ASYMPTOTICS FOR LINEAR PREDICTORS OF STRONGLY DEPENDENT TIME SERIES

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ABSTRACT

This paper assesses the performance of finite-sample predictors as compared to forecasts based on the infinite past, in the context of long-memory processes. We establish the rate at which the autoregressive expansion based on a finite number of past observations for a large class of long-memory processes, including the popular fractional autoregressive moving average model, converges in mean square to the best linear predictor given the entire infinite past, as the number of observations increases to infinity.

1. INTRODUCTION

Let (X_t), $t \in \mathbb{Z}$, be a real, zero-mean, purely nondeterministic weakly stationary process with covariance function (γ_t) and spectral density f. Long-memory in (X_t) occurs when (γ_t) tends to zero at infinity like a power function and so slowly that $\sum_{t=0}^{\infty} |\gamma_t| = \infty$. In the frequency domain, long-range dependence corresponds to the blow-up of f at the origin. Long-memory appears quite frequently in fields as diverse as hydrology and economics, see for example the surveys by Beran (1994), Baillie (1996) and Doukhan et al. (2003). If the probabilistic aspects of long-range dependence (central limits theorems and theirs connections) as well as the statistical aspects related to its detection and its estimation (parametric, semi-parametric, and non-parametric methods) have been well investigated the last twenty years, optimal prediction of long-memory processes has been less studied although it poses several fundamental problems to time series analysts.

First, since in practice only a finite sample is available, the past of X_t is truncated. For short-memory processes this truncation may not affect dramatically the prediction quality since the influence of an observation far into the past may be negligible. On the contrary, for long-memory processes the effect of a distant observation could be large, so that an important difference between $\widehat{X}_t$, the best mean square linear predictor of X_t based on the entire infinite past $\{X_s; s \le t-1\}$, and $\widehat{X}_{t,n}$, the best mean square linear predictor of X_t based on the finite past $\{X_{t-n}, \dots, X_{t-1}\}$, could be expected.

Second, the computation of $\widehat{X}_{t,n}$ is cumbersome since no highly efficient methods for calculating the coefficients $a_{k,n}$ in

$$\widehat{X}_{t,n} = \sum_{k=1}^{n} a_{k,n} X_{t-k},$$

seem to be available yet. For example, while the arithmetic complexity of the innovations algorithm is order n for short-memory autoregressive moving average (ARMA) processes (Brockwell and Davis, 1991, section 5.3), for long-memory processes, all the available methods up to date, including the Durbin-Levinson algorithm, have arithmetic complexity n^2.

Third, the computation of the coefficients $a_{k,n}$ requires the evaluation of (γ_t). But, as described by Sowell (1992), this is highly demanding computationally in the case of a fractionally integrated ARMA (ARFIMA) time series, since this requires the calculation of the Gaussian hypergeometric series.

Apart from the difficulties for calculating the predictors themselves, one-step finite past predictors are also necessary for computing maximum likelihood estimates. For instance, Hasslett and Raftery (1989) have used a combination of optimal and approximate predictors to obtain quasi-maximum likelihood estimates. In the same line, Bhansali and Kokoszka (2003), among others, have approximated $\widehat{X}_{t,n}$ by truncating the AR(∞) expansion of $\widehat{X}_t$,

$$\widehat{X}_t = \sum_{k=1}^{\infty} a_k X_{t-k}, \tag{1}$$

where the series is mean square convergent and (a_k) are the AR(∞) parameters of (X_t). The nth partial sum of (1) gives the predictor

$$S_{t,n} = \sum_{k=1}^{n} a_k X_{t-k},$$

which is suboptimal compared to $\widehat{X}_{t,n}$, but is easier to calculate because the coefficients a_k do not depend on n. This

*This research was supported by grant ECOS/CONICYT C03E07.

technique was also used by Beran (1994, section 5.6) to obtain approximate estimates for an ARFIMA process. Other applications of the AR truncation of an ARFIMA process have been considered by Shumway and Stoffer (2000, section 2.10).

