PARAMETRIC ESTIMATION OF MULTI-DIMENSIONAL AFFINE TRANSFORMATIONS IN THE PRESENCE OF NOISE: A LINEAR SOLUTION

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ABSTRACT

We consider the general framework of planar object registration and recognition based on a set of known templates. While the set of templates is known, the tremendous set of possible affine transformations that may relate the template and the observed signature, makes any detection and recognition problem ill-defined unless this variability is taken into account. Given a noisy observation on one of the known objects, subject to an unknown affine transformation of it, our goal is to estimate the deformation that transforms some prechosen representation of this object (template) into the current observation. We propose a method that employs a set of nonlinear operators to replace the original high dimensional and non-linear problem by an *equivalent linear least-squares problem***, expressed in terms of the unknown affine transformation parameters. The proposed solution is** *unique* **and is applicable to any affine transformation** *regardless of the magnitude of the deformation***.**

1. INTRODUCTION

This paper is concerned with the general problem of object recognition and registration based on a set of known templates. However, while the set of templates is known, the variability associated with the object, such as its location and pose in the observed scene, or its deformation are unknown *a-priori*, and only the group of actions causing this variability in the observation, can be defined. This huge variability in the object signature (for any single object) due to the tremendous set of possible deformations that may relate the template and the observed signature, makes any detection and recognition problem ill-defined unless this variability is taken into account. In other words, implicit or explicit registration of the observed object signature with respect to any template in an indexed set is an inherent and essential part of the solution to any detection and recognition problem.

To enable a rigorous treatment of the problem we begin by defining the "similarity criterion". Let G be a group and S be a set (a function space in our case), such that G acts as a transformation group on S. The action of G on S is defined by $G \times S \to S$ such that for every $\phi \in G$ and every $s \in S$, $(\phi, s) \to s \circ \phi$ (composition of functions on the right), where $s \circ \phi \in S$. From this point of view, given two functions h and g on the same orbit, the initial task (that enables recognition in a second stage), is to find the element ϕ in G that makes h and g identical in the sense that $h = g \circ \phi$.

In this paper we concentrate on parametric modelling and estimation of affine transformations in the presence of noise. This problem is a special case of the general problem of estimating deformations belonging to the homeomorphism group. Theoretically, in the absence of noise, the solution to the recognition problem is obtained by applying each of the deformations in the group to the template, followed by comparing the result to the observed realization. In the absence of noise, application of one of the deformations to the template yields an image, identical to the observation. Thus the procedure of searching for the deformation that transforms g into h is achieved, in principle, by a mapping from the group (the affine group, in our case) to the space of functions defined by the orbit of g. However, as the number of such possible deformations is infinite, this direct approach is computationally prohibitive. Hence, more sophisticated methods capable of handling noisy observations are essential.

2. ESTIMATION OF MULTIDIMENSIONAL AFFINE TRANSFORMATIONS: PROBLEM DEFINITION

The basic problem addressed in this paper is the following: Given two bounded, Lebesgue measurable functions h, g with compact supports (and with no affine symmetry, as rigorously defined below) such that $h : R^n \to R$, $g : R^n \to R$ where

$$h(\mathbf{x}) = g(\mathbf{A}\mathbf{x}) + \eta(\mathbf{x}) \ , \quad \mathbf{A} \in GL_n(R), \ \mathbf{x} \in R^n \qquad (1)$$

and $\eta(\mathbf{x})$ denotes the observation noise, find the matrix $\mathbf{A}$.

Let $M(R^n, R)$ denote the space of compact support, bounded, and Lebesgue measurable functions from R^n to R. Let $N \subset M(R^n, R)$ denote the set of measurable functions with an affine symmetry (or affine invariance), *i.e.*, $N = \{f \in M(R^n, R) | \exists \mathbf{A} \in \mathrm{GL}_n(R), \mathbf{A} \neq \mathbf{I} \text{ such that} f(\mathbf{x}) = f(\mathbf{A}\mathbf{x}) \text{ for every } \mathbf{x} \in R^n\}$. Let $M_{Aff}(R^n, R) \triangleq M(R^n, R) \setminus N$ denote the set of compact support and bounded Lebesgue measurable functions with no affine symmetry. Clearly $M_{Aff}(R^n, R)$ is closed with respect to the affine group operation, hence, if $g \in M_{Aff}(R^n, R)$ then its entire orbit is also is $M_{Aff}(R^n, R)$.

