

BOOTSTRAP BASED SPECTRAL ANALYSIS

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ABSTRACT

The problem of confidence interval estimation for spectra, coherence functions and transfer functions is addressed. We propose bootstrap based methods as an alternative to asymptotic techniques. Bootstrap methods provide accurate confidence coverage, in particular when asymptotic techniques are invalid, i.e., for small sample sizes or non-Gaussian data.

1. INTRODUCTION

Non-parametric spectral density estimation has been extensively investigated in the signal processing as well as the statistical literature. However, assessing accuracy when little is known about the statistical distributional properties of the signal or when only a small number of observations are available and large sample theory does not apply has found only little coverage. The issue is of great importance because many signals encountered in real-life applications are non-Gaussian and/or non-stationary. Non-stationarity can be relaxed by limiting the number of observations and assuming (quasi) stationarity within the observation interval. This leads to the problem of spectral resolution and thus, the choice of the kernel bandwidth becomes fundamental. Although this has been largely investigated [1, 2], most results are asymptotic and inapplicable when the signal is non-Gaussian. Small sample size based results on the accuracy of estimators for coherence functions and transfer functions are limited, see for example, the work by Thomson and Chave [3] who use the jackknife to estimate errors for spectrum, coherence and transfer functions estimates.

The objective in this paper is to propose a set of bootstrap [4, 5] based methods for accuracy analysis of spectral densities, coherence functions and transfer functions when little is known about the distribution of the data and the sample size is small.

2. PRELIMINARIES

Consider Figure 1 in which $Y(t)$ is the output of a linear time-invariant filter plus noise where $X(t)$ and $\mathcal{E}(t)$ are as-

sumed to be uncorrelated. Let the real, 2 vector-valued process $(X(t), Y(t))$ for $t = 0, \pm 1, \dots$ be stationary with second order spectral densities $C_{XX}(\omega)$, $C_{YX}(\omega)$ and $C_{YY}(\omega)$ for $-\infty < \omega < \infty$. The transfer function $H(\omega)$ is given by

$$H(\omega) = \frac{C_{YX}(\omega)}{C_{XX}(\omega)}, \quad -\infty < \omega < \infty. \quad (1)$$

The spectral density of the noise process $\mathcal{E}(t)$ in Figure 1 is given by $C_{\mathcal{E}\mathcal{E}}(\omega) = (1 - |R_{YX}(\omega)|^2)C_{YY}(\omega)$, where $|R_{YX}(\omega)|^2$ is the coherence function, also called the magnitude-squared coherence function, of $Y(t)$ and $X(t)$, $t = 0, \pm 1, \dots$, at frequency ω and is defined by

$$|R_{YX}(\omega)|^2 = \frac{|C_{YX}(\omega)|^2}{C_{XX}(\omega)C_{YY}(\omega)}, \quad -\infty < \omega < \infty \quad (2)$$

It measures the extent to which $Y(t)$ is determinable from $X(t)$, $t = 0, \pm 1, \dots$, at frequency ω by linear time-invariant operations and is bounded between 0 and 1.

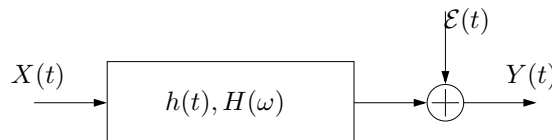


Fig. 1. Linear time-invariant filtering plus noise.