When n tends to infinity, $a_{k,n}$ converges to a_k for all $k \geq 1$ (Pourahmadi, 2001, Theorem 7.14), and both $\widehat{X}_{t,n}$ and $S_{t,n}$ converge in mean square to $\widehat{X}_t$. To the best of our knowledge, the rate of convergence of $S_{t,n}$ to $\widehat{X}_t$ has not been studied in detail. This paper provides an asymptotic formula for $\mathrm{Var}(\widehat{X}_t - S_{t,n})$ that applies to a large class of long-memory processes, including the ARFIMA model and some processes discussed by Giraitis and Surgailis (2002) that are useful in econometrics and finance to model long-memory both in conditional mean and in conditional variance. If d denotes the long-memory parameter of the process, $d \in (0, 1/2)$, we establish that $\mathrm{Var}(\widehat{X}_t - S_{t,n}) \sim d\tan(\pi d)/\pi n$ for large n. Besides its intrinsic theoretical interest for the prediction of a long-memory process, this simple formula has several practical consequences. For instance, it shows that the quality of the AR(∞) truncation depends only on d and n asymptotically and is therefore independent of any short-memory component of the model. Moreover, the rate of convergence is of order $1/n$ and thus does not depend on d. In section 4, simulation results with ARFIMA processes show the good precision of this formula, even for small values of n. This suggests that it may be helpful to determine in practice an appropriate truncation size n to achieve a given level of approximation error.

The remaining of the paper is structured as follows. Section 2 gives an overview of the problem. Section 3 contains the main results and section 4 illustrates the usefulness of the theoretical results by means of numerical experiments. The conclusion is presented in section 5.

2. BACKGROUND

Throughout the paper, $L_2 = L_2(\Omega, \mathcal{F}, \mathsf{P})$ is the Hilbert space of square integrable real random variables defined on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$, endowed with the inner product $\langle X, Y\rangle = \mathrm{E}\,XY$ and norm $\|X\| = \sqrt{\mathrm{E}\,X^2}$, where E stands for the expectation operator. In what follows, convergence of a series of random variables will always be in L_2.

As (X_t) is purely nondeterministic, its Wold decomposition reduces to

$$X_t = \sum_{k=0}^{\infty} c_k \epsilon_{t-k},$$

where (ϵ_t) is the innovation process of (X_t), i.e., $\epsilon_t = X_t - \widehat{X}_t$, the coefficients (c_k) are called the MA(∞) parameters of (X_t), $c_0 = 1$, and $\sum_{k=0}^{\infty} c_k^2 < \infty$. The AR($\infty$) parameters (a_k) of (X_t) are defined recursively by

$$a_0 = -1, \qquad a_k = -\sum_{i=0}^{k-1} a_i c_{k-i}, \qquad (k \geq 1), \qquad (2)$$

and both (a_k) and (c_k) only depend on the Fourier coefficients of $\ln f$. When the sequence (a_k) is summable, the series in (1) converges absolutely in L_2.

The mean square behaviors of $\widehat{X}_{t,n}$ and $S_{t,n}$ as n increases are characterized by the rates of convergence to zero of

$$\delta_n = \mathrm{Var}(X_t - \widehat{X}_{t,n}) - \mathrm{Var}(\epsilon_t) = \mathrm{Var}(\widehat{X}_t - \widehat{X}_{t,n}),$$
$$r_n = \mathrm{Var}(X_t - S_{t,n}) - \mathrm{Var}(\epsilon_t) = \mathrm{Var}(\widehat{X}_t - S_{t,n}). \quad (3)$$

It results directly from these definitions that $\delta_n \leq r_n$. Without lost of generality, we will suppose in the following that (X_t) is normalized such that $\mathrm{Var}(\epsilon_t) = 1$.