In the following sections we show that the problem of finding the parameters of the unknown affine transformation, whose direct solution requires a highly complex search in a function space, can be formulated as a *parameter estimation problem*. Moreover, it is shown that the original problem can be formulated in terms of an *equivalent* problem which is expressed in the form of a *linear* system of equations in the unknown parameters of the affine transformation. A least squares solution of this linear system of equations provides *the* unknown transformation parameters.

3. AN ALGORITHMIC SOLUTION

In this section we provide a constructive proof showing that given a noisy observation $h(\mathbf{x}) \in M_{Aff}(R^n, R)$ and an observation on $g(\mathbf{x}) \in M_{Aff}(R^n, R)$ where $h(\mathbf{x}) = g(\mathbf{A}\mathbf{x}) + \eta(\mathbf{x})$, $\mathbf{A}$ can be *uniquely* determined.

Let, $\mathbf{x}, \mathbf{y} \in R^n$, *i.e.*, $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$, $\mathbf{y} = [y_1, y_2, \ldots, y_n]^T$. Thus,

$$\mathbf{y} = \mathbf{A}\mathbf{x} \ \ , \mathbf{x} = \mathbf{A}^{-1}\mathbf{y} \tag{2}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad \mathbf{A}^{-1} = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix}$$

Since $\mathbf{A} \in GL_n(R)$, also $\mathbf{A}^{-1} \in GL_n(R)$. It is therefore possible to solve for $\mathbf{A}^{-1}$ and the solution for $\mathbf{A}$ is guaranteed to be in $GL_n(R)$. Moreover, as shown below, in the proposed procedure the transformation Jacobian is evaluated first, and by a different procedure than the one employed to estimate the elements of $\mathbf{A}^{-1}$. Hence, a non-zero Jacobian guarantees the existence of an inverse to the transformation matrix.

Let $f \in M_{Aff}(R^n, R)$ and let μ_n denote the Lebesgue measure on R^n. Define the notation

$$\int_{R^n} f \triangleq \int_{R^n} f d\mu_n$$

Note that in the following derivation it is assumed that the functions are bounded and have compact support, as they are measurable but not necessarily continuous. It is further assumed that $\mathbf{A} \in GL_n(R)$ has a positive determinant.

The first step in the solution is to find the Jacobian of the linear transformation $\mathbf{A}$. Applying a family of Lebesgue measurable, left-hand compositions $\{w_\ell\} : R \rightarrow R$ to the known relation $h(\mathbf{x}) = g(\mathbf{A}\mathbf{x}) + \eta(\mathbf{x})$ and integrating over both sides of the equality, we obtain

$$\int_{R^n} w_k \circ h(\mathbf{x}) = \int_{R^n} w_k \circ g(\mathbf{A}\mathbf{x}) + \epsilon_k = \ \mathbf{A}^{-1} \int_{R^n} w_k \circ g(\mathbf{y}) + \epsilon_k \tag{3}$$

where ϵ_k is the error term due to the noise contribution.

Thus, (3) produces the overdetermined system

$$\begin{pmatrix} \int_{R^n} w_1 \circ h(\mathbf{x}) \\ \vdots \\ \int_{R^n} w_p \circ h(\mathbf{x}) \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} \int_{R^n} w_1 \circ g(\mathbf{y}) \\ \vdots \\ \int_{R^n} w_p \circ g(\mathbf{y}) \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_p \end{pmatrix} \tag{4}$$

which by a least squares solution provides an estimate for $|\mathbf{A}|$.

In the second stage we prove that, provided that g is "rich" enough in a sense we rigorously define below, $\mathbf{A}$ can be *uniquely* estimated by establishing an overdetermined system of linear equations in the n unknown elements in each of its rows. More specifically, let $(\mathbf{A})_k$ denote the kth row of $\mathbf{A}$. Applying a family of Lebesgue measurable, left-hand compositions $\{w_\ell\} : R \rightarrow R$ to the known relation $h(\mathbf{x}) = g(\mathbf{A}\mathbf{x}) + \eta(\mathbf{x})$ and integrating over both sides of the equality, we obtain

$$\int_{R^n} x_k w_p \circ h(\mathbf{x}) = \int_{R^n} x_k w_p \circ [g(\mathbf{A}\mathbf{x}) + \eta(\mathbf{x})] \tag{5}$$

Let $\epsilon^g_{p,k}$ denote the error term in each equation due to the presence of the noise, *i.e.*,

$$\epsilon^g_{p,k} = \int_{R^n} x_k \{w_p \circ [g(\mathbf{A}\mathbf{x}) + \eta(\mathbf{x})] - w_p \circ g(\mathbf{A}\mathbf{x})\} \tag{6}$$