3. SPECTRAL ESTIMATION

Assume that we are given independent data records $X(t, l)$ and $Y(t, l)$ for $t = 0, \dots, T-1$ and $l = 1, \dots, n$. Define the finite Fourier transforms of the input and output data by respectively $d_X(\omega, l) = \sum_{t=0}^{T-1} w(t/T)X(t, l)e^{-j\omega t}$ and $d_Y(\omega, l) = \sum_{t=0}^{T-1} w(t/T)Y(t, l)e^{-j\omega t}$ for $-\infty < \omega < \infty$, where $w(\hat{t})$, $\hat{t} \in \mathbb{R}$, is a window that is bounded, is of bounded variation and vanishes for $|\hat{t}| > 1$. Let

$$\hat{C}_{XX}(\omega) = \sum_{l=1}^n \frac{|d_X(\omega, l)|^2}{nT}, \quad \hat{C}_{YY}(\omega) = \sum_{l=1}^n \frac{|d_Y(\omega, l)|^2}{nT}$$

and

$$\hat{C}_{YX}(\omega) = \frac{1}{nT} \sum_{l=1}^n d_Y(\omega, l) \bar{d}_X(\omega, l)$$

be estimates of $C_{XX}(\omega)$, $C_{YY}(\omega)$ and $C_{YX}(\omega)$, $-\infty < \omega < \infty$, respectively, obtained by averaging n periodograms, where we denoted by $\bar{d}_X(\omega, l)$ the complex conjugate of $d_X(\omega, l)$. Estimates for $H(\omega)$ and $|R_{YX}(\omega)|^2$ are respectively obtained by

$$\hat{H}(\omega) = \frac{\hat{C}_{YX}(\omega)}{\hat{C}_{XX}(\omega)}, \quad -\infty < \omega < \infty, \quad (3)$$

and

$$|\hat{R}_{YX}(\omega)|^2 = \frac{|\hat{C}_{YX}(\omega)|^2}{\hat{C}_{XX}(\omega)\hat{C}_{YY}(\omega)}, \quad -\infty < \omega < \infty. \quad (4)$$

Most methods available for assessing the distribution of the spectral estimators above assume that either the distribution of the process is Gaussian or the data size is sufficiently large so as to make use of asymptotic theory. The focus of this paper is on accuracy analysis when little is known about the distribution of the process and the data size is small.

4. CONFIDENCE INTERVAL ESTIMATION

In this section we will describe bootstrap methods for confidence interval estimation for spectra, coherence functions and transfer functions. Our analysis makes no assumption on the distributional properties of the signals and/or noise and we do not assume a large sample size.

4.1. Confidence Intervals for Spectra

There exist several bootstrap methods for approximating the distribution of spectral density estimates and confidence intervals for $C_{XX}(\omega)$, say, based on an estimate $\hat{C}_{XX}(\omega)$. In [6] we used the residual based method for bootstrapping kernel spectral density estimates, see also [7, 5]. We explored the fact that the spectral density is a scale parameter of the asymptotic distribution of $I_{XX}(\omega)$. Thus, we interpreted the spectral estimation problem as an approximate multiplicative regression problem, where

$$I_{XX}(\omega_k) = C_{XX}(\omega_k) \cdot \varepsilon_k, \quad k = 1, \dots, N_0,$$

and $\omega_k = 2\pi k/T$ and N_0 is the largest integer less than or equal to $T/2$. For averaged periodograms, we devise the procedure in Table 1 below. Note that in **Step 5** we set $I_{XX}^*(0) = 0$, the periodogram at $\omega_k = 0$ of a mean corrected sample.

Note that the multiplicative regression approach is not necessary if one has multiple independent data records for the estimation of the spectrum. As shown below for coherence functions and transfer functions one can resample

Table 1. Confidence interval estimation for spectra.

Step 0. Data Collection. Collect the data $X(t, l)$ for $t = 0, \dots, T-1$ and $l = 1, \dots, n$, and detrend it by subtracting the sample mean.