The asymptotic behaviors of δ_n and r_n differ whether (X_t) is a short or a long-memory process. A typical short-memory process is the ARMA model, and a general class of long-memory processes is the ARFIMA model. This model was introduced by Granger and Joyeux (1980) and Hosking (1981) and has been used to describe long-memory phenomena in a wide variety of scientific disciplines, from hydrology to economics. More precisely, (X_t) is called an long-memory ARFIMA(p, d, q) process if (X_t) satisfies the difference equation

$$\phi(B)(1-B)^d X_t = \theta(B) Z_t, \qquad (4)$$

where B is the backward shift operator, $BX_t = X_{t-1}$, (Z_t) is a sequence of uncorrelated random variables in L_2 ($\mathrm{Var}(Z_t) = 1$ without lost of generality), $d \in (0, 1/2)$, ϕ and θ are polynomials of degrees p, q respectively without common zeros and without zeroes in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$. Under these conditions, the (unique) stationary solution (X_t) of (4) is purely nondeterministic, (Z_t) is the innovation process of (X_t), $\widehat{X}_t$ has the AR(∞) representation (1), the MA(∞) and AR(∞) parameters of (X_t) are the coefficients in the Taylor series expansion of $h(z) = (1-z)^{-d}\theta(z)/\phi(z)$ and $-1/h(z)$ for $|z| < 1$, respectively, $\gamma_n \sim C_1 n^{2d-1}$ as $n \to \infty$ and $f(\lambda) \sim C_2 |\lambda|^{-2d}$ as $\lambda \to 0$, where C_1, C_2 are some constants, see for instance Brockwell and Davis (1991, section 13.2). Then (γ_t) is not summable and f has a pole at the origin. If $\phi(z) = \theta(z) = 1$, (X_t) is called a fractional noise. Lastly, when $d = 0$ in (4), (X_t) is the usual ARMA(p, q) short-memory process, $\gamma_n = O(\alpha^n)$ for some $\alpha \in (0, 1)$ and f is bounded and bounded away from zero.

The convergence rate of δ_n for different classes of spectral densities has been considered by many authors. Typically, if f satisfies the boundedness condition

$$0 < c \leq f \leq C < \infty, \qquad (5)$$

and some additional smoothness conditions, δ_n may converge at least exponentially: $\delta_n = O(\alpha^n)$ for some $\alpha \in (0,1)$ (Grenander and Rosenblatt, 1954, Theorem 4), or δ_n may converge at least hyperbolically: $\delta_n = O(n^{-2p})$ with $p > 1/2$, p non-integer (Ibragimov, 1964, Theorem 1). For instance, if (X_t) is an ARMA process, δ_n converges at least exponentially. If f does not comply with (5) but takes the form

$$f(\lambda) = f_1(\lambda) \prod_{j=1}^{m} |e^{i\lambda_j} - e^{i\lambda}|^{-2d_j},$$

where f_1 is positive and satisfies a Lipschitz condition of order $\alpha \geq 1/2$, (λ_j) are distinct values in $[-\pi, \pi]$, and all $d_j < 1/2$, $d_j \neq 0$, then Ibragimov and Solev (1968) have shown that $\delta_n \asymp 1/n$, i.e., $c/n \leq \delta_n \leq C/n$ for some positive constants c and C. Under different smoothness conditions on f_1, Ginovian (1999) has obtained similar results. Now, Inoue (2000, Theorem 6.4) has established the precise rate of convergence $\delta_n \sim d^2/n$ for a particular class of processes satisfying $\gamma_n \sim n^{2d-1} l(n)$ where $d \in (0, 1/2)$ and $l \in \mathcal{R}_0$, with $\mathcal{R}_0$ the set of slowly varying functions at infinity: the set of positive, measurable functions l, defined on some neighborhood $[A, \infty)$ of infinity, such that, for any $\lambda > 0$, $l(\lambda x)/l(x) \to 1$ as $x \to \infty$ (Bingham et al., 1987, chapter I). Since the class of processes considered by Inoue (2000) does not contain the ARFIMA model, Inoue (2002, Theorem 4.3) has shown that the asymptotic behavior $\delta_n \sim d^2/n$ holds also for ARFIMA processes with $d \in (0, 1/2)$.