Rewriting (5) using (6) we have

$$\begin{aligned} \int_{R^n} x_k w_p \circ h(\mathbf{x}) &= \mathbf{A}^{-1} \int_{R^n} ((\mathbf{A}^{-1})_k \mathbf{y}) w_p \circ g(\mathbf{y}) + \epsilon^g_{p,k} \\ &= \mathbf{A}^{-1} \sum_{i=1}^{n} q_{ki} \int_{R^n} y_i w_p \circ g(\mathbf{y}) + \epsilon^g_{p,k} \\ & \qquad p = 1, \ldots, P \end{aligned} \tag{7}$$

Let

$$\mathbf{G} = \begin{bmatrix} \int_{R^n} w_1 \circ g(\mathbf{y}) & \int_{R^n} y_1 w_1 \circ g(\mathbf{y}) & \cdots & \int_{R^n} y_n w_1 \circ g(\mathbf{y}) \\ \vdots & \ddots & \vdots & \\ \int_{R^n} w_P \circ g(\mathbf{y}) & \int_{R^n} y_1 w_P \circ g(\mathbf{y}) & \cdots & \int_{R^n} y_n w_P \circ g(\mathbf{y}) \end{bmatrix}$$

Rewriting (7) in a matrix form

$$\mathbf{G} \begin{bmatrix} q_{k1} \\ \vdots \\ q_{kn} \end{bmatrix} - \begin{bmatrix} |\mathbf{A}| \int_{R^n} x_k (w_1 \circ h(\mathbf{x})) \\ \vdots \\ |\mathbf{A}| \int_{R^n} x_k (w_P \circ h(\mathbf{x})) \end{bmatrix} = \begin{bmatrix} \epsilon^g_{1,k} \\ \vdots \\ \epsilon^g_{P,k} \end{bmatrix} \tag{8}$$

The system (8) represents a linear regression problem where the noise sequence $\{\epsilon^g_{p,k}\}$ is, in general, non-stationary since its statistics depend on the choice of w_p for each p. The regressors are functions of w_p and the template g, and hence are *deterministic*. Provided that the sequence of composition functions $\{w_p\}_{p=1}^P$ is chosen such that the resulting regressors matrix is full rank, the system (8) is solved by a linear least squares method such that the l_2 norm of the noise vector is minimized. Similar system of equations is solved for each k to obtain all the elements of $\mathbf{A}$. Hence we have the following:

Theorem 1 *Let $\mathbf{A} \in GL_n(R)$. Assume $h, g \in M_{Aff}(R^n, R)$ such that $h(\mathbf{x}) = g(\mathbf{A}\mathbf{x}) + \eta(\mathbf{x})$. Given measurements of h and g, then $\mathbf{A}$ can be uniquely determined if there exists a set of Lebesgue measurable functions $\{w_\ell\}_{\ell=1}^p$, $p \geq n$, such that the matrix $\mathbf{G}$ defined above, is full rank.*

The dependence of the noise sequence $\{\epsilon^g_{p,k}\}$ on the choice of w_p suggests that different choices of the composition sequence $\{w_p\}_{p=1}^P$ may provide different solutions. We shall be first interested in systems such that for each p, the linear constraint imposed by w_p is such that the "effective noise" that corresponds to each w_p is zero mean.

3.1. Construction of Linear Constraints with Zero-Mean Error Terms

Consider the case where we choose $w_p(x) = \sum_k \alpha_k^p x^k$. We next evaluate the mean term of the Jacobian and the mean term of each of the linear constraints, (7), on the entries of $\mathbf{A}^{-1}$. Thus, correction terms can be introduced such that the non-zero-mean error terms are replaced by zero mean error terms. To simplify the notation we will take advantage of the linear structure of $w_p(x)$, and analyze only the case where $w_p(x) = x^p$ and the generalization is straightforward. Thus, in this case

$$\begin{aligned}\int_{R^n} h^p(\mathbf{x}) &= \int_{R^n} (g(\mathbf{A}\mathbf{x}) + \eta(\mathbf{x}))^p \\ &= \int_{R^n} \sum_{i=0}^{p} \binom{p}{i} g^i(\mathbf{A}\mathbf{x})\,(\eta(\mathbf{x}))^{p-i} \\ &= |\mathbf{A}^{-1}| \sum_{i=0}^{p} \binom{p}{i} \int_{R^n} g^i(\mathbf{y}) \left(\eta(\mathbf{A}^{-1}(\mathbf{y}))\right)^{p-i} \end{aligned} \quad (9)$$

