matrix with the zero mean and the variance σ^2 (Assumed to be 1 in this paper), and the channel correlation matrix $W = H^H H$ $(N_t \geq N_r)$ or $W = HH^H$ $(N_t \leq N_r)$ becomes a Wishart matrix [8] with the density function shown by next equation:

$$
p(W) = \frac{|W|^{S-R} \exp(-\mathrm{tr}W)}{\Gamma_R(S)}
$$

where $\Gamma_R(S)$ denotes the multivariate gamma function defined as follows:

$$
\Gamma_R(z) = \int_{A>0} |A|^{z-R} e^{-\mathrm{tr}A}\, dA.
$$

The joint density function of ordered eigenvalues of this matrix is expressed by next equation [4]

$$
\begin{aligned}
p(\lambda_0, \cdots, \lambda_{R-1}) &= \prod_{r=1}^{R} \frac{\lambda_{r-1}^{S-R}}{(R-r)!(S-r)!} \\
&\times \prod_{r<s}^{R-1} (\lambda_r - \lambda_s)^2 \exp\left(-\sum_{r=0}^{R-1} \lambda_r\right)
\end{aligned} \tag{2}
$$

The marginal density function of each eigenvalue is derived by integrating above equation, and the closed form expression for the unordered case is already given in [3]. In the ordered case, however, since the extent of integration changes depending on the variable, the marginal distribution is not easy to calculate and seems to be not explicitly shown except for the case of the largest eigenvalue. The marginal distribution function of the largest eigenvalue λ_0 is shown by next equation [5, 6],

$$
P(\lambda_0 \in [0, \lambda]) = \frac{|G(\lambda)|}{\prod_{r=0}^{R-1} (R-r)!\,(S-r)!} \tag{3}
$$

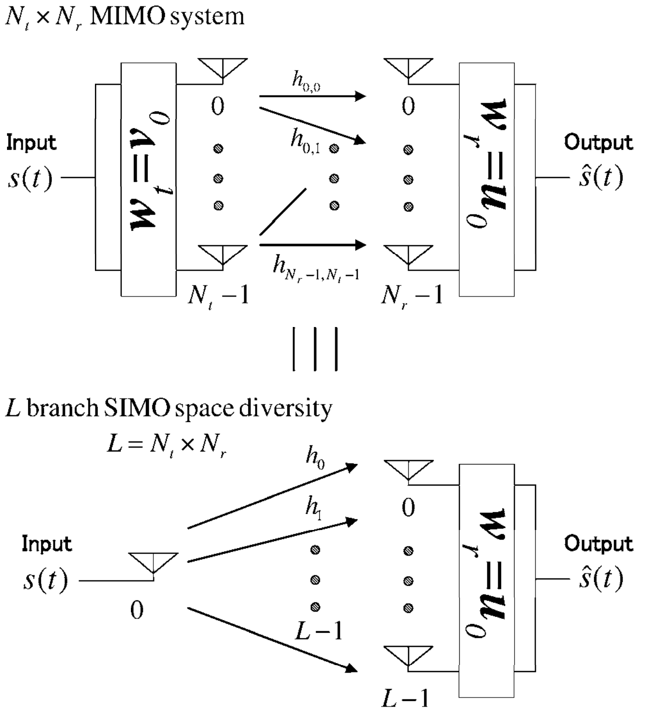


Fig. 3. Approximation of the marginal distribution of eigenvalues in MIMO system based on SIMO space diversity theory.