Step 1. Initial Estimate. Compute for $k = 1, \dots, N_0$

$$\hat{C}_{XX}(\omega_k) = \frac{1}{n} \sum_{l=1}^n I_{XX}(\omega_k, l).$$

Step 2. Compute Residuals. Calculate the residuals

$$\hat{\varepsilon}_k = \frac{I_{XX}(\omega_k)}{\hat{C}_{XX}(\omega_k)}, \quad k = 1, \dots, N_0.$$

Step 3. Rescaling. Rescale the empirical residuals to

$$\tilde{\varepsilon}_k = \frac{\hat{\varepsilon}_k}{\hat{\varepsilon}_\bullet}, \quad k = 1, \dots, N_0, \quad \hat{\varepsilon}_\bullet = \frac{1}{N_0} \sum_{k=1}^{N_0} \hat{\varepsilon}_k.$$

Step 4. Resampling. Draw independent bootstrap residuals $\tilde{\varepsilon}_1^*, \dots, \tilde{\varepsilon}_N^*$ from the empirical distribution of $\tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_N$.

Step 5. Bootstrap Estimates. Calculate

$$\begin{aligned} I_{XX}^*(\omega_k) &= I_{XX}^*(-\omega_k) = \hat{C}_{XX}(\omega_k) \tilde{\varepsilon}_k^*, \\ \hat{C}_{XX}^*(\omega) &= \sum_{l=1}^n I_{XX}^*(\omega_k, l). \end{aligned}$$

Step 6. Confidence Bands Estimation. Repeat **Steps 4-5** a large number of times and find c_U^* (and proceed similarly for c_L^*) such that

$$\Pr_* \left(\sqrt{n} \frac{\hat{C}_{XX}^*(\omega_k) - \hat{C}_{XX}(\omega_k)}{\hat{C}_{XX}(\omega_k)} \leq c_U^* \right) = \alpha,$$

that is $\{1 + c_U^* n^{-1/2}\}^{-1} \hat{C}_{XX}(\omega_k)$ is the lower bound of an $(1 - 2\alpha)\%$ -confidence interval for $C_{XX}(\omega_k)$.

the frequency data to construct confidence intervals. More details on confidence interval estimation for spectra can be found in [6]. Theoretical analysis and bandwidth optimization with the bootstrap is treated in [8].

4.2. Confidence Intervals for Coherence Functions

In [9] we discussed confidence interval estimation for coherence functions and proposed a bootstrap approach as an alternative to asymptotic techniques such as in [10, 11]. We compared the results produced by the three methods in view of confidence level and lower and upper tail coverages under different settings such as signal and noise distributions. Extensive simulations have shown that for a large coherence the bootstrap approach is more accurate in that it maintains the nominal confidence level, also in non-Gaussian environ-

ments. The approach is reproduced in Table 2.

Table 2. Confidence interval estimation for coherence functions.

