ASYMPTOTIC NORMALITY OF THE SAMPLE COVARIANCES OF EVANESCENT FIELDS IN NOISE

Mark Kliger and Joseph M. Francos

Electrical & Computer Engineering Dept.
Ben-Gurion University
Beer Sheva 84105, Israel

ABSTRACT

We consider the asymptotic properties of the sample covariance sequence of a field composed of the sum of evanescent components and a purely-indeterministic component. In this framework a Bartlett-type formula for the covariance function of the sample covariances of a horizontal evanescent field observed in the presence of a purely-indeterministic field, is derived. The asymptotic normality of the sample covariances is established.

Keywords: Homogeneous random fields, sample covariance, asymptotic normality.

1. INTRODUCTION

It is well known, [2], that any 2-D regular and homogeneous discrete random field can be represented as a sum of two mutually orthogonal components: a purely-indeterministic field and a deterministic one. The deterministic component is further orthogonally decomposed into a harmonic field and a countable number of mutually orthogonal evanescent fields.

In this paper we consider the asymptotic properties of the sample autocovariance sequence of a field composed of the sum of evanescent components and a purely-indeterministic component.

Asymptotic normality of the sample autocovariances for *linear processes* was first introduced by Bartlett (see Brockwell and Davis [1], Propositions 7.3.1-7.3.4). The asymptotic normality of the sample autocovariances for real multivariate time series with *continuous spectra* was established by Hannan ([4], pp. 209-212). Stoica *et. al.* [11] proved the asymptotic normality of the sample autocovariances of a time series where the observed series is the sum of i.i.d. noise samples and a finite number of real sinusoids. Li *et. al.* [8] proved the asymptotic normality of the sample autocovariances for a time series with *mixed spectra*, where the observed *mixed-spectrum* process is the sum of a stationary process with a continuous spectra, and a finite number of real sinusoids. The results in [8] generalize the previous results of Mackisack and Poskit [10], as well as those of Stoica *et. al.* [11], to the case of *colored* noise, and fully extend the classical results of Bartlett to the general case of a mixed-spectrum time series. This problem arises in many frequency estimation procedures (e.g. [6]), which aim to estimate the unknown frequencies of a sinusoidal component, based on sample autocovariances of a noisy data record. Finally, Li [9] completed the generalization of Bartlett's result to the case of *complex multivariate* time series with *mixed spectra*.

To the best of our knowledge, the problem of analyzing the asymptotic properties of the sample autocovariance sequence of homogenous random fields, remains an open problem. The asymptotic normality of the sample autocovariance sequence of a homogenous field with no evanescent components (*i.e.*, the field is composed of only harmonic and purely-indeterministic components), can be shown based on the results of [8]. In this paper we derive a Bartlett-type formula for a *horizontal evanescent* component, observed in the presence of a purely-indeterministic component. Finally, we establish the asymptotic normality of the sample autocovariance function of the observed field.

2. PROBLEM DEFINITION

Recall from [2], [3], the definitions of an evanescent component. Let O denote the set of all possible RNSHP (*rational non-symmetrical half-plane*) definitions on the 2-D lattice and let a,b be two coprime integers, that define the slope of the RNSHP. The evanescent field corresponding to the RNSHP defined by $(a,b) \in O$ is given by

$$\begin{aligned} e_{(a,b)}(n,m) &= \sum_{i=1}^{I^{(a,b)}} e_i^{(a,b)}(n,m) \\ &= \sum_{i=1}^{I^{(a,b)}} s_i^{(a,b)}(na+mb)\cos\Big(\nu_i^{(a,b)}(nc+md)\Big) \\ &+ t_i^{(a,b)}(na+mb)\sin\Big(\nu_i^{(a,b)}(nc+md)\Big) \qquad (1) \end{aligned}$$

where c and d are coprime integers such that $|ad - bc| = 1$. For the case where $(a, b) = (0, 1)$ we have $(c, d) = (1, 0)$, and for $(a, b) = (1, 0)$ we have $(c, d) = (0, 1)$. The 1-D purely-indeterministic processes $\{s_i^{(a,b)}(na + mb)\}$, $\{s_j^{(a,b)}(na+mb)\}$, $\{t_k^{(a,b)}(na+mb)\}$, $\{t_\ell^{(a,b)}(na+mb)\}$ are mutually orthogonal for all $i, j, k, \ell, i \neq j, k \neq \ell$, and for all i the processes $\{s_i^{(a,b)}(na+mb)\}$ and $\{t_i^{(a,b)}(na+mb)\}$ have an identical autocorrelation function.