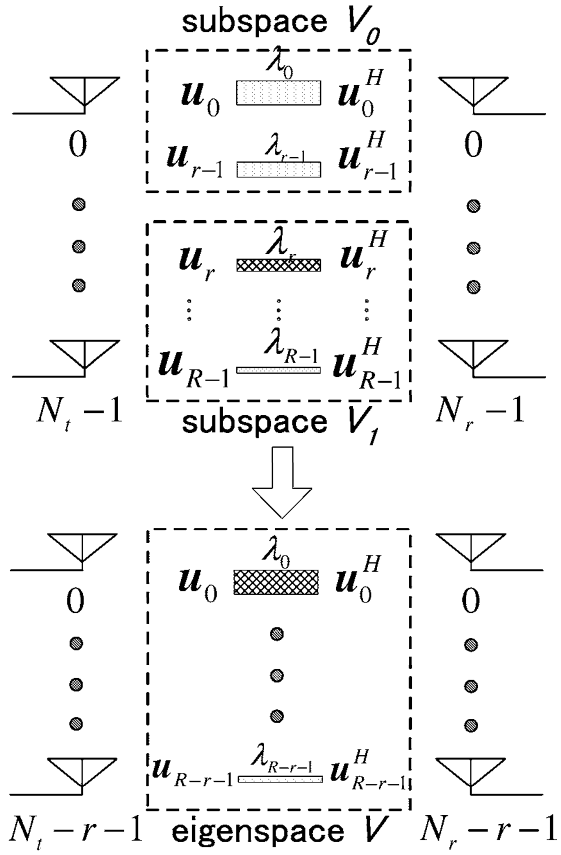


Fig. 4. The subspace V_1 with the largest eigenvalue λ_r could be ascribed to eigenspace V of $(N_t - r) \times (N_r - r)$ MIMO system.

but $G(\lambda) \in \mathbb{R}^{R\times R}$ is a matrix with an (m, n) element of incomplete gamma function $\gamma(S-R+m+n-1, \lambda)$ [9], and has a complicated form not easy to use for the calculations such as capacity and bit error rate (BER). Moreover, the computational load drastically increases in proportion to $R!$ for the expansion of the determinant.

4. APPROXIMATION METHOD

In this section, an approximation formula of the marginal distribution of all the eigenvalues in MIMO systems is proposed based on the theory of space diversity in SIMO systems using MRC scheme.

In reference [7], a simple approximation method of the statistical distribution of the largest eigenvalue has been given. Here, the MRC transmission in MIMO systems are connected with the theory of space diversity in MRC-based SIMO systems (which is also referred to as array antennas).

Let us consider a SIMO system with L diversity branches. Assuming that the means of SNR for all branches are same, the statistical distribution of the total output SNR γ becomes as follows [10]:

$$p(\gamma) = \frac{1}{(L-1)!}\frac{\gamma^{L-1}}{\overline{\gamma}^L}\exp\left(-\frac{\gamma}{\overline{\gamma}}\right)$$

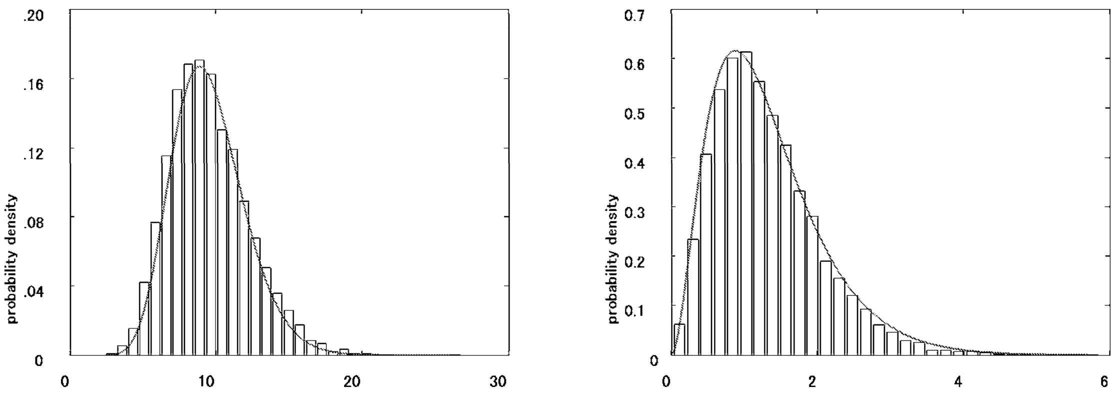
$$P(\gamma) = 1 - \exp\left(-\frac{\gamma}{\overline{\gamma}}\right)\sum_{\ell=0}^{L-1}\frac{(\gamma/\overline{\gamma})^\ell}{\ell!}$$

where $\overline{\gamma}$ denotes the mean of SNR.