<p>Step 0. Data Collection. Conduct the experiment and calculate the frequency data $d_X(\omega, 1), \dots, d_X(\omega, n)$ and $d_Y(\omega, 1), \dots, d_Y(\omega, n)$.</p> <p>Step 1. Resampling. Using a pseudo random number generator, draw a random sample, $\mathcal{X}^*(\omega)$ (of the same size), with replacement, from $\mathcal{X}(\omega) = \{(d_X(\omega, 1), d_Y(\omega, 1)), \dots, (d_X(\omega, n), d_Y(\omega, n))\}$.</p> <p>Step 2. Bootstrap Estimates. From $\mathcal{X}^*(\omega)$, calculate $\hat{R}_{YX}^*(\omega) ^2$, the bootstrap analogue of $\hat{R}_{YX}(\omega) ^2$ given in Eq. (4), replacing the estimates by their bootstrap counterparts and form $h(\hat{R}_{YX}(\omega) ^2)$ and $h(\hat{R}_{YX}^*(\omega) ^2)$.</p> <p>Step 3. Repetition. Repeat Steps 1-2 a large number of times to obtain a total of N bootstrap statistics $\hat{R}_{YX}^{*1}(\omega) ^2, \dots, \hat{R}_{YX}^{*N}(\omega) ^2$.</p> <p>Step 4. Distribution Function Estimation. Sort the variance stabilised bootstrap estimates in increasing order to obtain $h^{(1)}(\hat{R}_{YX}^*(\omega) ^2) \leq \dots \leq h^{(N)}(\hat{R}_{YX}^*(\omega) ^2)$ and approximate the density function of $h(\hat{R}_{YX}(\omega) ^2) - h(\hat{R}_{YX}^*(\omega) ^2)$ by the density function of $h(\hat{R}_{YX}^*(\omega) ^2) - h(\hat{R}_{YX}(\omega) ^2)$.</p> <p>Step 5. Confidence Bands Estimation. For a desired $(1 - \alpha)100\%$ bootstrap confidence interval, find critical points of the bootstrap distribution of $h(\hat{R}_{YX}^*(\omega) ^2) - h(\hat{R}_{YX}(\omega) ^2)$, $h^{(q_1)}(\hat{R}_{YX}^*(\omega) ^2)$ and $h^{(q_2)}(\hat{R}_{YX}^*(\omega) ^2)$, say, where $q_1 = \lfloor (N+1)\alpha/2 \rfloor$ and $q_2 = N - q_1 + 1$. The confidence interval for $\hat{R}_{YX}(\omega) ^2$ is obtained as $(h^{-1}(h(\hat{R}_{YX}(\omega) ^2) - h^{(q_2)}(\hat{R}_{YX}^*(\omega) ^2)), h^{-1}(h(\hat{R}_{YX}(\omega) ^2) - h^{(q_1)}(\hat{R}_{YX}^*(\omega) ^2)))$.</p>

4.3. Confidence Intervals for Transfer Functions

The method for confidence interval estimation for transfer functions with the bootstrap is similar to that for coherence functions in Table 2. In Table 3, as in Table 2, we used a variance stabilising transformation h in order to get more accurate confidence intervals [5]. An estimate of h can be obtained with the bootstrap (see for example [5]).

We note that we could have performed the bootstrapping in a different way. The alternative approach would be to explore a linear regression as a result of the model given in Figure 1. Omitting the error term $o_{a.s.}(1)$, which tends to zero almost surely as $T \rightarrow \infty$ [1], we have

$$d_Y(\omega) = H(\omega)d_X(\omega) + d_{\mathcal{E}}(\omega).$$

Table 3. Confidence interval estimation for transfer functions.

<p>Step 0. Data Collection. Conduct the experiment and calculate the frequency data $d_X(\omega, 1), \dots, d_X(\omega, n)$ and $d_Y(\omega, 1), \dots, d_Y(\omega, n)$.</p> <p>Step 1. Resampling. Using a pseudo random number generator, draw a random sample, $\mathcal{X}^*(\omega)$ (of the same size), with replacement, from $\mathcal{X}(\omega) = \{(d_X(\omega, 1), d_Y(\omega, 1)), \dots, (d_X(\omega, n), d_Y(\omega, n))\}$.</p> <p>Step 2. Bootstrap Estimates. From $\mathcal{X}^*(\omega)$, calculate $\hat{H}^*(\omega)$, the bootstrap analogue of $\hat{H}(\omega)$ given in Eq. 3, replacing the estimate by their bootstrap counterparts and form $h(\hat{H}(\omega))$ and $h(\hat{H}^*(\omega))$.</p> <p>Step 3. Repetition. Repeat Steps 1-2 a large number of times to obtain a total of N bootstrap statistics $\hat{H}^{*1}(\omega), \dots, \hat{H}^{*N}(\omega)$.</p> <p>Step 4. Distribution Function Estimation. Sort the variance stabilised bootstrap estimates in increasing order to obtain $h^{(1)}(\hat{H}^*(\omega)) \leq \dots \leq h^{(N)}(\hat{H}^*(\omega))$ and approximate the density function of $h(\hat{H}(\omega)) - h(\hat{H}(\omega))$ by the density function of $h(\hat{H}^*(\omega)) - h(\hat{H}(\omega))$.</p> <p>Step 5. Confidence Bands Estimation. For a desired $(1 - \alpha)100\%$ bootstrap confidence interval, find critical points of the bootstrap distribution of $h(\hat{H}^*(\omega)) - h(\hat{H}(\omega))$, $h^{(q_1)}(\hat{H}^*(\omega))$ and $h^{(q_2)}(\hat{H}^*(\omega))$, say, where $q_1 = \lfloor (N+1)\alpha/2 \rfloor$ and $q_2 = N - q_1 + 1$. The confidence interval for $H(\omega)$ is obtained as $(h^{-1}(h(\hat{H}(\omega)) - h^{(q_2)}(\hat{H}^*(\omega))), h^{-1}(h(\hat{H}(\omega)) - h^{(q_1)}(\hat{H}^*(\omega))))$.</p>