The study of the asymptotic behavior of r_n is relevant only when the AR(∞) expansion (1) exists. If f satisfies (5), this is actually the case and $\delta_n \leq r_n \leq \delta_n C/c$ (Bondon and Palma, 2004, Theorem 2). Hence, δ_n and r_n have convergence rates of the same order. If f is unbounded or has zeros, the series (1) may not converge in mean square and the relation $r_n \asymp \delta_n$ does not necessarily hold. In this case, the asymptotic behavior of r_n does not derive from those of δ_n. As far as we know, the asymptotic behavior of r_n has not been studied for strongly dependent processes. This, despite the great attention that AR truncation approaches have attracted in the long-memory context.

3. MAIN RESULTS

We establish the rate of convergence to zero of r_n as $n \to \infty$ for the class of processes (X_t) whose MA(∞) and AR(∞) parameters satisfy, respectively,

$$c_n \sim n^{-(1-d)} l(n) \Gamma(d)^{-1}, \qquad (n \to \infty), \qquad (6)$$

$$a_n \sim -n^{-(1+d)} l(n)^{-1} \Gamma(-d)^{-1}, \qquad (n \to \infty), \qquad (7)$$

where $l \in \mathcal{R}_0$, $d \in (0, 1/2)$, and Γ is the Gamma function. This class includes the ARFIMA model, as shown by Theorem 1 which is a consequence of Kokoszka and Taqqu (1995, Corollary 3.1).

Theorem 1. The MA(∞) and AR(∞) parameters of an ARFIMA process with $d \in (0, 1/2)$ satisfy (6) and (7), respectively, with the constant function $l(x) = \theta(1)/\phi(1)$.

We establish in Theorem 2 the long-memory character of the processes satisfying (6).

Theorem 2. Let $l \in \mathcal{R}_0$ and $d \in (0, 1/2)$. Condition (6) implies that

$$\gamma_n \sim n^{2d-1} l(n)^2 \frac{\Gamma(1-2d)}{\Gamma(1-d)\Gamma(d)}, \qquad (n \to \infty). \qquad (8)$$

If, in addition, $l_0(k) = \gamma_n n^{1-2d}$ is quasi-monotone slowly varying at infinity, we have

$$f(\lambda) \sim \frac{1}{2\pi} \lambda^{-2d} l_0(1/\lambda) \frac{\Gamma(1-d)\Gamma(d)}{\Gamma(1-2d)}, \quad (\lambda \to 0_+). \quad (9)$$

Proof. (8) results from the equality $\gamma_n = \sum_{k=0}^{\infty} c_k c_{k+|n|}$, (6), and a technical lemma in Inoue (1997). (9) is obtained using an Abelian-type result in Bingham et al. (1987). □

Either (8) or (9) where $d \in (0, 1/2)$ is a common definition of long-range dependence, see Taqqu (2003). On the other hand, a process with completely monotone covariance function satisfying (8) also satisfies (6) and (7) (Inoue, 2000, Theorems 5.1 and 7.3).

Under condition (7), the sequence (a_k) is summable, and therefore, the series in (1) converges absolutely in L_2. Furthermore, we deduce from Bingham et al. (1987, Proposition 1.5.10) that

$$\sum_{k=n+1}^{\infty} |a_k| \sim n^{-d} l(n)^{-1} \Gamma(1-d)^{-1}, \quad (n \to \infty). \quad (10)$$

According to (3),

$$r_n = \operatorname{Var}\Big(\sum_{k=n+1}^{\infty} a_k X_{t-k} \Big) \leq \gamma_0 \left(\sum_{k=n+1}^{\infty} |a_k| \right)^2,$$

and then (10) implies that $r_n = O(n^{-2d} l(n)^{-2})$ as $n \to \infty$, which in turn implies that $r_n = o(n^{-2d+\epsilon})$ for any $\epsilon > 0$. In the case of a short memory process of the ARMA type, we have $|a_n| = O(\alpha^n)$ for some $\alpha \in (0,1)$, and then $r_n = O(\alpha^n)$.

Theorem 3 states that conditions (6) and (7) imply the precise convergence rate (11) for r_n.