Evaluating the expected values of both sides of (9) we have

$$\begin{aligned} E\left[\int_{R^n} h^p(\mathbf{x})\right] &= |\mathbf{A}^{-1}| \sum_{i=0}^{p} \binom{p}{i} \int_{R^n} g^i(\mathbf{y}) \sigma_{p-i} \\ = |\mathbf{A}^{-1}| \int_{R^n} g^p(\mathbf{y}) &+ |\mathbf{A}^{-1}| \sum_{i=0}^{p-1} \binom{p}{i} \int_{R^n} g^i(\mathbf{y}) \sigma_{p-i} \end{aligned} \quad (10)$$

where the first term in the sum is precisely the term obtained in the absence of noise and with $w_p(x) = x^p$, while the second is an added mean term due to the noise contribution.

Let $\epsilon_p^{J,g}$ denote the error term in the Jacobian constraint equation, due to the presence of the noise for the choice $w_p(x) = x^p$, *i.e.,*

$$\begin{aligned} \epsilon_p^{J,g} &= \int_{R^n} \{[g(\mathbf{A}\mathbf{x}) + \eta(\mathbf{x})]^p - g^p(\mathbf{A}\mathbf{x})\} \\ &= |\mathbf{A}^{-1}| \sum_{i=0}^{p-1} \binom{p}{i} \int_{R^n} g^i(\mathbf{y}) \sigma_{p-i} + \tilde{\epsilon}_p^{J,g} \end{aligned} \quad (11)$$

where $\tilde{\epsilon}_p^{J,g}$ is a zero mean random variable. We can therefore rewrite (9) as

$$\begin{aligned} &\int_{R^n} h^p(\mathbf{x}) = \\ &|\mathbf{A}^{-1}| \left(\int_{R^n} g^p(\mathbf{y}) + \sum_{i=0}^{p-1} \binom{p}{i} \int_{R^n} g^i(\mathbf{y}) \sigma_{p-i} \right) + \tilde{\epsilon}_p^{J,g} \end{aligned} \quad (12)$$

The system (12) represents a linear regression problem in $|\mathbf{A}^{-1}|$ where the observation noise is zero mean. The regressors are functions of w_p, the template g, and the known statistics of the noise. Hence the regressors are *deterministic*. The system (12) is solved by a linear least squares method such that $\sum_{p=1}^{P} |\tilde{\epsilon}_p^{J,g}|^2$ is minimized.

Following a similar procedure we next obtain zero-mean linear constraints on the entries of $\mathbf{A}^{-1}$:

$$\begin{aligned} \int_{R^n} x_k h^p(\mathbf{x}) &= \int_{R^n} x_k \left(g(\mathbf{A}\mathbf{x}) + \eta(\mathbf{x})\right)^p \\ &= \int_{R^n} x_k \sum_{i=0}^{p} \binom{p}{i} g^i(\mathbf{A}\mathbf{x})\,(\eta(\mathbf{x}))^{p-i} \\ &= \int_{R^n} |\mathbf{A}^{-1}| \left(\sum_{j=1}^{n} q_{kj} y_j\right) \sum_{i=0}^{p} \binom{p}{i} g^i(\mathbf{y}) \left(\eta(\mathbf{A}^{-1}(\mathbf{y}))\right)^{p-i} \\ &= |\mathbf{A}^{-1}| \sum_{j=1}^{n} q_{kj} \sum_{i=0}^{p} \binom{p}{i} \left(\int_{R^n} y_j \, g^i(\mathbf{y}) \left(\eta(\mathbf{A}^{-1}(\mathbf{y}))\right)^{p-i} \right) \end{aligned} \quad (13)$$

Evaluating the expected values of both sides of (13) we have

$$\begin{aligned} E \int_{R^n} x_k h^p(\mathbf{x}) &= |\mathbf{A}^{-1}| \sum_{j=1}^{n} q_{kj} \sum_{i=0}^{p} \binom{p}{i} \int_{R^n} y_j g^i(\mathbf{y}) \sigma_{p-i} \\ = |\mathbf{A}^{-1}| &\sum_{j=1}^{n} q_{kj} \int_{R^n} y_j g^p(\mathbf{y}) \\ + |\mathbf{A}^{-1}| &\sum_{j=1}^{n} q_{kj} \left(\sum_{i=0}^{p-1} \binom{p}{i} \int_{R^n} y_j g^i(\mathbf{y}) \sigma_{p-i} \right) \end{aligned} \quad (14)$$

where the first term in the sum is precisely the term obtained in the absence of noise and with $w_p(x) = x^p$, while the second is an added mean term due to the noise contribution.