We would first estimate $H(\omega)$ through

$$\hat{H}(\omega) = \frac{\hat{C}_{YX}(\omega)}{\hat{C}_{XX}(\omega)}$$

and define residuals

$$d_{\hat{\mathcal{E}}}(\omega) = d_Y(\omega) - \hat{H}(\omega)d_X(\omega).$$

Given n independent data records, we would perform two totally independent resampling operations. We would draw, with replacement $\{d_X^*(\omega, 1), \dots, d_X^*(\omega, n)\}$ from the random sample $\{d_X(\omega, 1), \dots, d_X(\omega, n)\}$ and $\{d_{\hat{\mathcal{E}}}^*(\omega, 1), \dots, d_{\hat{\mathcal{E}}}^*(\omega, n)\}$ from the residual data $\{d_{\hat{\mathcal{E}}}(\omega, 1), \dots, d_{\hat{\mathcal{E}}}(\omega, n)\}$. Bootstrap data for $d_Y(\omega)$ is obtained via the regression

$$d_Y^*(\omega) = \hat{H}(\omega)d_X^*(\omega) + d_{\hat{\mathcal{E}}}^*(\omega),$$

where the joint distribution of $\{(d_X^*(\omega, l), d_Y^*(\omega, l)), 1 \leq l \leq n\}$, conditional on the sample of the measurement data $\mathcal{X}(\omega) = \{(d_X(\omega, 1), d_Y(\omega, 1)), \dots, (d_X(\omega, n), d_Y(\omega, n))\}$

is the bootstrap estimate of the unconditional joint distribution of $\mathcal{X}(\omega)$.

The above approach enables us to replicate estimates of $\hat{H}(\omega)$ with the bootstrap and estimate confidence bounds. However, the approach relies on the model of Figure 1. In the absence of any model, we propose the use of Table 2, which makes use of the input-output data only.

Results for spectra have been reported in [6, 8] and comparisons with the χ^2 approximation have been performed. Bootstrapping coherence functions has been discussed in [9] and results were compared with a recently suggested iterative method [11]. In this contribution we will restrict our results to transfer functions.

5. THE EXPERIMENT

We consider a linear time-invariant (LTI) system as depicted in Figure 1, modelled by a finite impulse response (FIR) filter of second order whose frequency transfer function is given by

$$H(\omega) = 1 - 0.5e^{-j\omega} + 1.5e^{-j2\omega}.$$

The LTI system is driven by white noise $X(t)$, $t = 0, \pm 1, \dots$, with spectral density $C_{XX}(\omega) = \sigma_X^2$. Independent white noise $\mathcal{E}(t)$, with spectral density $\sigma_{\mathcal{E}}^2$, is added to the output of the system to generate $Y(t)$, $t = 0, \pm 1, \dots$ (See Figure 1). It can easily be shown that in this case the coherence $|R_{YX}(\omega)|^2$ is given by

$$|R_{YX}(\omega)|^2 = \frac{1}{1 + |H(\omega)|^{-2} \sigma_{\mathcal{E}}^2 / \sigma_X^2}.$$

We generate independently $X(t, l)$ and $\mathcal{E}(t, l)$, $t = 0, \dots, T-1$ for $l = 1, \dots, n$ independent data stretches. We filter $X(t, l)$, add $\mathcal{E}(t, l)$ to obtain $Y(t, l)$, $t = 0, \dots, T-1$ for $l = 1, \dots, n$. Then, we estimate $H(\omega)$ according to Eq. (3). We compute confidence intervals using Table 3.