Theorem 3. Let $l \in \mathcal{R}_0$ and $d \in (0, 1/2)$. Conditions (6) and (7) imply that

$$r_n \sim \frac{d \tan(\pi d)}{\pi n}, \qquad (n \to \infty). \qquad (11)$$

Remark 1. According to (11), r_n increases to infinity as the long-memory parameter d tends to $1/2$. On the other hand, for any ARFIMA(p, d, q) process with $d \in (0, 1/2)$, its results from Inoue (2002, Theorem 4.3) that $\delta_n \sim d^2/n$, and then δ_n remains bounded as d tends to $1/2$.

4. NUMERICAL EXPERIMENTS

Since the calculation of exact predictors and parameter estimates in the context of long-memory models is highly demanding computationally, many authors have explored alternative methods involving AR approximations. In this section, we discuss the practical usefulness of (11) to assess the quality of these AR approximation-based methods. To study the accuracy of (11) as a function of the truncation size n, we need to calculate the exact value of r_n. For this, we deduce from (1) that

$$\sum_{k=n+1}^{\infty} a_k X_{t-k} = -\epsilon_t - \sum_{k=0}^{n} a_k X_{t-k},$$

which implies that

$$r_n = \sum_{k=0}^{n} \sum_{l=0}^{n} a_k a_l \gamma_{l-k} - 1. \tag{12}$$

Consider the ARFIMA model (4) with $d \in (0, 1/2)$. It is well known that (X_t) can be regarded as the ARMA(p, q) process $\phi(B)X_t = \theta(B)Y_t$ driven by the fractional noise $Y_t = (1 - B)^{-d} Z_t$. Let (a_k^Y) and (γ_k^Y) be the AR(∞) parameters and the covariance function of (Y_t). Then,

$$a_k = \sum_{j=0}^{k} \widetilde{a}_j a_{k-j}^Y, \qquad (k \geq 0), \tag{13}$$

$$\gamma_k = \sum_{j=-\infty}^{\infty} \widetilde{\gamma}_j \gamma_{k-j}^Y, \qquad (k \geq 0), \tag{14}$$

where $\widetilde{a}_j$ is the j-th coefficient in the Taylor series expansion of $\phi(z)/\theta(z)$ for $|z| < 1$, $(\widetilde{\gamma}_j)$ is the covariance function of the ARMA(p, q) model $\phi(B)X_t = \theta(B)W_t$ where (W_t) is an uncorrelated sequence with zero mean and unit variance, and

$$a_k^Y = -\frac{\Gamma(k - d)}{\Gamma(k + 1)\Gamma(-d)}, \qquad (k \geq 0),$$

$$\gamma_k^Y = \frac{\Gamma(k + d)\Gamma(1 - 2d)}{\Gamma(k + 1 - d)\Gamma(1 - d)\Gamma(d)}, \qquad (k \geq 0),$$

see Brockwell and Davis (1991, section 13.2). Exact numerical values of $\widetilde{a}_j$ and $\widetilde{\gamma}_j$ can be obtained recursively. Since $\widetilde{\gamma}_j$ converges to zero at least exponentially as $|j|$ tends to infinity and $|\gamma_k^Y| \leq \gamma_0^Y$, a very good approximation of γ_k can be obtained by truncating the series (14). Inserting in (12) the numerical values given by (13) and a truncated version of (14), r_n can be calculated for any ARFIMA(p, d, q) process.

In the simulations (X_t) is the ARFIMA$(1, d, 1)$ model defined by the difference equation

$$(1 - \phi B)(1 - B)^d X_t = (1 + \theta B) Z_t.$$

Let

$$\alpha(n, \phi, d, \theta) = \frac{\pi n r_n}{d \tan(\pi d)}.$$

Figure 1 shows $\alpha(n, \phi, d, \theta)$ for $n \in \{1, \dots, 30\}$ and several values of (ϕ, d, θ). Observe that $\alpha(n, \phi, d, \theta)$ tends to 1 rapidly in all cases. For example, we have $|\alpha(n, \phi, d, \theta) - 1| \leq 0.2$ for any $n \geq 13$ and the combination of parameters considered.