The evanescent component with parameters $(a, b) = (1, 0)$ and $(c, d) = (0, 1)$ is usually called *horizontal evanescent*. We consider the case where a horizontal evanescent component is observed in the presence of a purely-indeterministic component.

Let $\{y(n, m)\}$, $(n, m) \in D$ where $D = \{(i, j) \mid 0 \leq i \leq N - 1, 0 \leq j \leq M - 1\}$ be the observed 2-D real valued homogeneous random field such that,

$$y(n, m) = e_{(1,0)}(n, m) + w(n, m). \tag{2}$$

The purely-indeterministic component, $\{w(n, m)\}$ is a 2-D zero mean random field with a finite variance σ^2, which is assumed to be Gaussian. The field $\{e_{(1,0)}(n, m)\}$ is an evanescent random field, *i.e.*,

$$\begin{aligned} e_{(1,0)}(n, m) &= \sum_{i=1}^{I^{(1,0)}} \left[s_i^{(1,0)}(n) \cos\left(\nu_i^{(1,0)} m\right) \right. \\ &+ \left. t_i^{(1,0)}(n) \sin\left(\nu_i^{(1,0)} m\right) \right]. \end{aligned} \tag{3}$$

In the following, for simplicity, we omit the index (1,0). The processes $\{s_i(n)\}$ and $\{t_i(n)\}$ have an identical autocorrelation function given by

$$C_i(\tau) = \mathrm{E}[s_i(n + \tau)s_i(n)] = \mathrm{E}[t_i(n + \tau)t_i(n)]. \tag{4}$$

Assumption 1: The modulating 1-D processes $\{s_i(n)\}$ and $\{t_i(n)\}$ of each evanescent field are infinite order MA processes, *i.e.*,

$$s_i(n) = \sum_{j=-\infty}^{\infty} a_i(j)\xi_i(n - j) , \tag{5}$$

and

$$t_i(n) = \sum_{j=-\infty}^{\infty} a_i(j)\zeta_i(n - j) , \tag{6}$$

where $\{\xi_i(n)\}$, $\{\zeta_i(n)\}$ are i.i.d. 1-D zero mean processes, independent of each other and of $\{w(n, m)\}$. Both have an identical variance σ^2 such that

$$\mathrm{E}\left[\xi_i^4(n)\right] = \mathrm{E}\left[\zeta_i^4(n)\right] = \eta\sigma^4 < \infty \tag{7}$$

and $\sum_{j=-\infty}^{\infty} |a_i(j)| < \infty$.

For any two jointly homogeneous random fields $\{u(n, m)\}$ and $\{v(n, m)\}$ the *covariance function* between $\{u(n, m)\}$ and $\{v(n, m)\}$ is given by

$$C_{uv}(\tau, \rho) = \mathrm{E}[u(n + \tau, m + \rho)v(n, m)] . \tag{8}$$

The *sample covariance function* (SCF) of any two such fields is defined by

$$\tilde{C}_{uv}^{N,M}(\tau, \rho) = \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} u(n + \tau, m + \rho)v(n, m). \tag{9}$$

Similarly, the *sample covariance function* (SCF) of the stationary 1-D processes $\{u(n)\}$ and $\{v(n)\}$ is defined by

$$\tilde{C}_{uv}^{N}(\tau) = \frac{1}{N} \sum_{n=0}^{N-1} u(n + \tau)v(n). \tag{10}$$

In the following, for simplicity, we will omit the superscript notation N, M and the SCF will be denoted by $\tilde{C}_{uv}(\tau, \rho)$ (and similarly for $\tilde{C}_{uv}^N(\tau)$).

Using (4), the covariance function of $\{e(n, m)\}$ is given by

$$C_{ee}(\tau, \rho) = \mathrm{E}[e(n+\tau, m+\rho)e(n, m)] = \sum_{i=1}^{I} C_i(\tau) \cos(\nu_i\rho). \tag{11}$$

The spectral measure of the evanescent field is not absolutely continuous with respect to the Lebesgue measure on the torus, and therefore its covariance function is not absolutely summable with respect to τ and ρ. We also make the following assumption:

Assumption 2: The covariance function of the noise field $C_{ww}(\tau, \rho)$ is absolutely summable with respect to τ and ρ.

Note that Assumption 1 implies the absolute summability of the covariance function $C_i(\tau)$ with respect to τ.