Unless otherwise stated, we will use the following parameters: $\alpha = 5\%$, $N = 399$, $T = 64$, $n = 20$, $N_1 = 100$, $N_2 = 25$, $\text{SNR} = 0$ dB. All results presented in this section are based on 200 Monte Carlo replications. Consider first the case where the input signal and the noise are Gaussian distributed. Figure 2 and 3 show the 95% confidence bounds obtained for the magnitude and the phase of $H(\omega)$, respectively.

The estimated confidence levels over 200 replications are given in Figure 4 for the magnitude and in Figure 5 for the phase.

We now consider non-Gaussian noise. Let the input signal be Gaussian distributed and the noise be t_4 -distributed. In Figure 6 and Figure 7, we show respectively for the magnitude and the the phase of the transfer function the confidence bounds estimated by the bootstrap method described in Table 3.

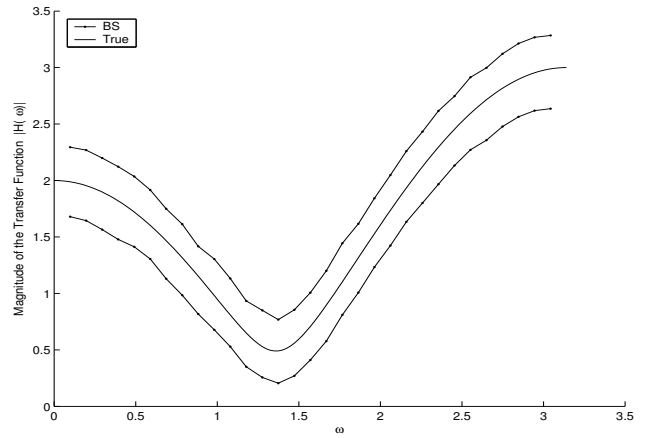


Fig. 2. Magnitude of the transfer function (solid) and estimated 95% confidence bounds with the bootstrap for Gaussian signal and noise.

Table 4. Lower and upper coverage probabilities and confidence level for a nominal 95% confidence interval over all frequencies

Magnitude	lower tail	confidence	upper tail
Gaussian	0.0415	0.9111	0.0474
non-Gaussian	0.0394	0.9094	0.0513
Phase	lower tail	confidence	upper tail
Gaussian	0.0535	0.9013	0.0635
non-Gaussian	0.0973	0.8453	0.0968

Table 4 shows the lower and upper tails and confidence level for a nominal 95% confidence interval for the magnitude and the phase of the transfer function over all frequencies in the Gaussian as well as the considered non-Gaussian case.

6. DISCUSSION

The sections above show how the bootstrap can be applied to the estimation of confidence intervals for spectra, coherence functions and transfer functions. The suggested methods do not assume large sample sizes or Gaussian data to be applicable. Results for spectra and coherence functions were presented in [6, 8, 9] and shown to be accurate in different scenarios. The experimental results presented here are limited to transfer functions. By way of simulations it was shown that the obtained confidence coverage for transfer functions is slightly biased downwards. This phenomenon is observed in the Gaussian as well as the non-Gaussian case. One should emphasize the fact that the sample size considered here is very low and consequently the estimation of $H(\omega)$ is inaccurate. Further analysis is be-