As shown in figure 3, the approximation of the statistical distribution of the largest eigenvalue in MIMO system is derived based on the SIMO system with the same number of branches, namely, by putting $L = N_tN_r$ and replacing the mean SNR $\overline{\gamma}$ by the per-branch mean of the largest eigenvalue $\tilde{\lambda}_0$ in above equations.

$$p(\lambda_0) \simeq \frac{1}{(N_tN_r-1)!}\frac{\lambda_0^{N_tN_r-1}}{\tilde{\lambda}_0^{N_tN_r}}\exp\left(-\frac{\lambda_0}{\tilde{\lambda}_0}\right) \quad (4)$$

$$P(\lambda_0) \simeq 1 - \sum_{\ell=0}^{N_tN_r-1}\frac{(\lambda_0/\tilde{\lambda}_0)^\ell}{\ell!}\exp\left(-\frac{\lambda_0}{\tilde{\lambda}_0}\right) \quad (5)$$



(a) Distribution functions $p(\lambda_0)$ (b) Distribution functions $p(\lambda_2)$

Fig. 5. Density functions of the largest and the third largest eigenvalues λ_0 and λ_2 of $(3, 5)$ MIMO communication system.

This paper attempts to extend the result of reference [7] for the approximation of marginal distribution of all the eigenvalues. But the extension of this method to the rest of eigenvalues $\lambda_1 \geq \cdots \geq \lambda_{R-1}$ is not a easy way, since only one eigenvalue could exist in SIMO systems. Consequently, the adoption of a new idea is requred to make the above scheme applicable to those eigenvalues.

After framing some hypothesis and executing trials, the following concept has been introduced for the efficient approximation of all the eigenvalues: First, to develop an equation for the second largest eigenvalue, define a matrix W_1 derived by subtracting the component corresponding to λ_0 from original $R \times R$ Wishart matrix. Here, it is found that the eigenvalue λ_1 is the largest eigenvalue after the true largest eigenvalue λ_0 is removed. Since the dimension of this matrix is decreased by 1 through above operation, W_1 becomes a Wishart matrix with $S - 1$ degrees of freedom derived from a channel matrix with a decreased dimension $R - 1$. Based on this idea, the distribution of λ_1 of $N_t \times N_r$ MIMO system is related to $(N_t - 1) \times (N_r - 1)$ MIMO system (figure 4), and could be estimated from the space diversity theory of $1 \times (N_t - 1)(N_r - 1)$ SIMO systems. Similarly, r-th eigenvalue of $N_t \times N_r$ MIMO system is related to the space diversity of $1 \times (N_t - r)(N_r - r)$ SIMO systems. As a result, the approximations of the statistical distributions are given as follows:

$$p(\lambda_r) \simeq \frac{1}{\{L(r)-1\}!} \frac{\lambda_r^{L(r)-1}}{\tilde{\lambda}_r^{L(r)}} \exp\left(-\frac{\lambda_r}{\tilde{\lambda}_r}\right) \quad (6)$$

$$P(\lambda_r) \simeq 1 - \sum_{\ell=0}^{L(r)-1} \frac{(\lambda_r/\tilde{\lambda}_r)^\ell}{\ell!} \exp\left(-\frac{\lambda_r}{\tilde{\lambda}_r}\right) \quad (7)$$

$$L(r) = (N_t - r)(N_r - r)$$

where $\tilde{\lambda}_r$ denotes the mean of r-th eigenvalue per one diversity branch given by next equation:

$$\tilde{\lambda}_r = \frac{\overline{\lambda}_r}{L(r)} = \frac{1}{L(r)} \int_0^\infty \lambda_r p(\lambda_r)\, d\lambda_r.$$

For large $L(r)$, the next equation could be used:

$$p(\lambda_r) \simeq \frac{1}{\sqrt{2\pi}\{L(r)-1\}^{L(r)-1/2}} \frac{\lambda_r^{L(r)-1}}{\tilde{\lambda}_r^{L(r)}} \times \exp\left(L(r) - 1 - \frac{\lambda_r}{\tilde{\lambda}_r}\right) \quad (8)$$