For a fractional noise, Figure 2 shows $\alpha(n, 0, d, 0)$ for $n \in \{1, \dots, 60\}$ and several values of d. We observe that $\alpha(n, 0, d, 0)$ tends to 1 more rapidly than in the case of an ARFIMA$(1, d, 1)$ process since we have $|\alpha(n, 0, d, 0) - 1| \leq 0.05$ for any $n \geq 10$ and the parameters d considered.

Figure 3 displays the value of n such that $|\alpha(n, 0, d, 0) - 1| = \epsilon$ for $d \in (0, 1/2)$ and $\epsilon \in \{2\%, \dots, 5\%\}$. We see that this value of n is quite small, even for small ϵ.

Due to the good precision of the asymptotic relation (11) observed in the numerical experiments, even for small values of n, it seems to be helpful to determine in practice a truncation size n such that r_n does not exceed a given percentage of the innovation variance.

5. CONCLUSION

In this paper, we have discussed the rate at which the truncated AR(∞) expansion converges in mean square to its exact value as the size of the past increases to infinity. The results are applicable to a large class of second-order stationary long-memory processes including the ARFIMA model. The considered processes are characterized by the asymptotic behavior of their MA(∞) and AR(∞) parameters. If these parameters behave like those of a fractional noise up to a slowly varying function, we have obtained an equivalence formula for the rate of convergence. This formula is proportional to the inverse of the size of the past, and the multiplicative constant depends only on the long-memory parameter. Simulations results have shown the good precision of this formula, and we expect it would be useful in practice to assess the performance of both estimation and prediction techniques in the context of long-memory modeling.

References

R. T. Baillie. Long memory processes and fractional integration in econometrics. *J. Econometrics*, 73(1):5–59, 1996.

J. Beran. *Statistics for Long-Memory Processes*, volume 61 of *Monographs on Statistics and Applied Probability*. Chapman and Hall, New York, 1994.

R. J. Bhansali and P. S. Kokoszka. Prediction of long-memory time series. In *Theory and Applications of Long-Range Dependence*, pages 355–367. Birkhäuser Boston, Boston, MA, 2003.

N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1987.

P. Bondon and W. Palma. Asymptotics for linear predictors of long-memory processes. Technical report, 2004.

P. J. Brockwell and R. A. Davis. *Time Series: Theory and Methods*. Springer Verlag, New York, second edition, 1991.

P. Doukhan, G. Oppenheim, and M. S. Taqqu, editors. *Theory and Applications of Long-Range Dependence*. Birkhäuser Boston Inc., Boston, MA, 2003.

M. S. Ginovian. Asymptotic behavior of the prediction error for stationary random sequences. *J. Contemp. Math. Anal.*, 34(1): 14–33, 1999.

L. Giraitis and D. Surgailis. ARCH-type bilinear models with double long memory. *Stochastic Process. Appl.*, 100:275–300, 2002.

C. W. J. Granger and R. Joyeux. An introduction to long-memory time series models and fractional differencing. *J. Time Ser. Anal.*, 1(1):15–29, 1980.

U. Grenander and M. Rosenblatt. An extension of a theorem of G. Szegö and its application to the study of stochastic processes. *Trans. Amer. Math. Soc.*, 76:112–126, 1954.

J. Hasslett and A. E. Raftery. Space-time modelling with long-memory dependence: Assessing ireland's wind power resource. invited paper with discussion. *J. Appl. Statist.*, 38:1–50, 1989.

J. R. M. Hosking. Fractional differencing. *Biometrika*, 68(1):165–176, 1981.

I. A. Ibragimov. On the asymptotic behavior of the prediction error. *Theory Probab. Appl.*, 9(4):627–633, 1964.

I. A. Ibragimov and V. N. Solev. The asymptotic behavior of the prediction error of a stationary sequence with a spectral density of special type. *Theory Probab. Appl.*, 13(4):703–707, 1968.

A. Inoue. Regularly varying correlation functions and KMO-Langevin equations. *Hokkaido Math. J.*, 26(2):457–482, 1997.

A. Inoue. Asymptotics for the partial autocorrelation function of a stationary process. *J. Anal. Math*, 81:65–109, 2000.