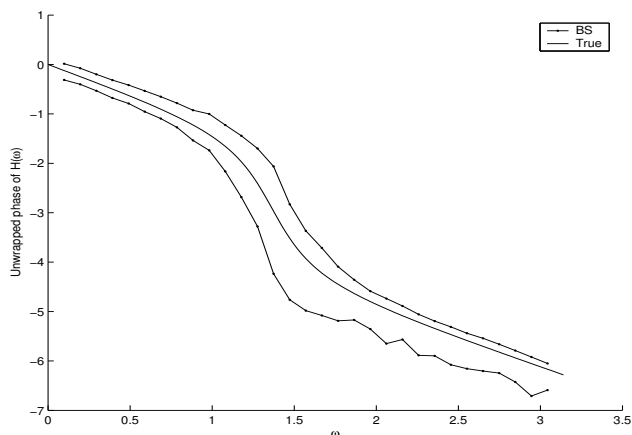


Fig. 3. Phase of the transfer function (solid) and estimated 95% confidence bounds with the bootstrap for a Gaussian signal and Gaussian noise.

ing carried out for different types of transfer functions and different sample sizes.

7. CONCLUDING REMARKS

We have discussed confidence interval estimation for spectra, coherence functions and transfer functions. We have suggested methods based on the bootstrap and shown by way of simulations that they are accurate even when the statistics of the random processes are unknown or the sample size is small. Theoretical results can be developed for coherence functions and transfer functions following the approach used in [8] but will be presented elsewhere.

8. ACKNOWLEDGEMENTS

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9. REFERENCES

- [1] D. R. Brillinger, *Time Series: Data Analysis and Theory*, Holden-Day, San Francisco, 1981.
- [2] M. B. Priestley, *Spectral Analysis and Time Series*, Academic Press London, 1981.
- [3] J. Thomson and D. Chave, "Jackknifed Error Estimates for Spectra, Coherence, and Transfer Functions," in *Advances in Spectrum Analysis and Array Processing*, S. Haykin, Ed., vol. 1, pp. 58–113. Prentice Hall, 1991.

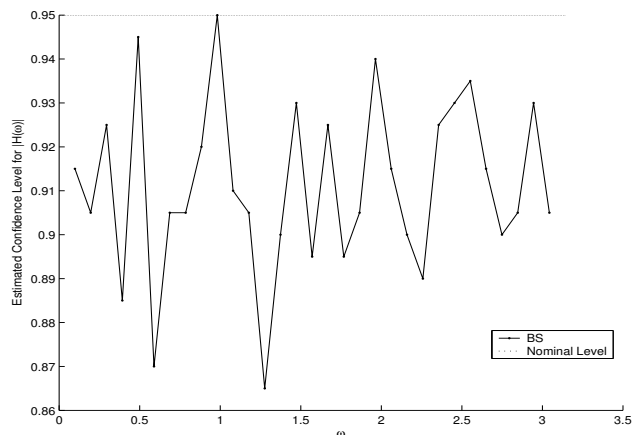


Fig. 4. Estimated confidence level with the bootstrap for the magnitude of the transfer function for a nominal level of 95% (dashed) for a Gaussian signal and Gaussian noise.

- [4] B. Efron and R. Tibshirani, *An Introduction to the Bootstrap*, Chapman and Hall, 1993.
- [5] A. M. Zoubir and D. R. Iskander, *Bootstrap Techniques for Signal Processing*, Cambridge University Press, 2004.
- [6] A. M. Zoubir, "Bootstrapping Kernel Spectral Density Estimates with Kernel Bandwidth Estimation," in *Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing, ICASSP-03*, Hong Kong, 2003, pp. 325–328.
- [7] J. Franke and W. Härdle, "On Bootstrapping Kernel Estimates," *The Annals of Statistics*, vol. 20, pp. 121–145, 1992.
- [8] A. M. Zoubir, "On the Accuracy of Spectral Density Estimation," in *Proceedings of the IEEE International Workshop on Statistical Signal Processing, SSP'03*, St. Louis, 2003, pp. 117–120.
- [9] A. M. Zoubir, "On Confidence Intervals for the Coherence Function," in *Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing, ICASSP-05*, Philadelphia, 2005.
- [10] L.D. Enochson and N.R. Goodman, "Gaussian Approximations to the Distribution of Sample Coherence," AFFDL–TR–65–57, Wright-Patterson Air Force Base, 1965.
- [11] S. Wang and M. Tang, "Exact Confidence Interval for Magnitude-Squared Coherence Estimates," *IEEE Signal Processing Letters*, vol. 11, pp. 326–329, 2004.