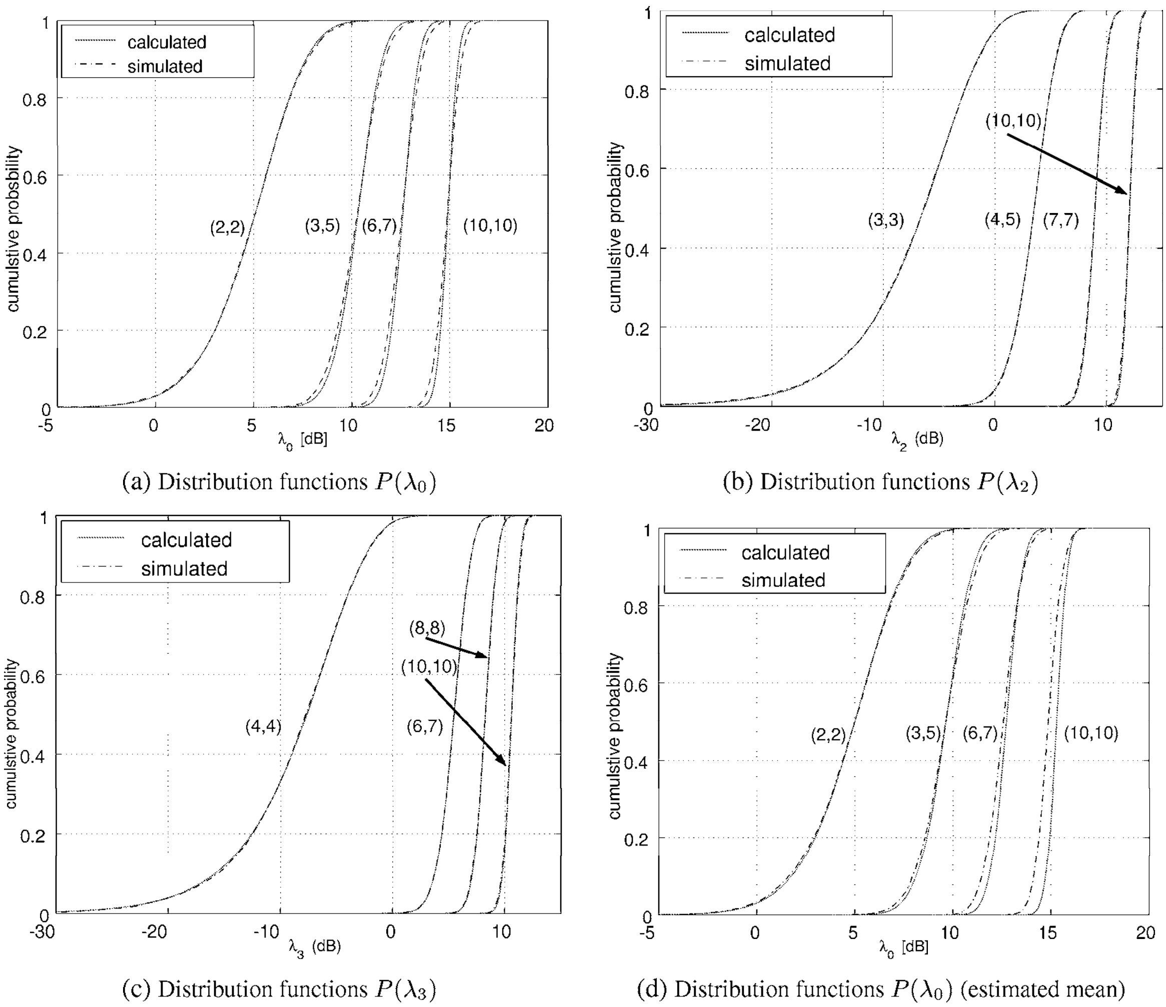
$$P(\lambda_r) \simeq 1 - \sum_{\ell=0}^{L(r)-1} \frac{1}{\sqrt{2\pi}\,\ell^{\ell+1/2}} \left(\frac{\lambda_r}{\tilde{\lambda}_r}\right)^\ell \exp\left(\ell - \frac{\lambda_r}{\tilde{\lambda}_r}\right) \quad (9)$$

For the mean of largest eigenvalue, experimental estimations are already given [2] and [7]:

$$\overline{\lambda}_0 = \begin{cases} N_t N_r \left(\dfrac{N_t + N_r}{N_t N_r + 1}\right)^{2/3} & N_t N_r \leq 250 \\ \left(\sqrt{N_t} + \sqrt{N_r}\right)^2 & N_t N_r > 250 \end{cases} \quad (10)$$

Unfortunately, estimation formulas for eigenvalue means $\overline{\lambda}_1$, $\cdots$, $\overline{\lambda}_{R-1}$ are not yet proposed. Therefore, the calculation of those mean eigenvalues required in above functions is achieved by generating samples of channel matrix H, and calculating the sample mean of them. A sufficient number of samples bring us a precise estimation, though this approach requires computational power for a large R.

The equations derived here provide not only the marginal distribution of all the eigenvalues but also advantages of the simple monomial form suitable for calculation of MIMO performance.



(a) Distribution functions $P(\lambda_0)$

(b) Distribution functions $P(\lambda_2)$

(c) Distribution functions $P(\lambda_3)$

(d) Distribution functions $P(\lambda_0)$ (estimated mean)

Fig. 6. Distribution functions of eigenvalues for (N_t, N_r) MIMO communication systems.