A. Inoue. Asymptotic behavior for partial autocorrelation functions of fractional ARIMA processes. *Ann. Appl. Probab.*, 12 (4):1471–1491, 2002.

P. S. Kokoszka and M. S. Taqqu. Fractional ARIMA with stable innovations. *Stochastic Process. Appl.*, 60(1):19–47, 1995.

M. Pourahmadi. *Foundations of Time Series Analysis and Prediction Theory*. Wiley, New York, 2001.

R. H. Shumway and D. S. Stoffer. *Time Series Analysis and its Applications*. Springer Verlag, New York, 2000.

F. Sowell. Maximum likelihood estimation of stationary univariate fractionally integrated time series models. *J. Econometrics*, 53 (1-3):165–188, 1992.

M. S. Taqqu. Fractional Brownian motion and long-range dependence. In *Theory and applications of long-range dependence*, pages 5–38. Birkhäuser Boston, Boston, MA, 2003.

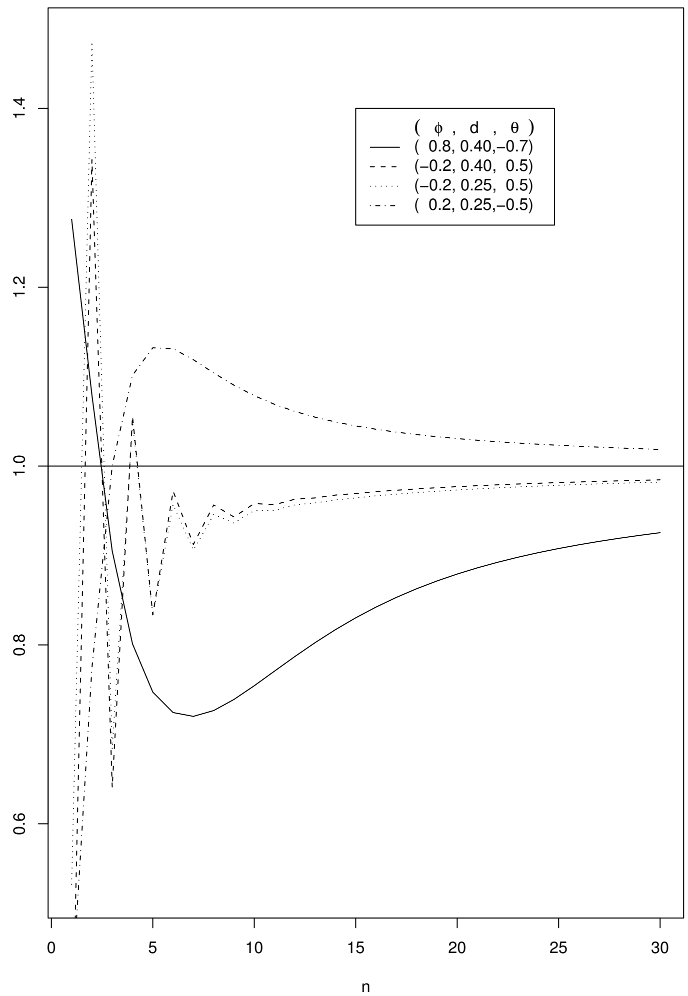

Fig. 1. Ratio $r_n/[d\tan(\pi d)/\pi n]$ for an ARFIMA$(1, d, 1)$ process with parameters (ϕ, d, θ).