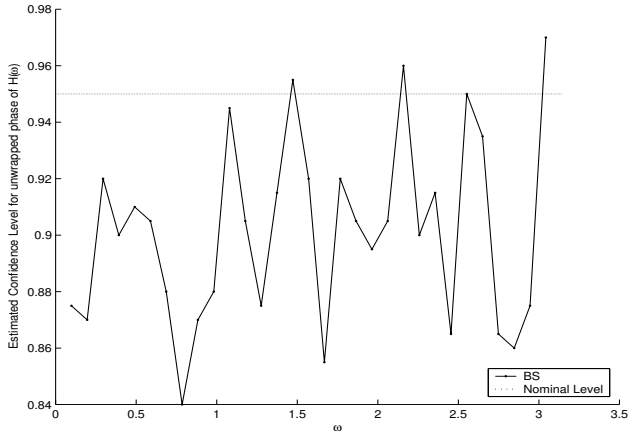


Fig. 5. Estimated confidence level with the bootstrap for the phase of the transfer function for a nominal level of 95% (dashed) for a Gaussian signal and Gaussian noise.

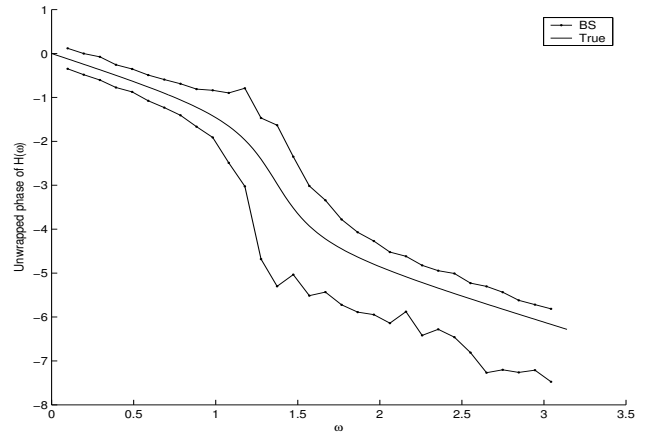


Fig. 7. Phase of the transfer function (solid) and estimated 95% confidence bounds with the bootstrap for Gaussian signal and t_4 -distributed noise.

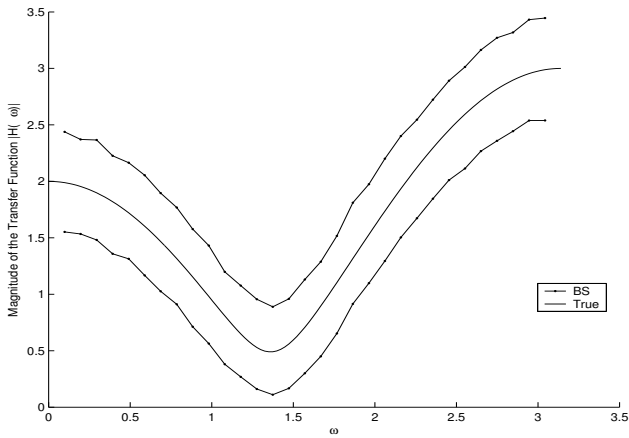


Fig. 6. Magnitude of the transfer function (solid) and estimated 95% confidence bounds with the bootstrap for Gaussian signal and t_4 -distributed noise.