5. SIMULATIONS

In this section, computer simulations are carried out for the verification of the effectiveness of the proposed method.

Figure 5 shows density functions of eigenvalues λ_0 and λ_2 of MIMO system with $(N_t, N_r) = (3, 5)$ derived using equation (8) and histogram obtained through Monte Carlo simulations using 10,000 samples of channel matrix H. The per-branch eigenvalue means $\overline{\lambda}_0$ and $\overline{\lambda}_2$ required in the approximation are calculated from those samples.

Figure 6 plots the curves of the distribution functions for the largest, the third largest, and the fourth largest eigenvalues denoted by λ_0, λ_2, and λ_3, by changing the number of transmitter and receiver antenna elements (N_t, N_r). The solid lines are calculated using equation (9), and broken lines are the results of Monte Carlo simulations. As the mean eigenvalue $\overline{\lambda}_r$ required in (9), the sample mean of 10,000 data are employed for (a) $\sim$ (c), and the estimation by equation (10) is used for the largest eigenvalue in (d). Those graphs show that the approximation proposed in this paper well coincides with the actual data. For the curve of the largest eigenvalue shown in figure 6 (d), the precision of the approximation degrades since the estimation formula (10) is used, but the error is less than 0.5dB which has a small influence on the peformance analysis of MIMO systems. Hence it could be concluded that the proposed equation gives a good approximation of the marginal distribution of ordered eigenvalues in MIMO channel correlation matrices.

Finally, let us consider an exmaple of the application of the proposed method. The curves of mean capacity in MIMO communication system with the same number of tramsmitter and receiver elements ($N_t = N_r$) are plotted in figure 7. The solid lines show the results of multimode

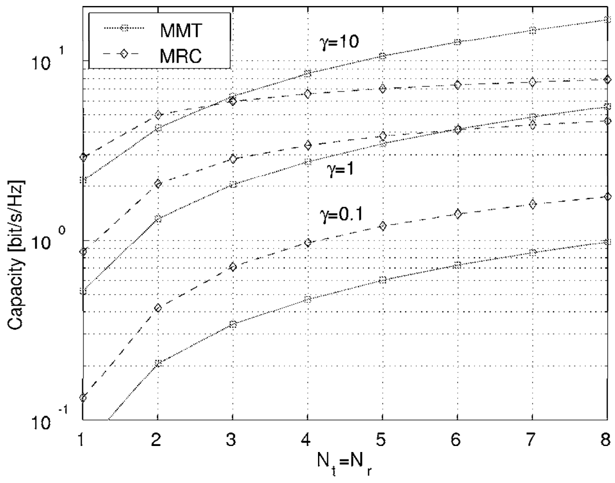


Fig. 7. Mean capacity of MIMO communication systems calculated using the proposed method.

transmission (MMT) with a power allocation so that the SNR γ_r of the eigenpath corresponding λ_r becomes

$$\gamma_r = \frac{\lambda_r}{\sum_m \lambda_m}\gamma$$

in equation (1), where γ denotes the SNR of SISO system under same condition. The broken lines plot the curves of MRC transmission. The results show the performance of above simplified MMT transmission sheme not adopting well known WF power allocation, and its comparison to MRC strategy. It could be seen that the MMT is superir to MRC in case of relatively high SNR as the number of antenna elements increase, and which is the same trend as WF-based MMT. Simultaneously, it could be observed that the advantage of MMT is decreased compared to WF scheme since the MRC transmission is better by condition. As shown here, the proposed method is useful for the theoretical analysis of MIMO systems.

6. CONCLUSIONS

In this paper, an approximation method of the statistical distribution of eigenvalues of correaltion matrices in i.i.d. MIMO Rayleigh channels is proposed. The approximation of the marginal distribution of ordered eigenvalues is achieved based on the theory of space diversity in SIMO systems adopting MRC. Eigenvalue means required in the approximations are derived using estimation formula or sample mean. The results of computer simulations show that the proposed approximation is, in spite of its simple mononial form, effective for the estimation of the actual eigenvalue distribution.

As a future work, the development of a simple procedure for the estimation of correct eigenvalue means is considered. In addition, the analysis in case of correlated MIMO channels also should be carried out. The derivation of the exact marginal function is also a theme to be investigated.

7. REFERENCES

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