Hence for the choice of $w_p(x) = x^p$, $\epsilon_{p,k}^g$ defined in (6) has the form

$$\begin{aligned} \epsilon_{p,k}^g &= \int_{R^n} x_k \{[g(\mathbf{A}\mathbf{x}) + \eta(\mathbf{x})]^p - g^p(\mathbf{A}\mathbf{x})\} \\ = |\mathbf{A}^{-1}| &\sum_{j=1}^{n} q_{kj} \left(\sum_{i=0}^{p-1} \binom{p}{i} \int_{R^n} y_j g^i(\mathbf{y}) \sigma_{p-i} \right) + \tilde{\epsilon}_{p,k}^g \end{aligned} \quad (15)$$

where $\tilde{\epsilon}_{p,k}^g$ is a zero mean random variable. Thus, (13) can be written in the form

$$\begin{aligned} &\int_{R^n} x_k h^p(\mathbf{x}) = \\ &|\mathbf{A}^{-1}| \sum_{j=1}^{n} q_{kj} \left\{ \int_{R^n} y_j g^p(\mathbf{y}) + \sum_{i=0}^{p-1} \binom{p}{i} \int_{R^n} y_j g^i(\mathbf{y}) \sigma_{p-i} \right\} \\ &+ \tilde{\epsilon}_{p,k}^g \end{aligned} \quad (16)$$

Thus the system (16) represents a linear regression problem, different from (7), where the observation noise is *non-stationary*, but with a zero mean. The regressors are functions of w_p, the template g, and the known statistics of the noise. Hence the regressors are *deterministic*. Provided that the resulting regressors matrix is full rank, the system (16) is solved by a linear least squares method such that $\sum_{p=1}^{P} |\tilde{\epsilon}_{p,k}^g|^2$ is minimized.

It should be noted however that the noise sequence $\tilde{\epsilon}^g_{p,k}$ is only approximately zero-mean, since in (14) it is assumed that $|\mathbf{A}^{-1}|$ is a deterministic constant. However, in the estimation procedure itself it is replaced by its estimate obtained from solving (12), which is a function of the noise sequence $\eta(\mathbf{x})$.

4. NUMERICAL EXAMPLE

The example illustrates the operation of the proposed algorithm on an airplane image. The template image dimensions are 1170×1750. It is shown in the bottom image of Figure 1. A typical observed deformed image is shown in the top image of the figure. The image coordinate system is $[-1, 1] \times [-1, 1]$. The deforming transformation is given by

$$\mathbf{A} = \begin{matrix} 1.5 & 0.6 \\ 0 & 1.1 \end{matrix}$$

In this example we have implemented the proposed solution using $w_p(x) = x^p$, $p = 1, \ldots, 3$. After being estimated, the deformation estimated for the given observation is applied to the stored template in order to obtain an estimate of the deformed object (middle image in Figure 1) which can be compared with the deformed observation shown in the upper image.

The bias of the estimates obtained in a series of 100 Monte-Carlo simulations of the foregoing scenario is

$$\begin{matrix} -0.0169 & -0.003 \\ -0.0144 & -0.00175 \end{matrix}$$

and the corresponding variances of the estimation error are given by

$$\begin{matrix} 0.0198 & 0.00003 \\ 0.028 & 0.00004 \end{matrix}$$

5. CONCLUSIONS

We have considered the problem of finding the affine transformation relating a given observation on a planar object with some pre-chosen template of this object. The direct approach for estimating the transformation is to apply each of the deformations in the affine group to the template in a search for the deformed template that matches the observation. We propose a method that employs a set of non-linear operators to replace this high dimensional problem by an *equivalent linear least squares problem*, expressed in terms of the unknown affine transformation parameters. The proposed solution is *unique and exact* and is applicable to any affine transformation *regardless of the magnitude of the deformation*.

6. REFERENCES

[1] R. Hagege and J. M. Francos, "Parametric Estimation of Two-Dimensional Affine Transformations," *Int. Conf. Acoust., Speech, Signal Processing*, Montreal 2004.

[2] R. Hagege and J. M. Francos, "Parametric Estimation of Multi-Dimensional Affine Transformations: Solving a Group-Theory Problem as a Linear Problem," submitted for publication.

Fig. 1. From top to bottom: Noisy observation on the deformed object; Estimated deformed object obtained by applying the deformation estimated from the observation to the template; Template.