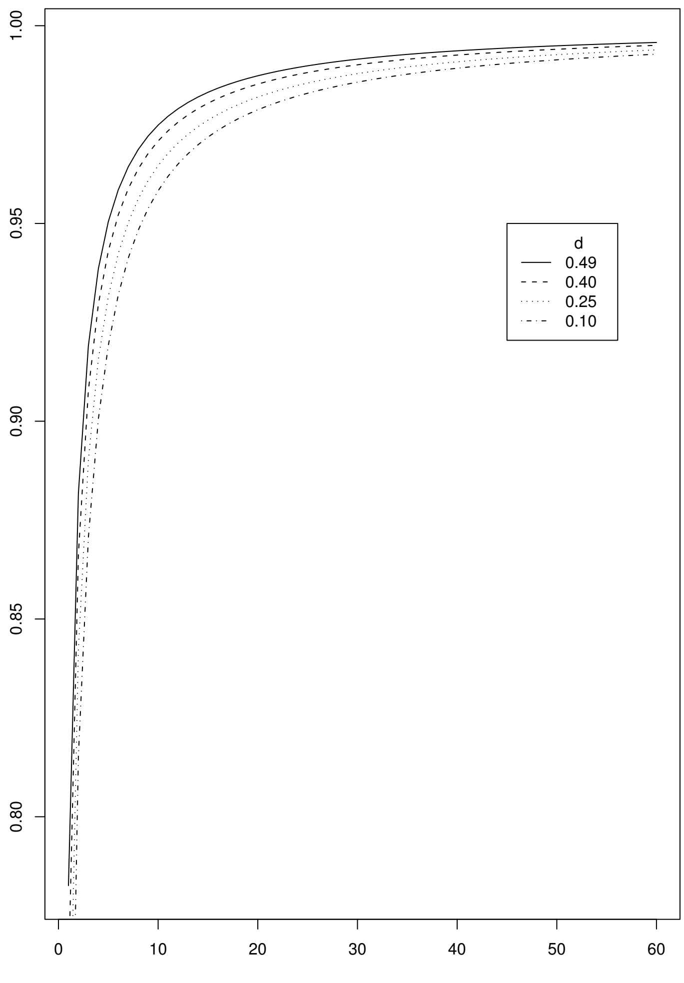

Fig. 2. Ratio $r_n/[d\tan(\pi d)/\pi n]$ for a fractional noise with long-memory parameter d.

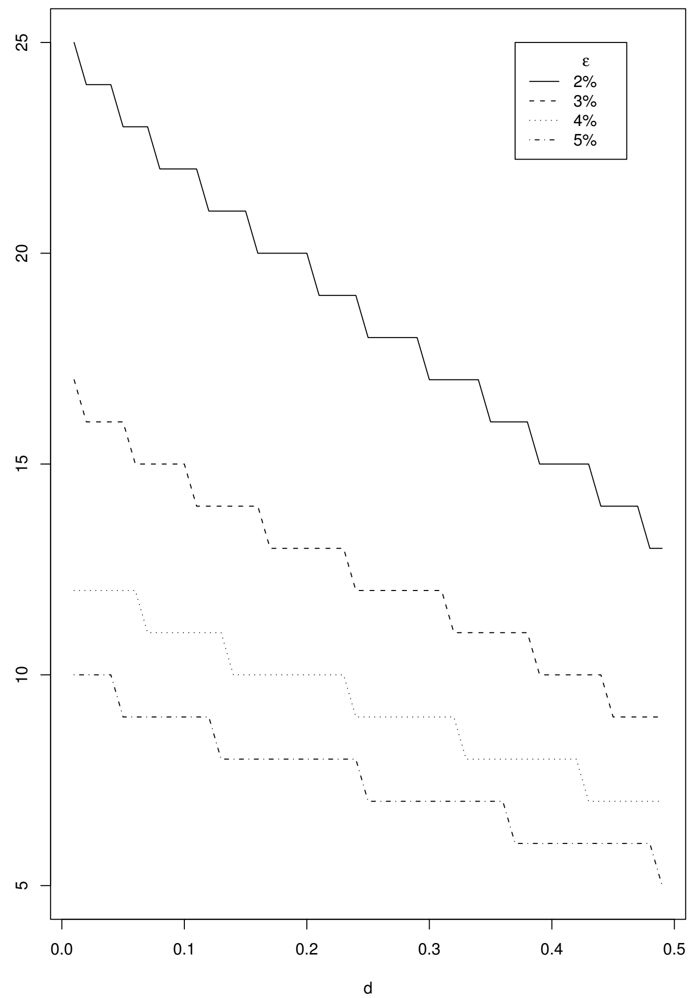

Fig. 3. Truncation size n such that $r_n/[d\tan(\pi d)/\pi n] = 1 \pm \epsilon$ for a fractional noise with long-memory parameter d.