The mutual orthogonality of the evanescent and noise fields ensures that,

$$C_{yy}(\tau, \rho) = C_{ee}(\tau, \rho) + C_{ww}(\tau, \rho). \tag{12}$$

The SACF of $\{y(n, m)\}$ is a function of the SACF and SCF of $\{e(n, m)\}$ and $\{w(n, m)\}$:

$$\begin{aligned} \tilde{C}_{yy}(\tau, \rho) = \tilde{C}_{ee}(\tau, \rho) + \tilde{C}_{ew}(\tau, \rho) \\ + \tilde{C}_{we}(\tau, \rho) + \tilde{C}_{ww}(\tau, \rho). \end{aligned} \tag{13}$$

It can be easily verified that $\tilde{C}_{yy}(\tau, \rho)$ is an unbiased estimator of $C_{yy}(\tau, \rho)$, *i.e.*

$$\mathrm{E}\left[\tilde{C}_{yy}(\tau, \rho)\right] = C_{yy}(\tau, \rho). \tag{14}$$

Let

$$\Delta\tilde{C}_{uv}(\tau, \rho) = \tilde{C}_{uv}(\tau, \rho) - C_{uv}(\tau, \rho). \tag{15}$$

In the next section we establish the asymptotic normality of the sample autocovariances of the observed field.

3. THE MAIN RESULT

Let $\{\Psi_i\}$ be a sequence of rectangles such that

$$\Psi_i = \{(n,m) \in \mathcal{Z}^2 \mid 0 \le n \le N_i - 1, 0 \le m \le M_i - 1\}.$$

Definition 1: The sequence of subsets $\{\Psi_i\}$ is said to tend to infinity (we adopt the notation $\Psi_i \to \infty$) as $i \to \infty$ if $\lim\limits_{i\to\infty} \min(N_i, M_i) = \infty$ and $0 < \lim\limits_{i\to\infty}(N_i/M_i) < \infty$. To simplify notations, we shall omit in the following the subscript i. Thus, the notation $\Psi(N,M) \to \infty$ implies that both N and M tend to infinity as functions of i, and at roughly the same rate.

To establish the main result we shall need the following proposition:

Proposition 1 *Let $\Psi(N,M) \to \infty$. Then,*

$$N^{\frac{1}{2}} \Delta\tilde{C}_{yy}(\tau,\rho) = N^{\frac{1}{2}} X(\tau,\rho) + o_P(1) \tag{16}$$

where

$$X(\tau,\rho) = \sum_{i=1}^{I} \Bigg[\delta_{\{\nu_i \ne 0\}} \{ \frac{1}{2}[\Delta\tilde{C}_{s_i s_i}(\tau) + \Delta\tilde{C}_{t_i t_i}(\tau)] \cos(\nu_i \rho)$$
$$+ \frac{1}{2}[\tilde{C}_{t_i s_i}(\tau) - \tilde{C}_{s_i t_i}(\tau)] \sin(\nu_i \rho) \} + \delta_{\{\nu_i = 0\}} \Delta\tilde{C}_{s_i s_i}(\tau) \Bigg] \tag{17}$$

and where for some set $\mathbf{A}$ in $[0, 2\pi)$, $\delta_{\mathbf{A}}$ is the indicator function

$$\delta_{\mathbf{A}} = \begin{cases} 1, & \nu \in \mathbf{A} \\ 0, & \nu \notin \mathbf{A} \end{cases} \tag{18}$$

Proof: See [7] for a detailed proof. ∎

Let g and h be arbitrary non-negative finite integers. Denote by $\Delta\tilde{\mathbf{C}}_{yy}$ the $(2h+1) \times (2g+1)$ matrix of sample covariances

$$\Delta\tilde{\mathbf{C}}_{yy} = \begin{bmatrix} \Delta\tilde{C}_{yy}(-h,-g) & \Delta\tilde{C}_{yy}(-h,-g+1) & \dots & \dots & \Delta\tilde{C}_{yy}(-h,g-1) & \Delta\tilde{C}_{yy}(-h,g) \\ \Delta\tilde{C}_{yy}(-h+1,-g) & \Delta\tilde{C}_{yy}(-h+1,-g+1) & \dots & \dots & \Delta\tilde{C}_{yy}(-h+1,g-1) & \Delta\tilde{C}_{yy}(-h+1,g) \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ \Delta\tilde{C}_{yy}(h,-g) & \Delta\tilde{C}_{yy}(h,-g+1) & \dots & \dots & \Delta\tilde{C}_{yy}(h,g-1) & \Delta\tilde{C}_{yy}(h,g) \end{bmatrix} \tag{19}$$

The main objective of this section is to establish the limiting distribution of the $(2g+1)(2h+1) \times 1$ vector $N^{\frac{1}{2}}\text{vec}(\Delta\tilde{\mathbf{C}}_{yy})$ as $\Psi(N,M) \to \infty$.

Define a matrix $\mathbf{X}$ in a similar way to (19). It is clear from (16) that as $\Psi(N,M) \to \infty$

$$N^{\frac{1}{2}} \Delta\tilde{\mathbf{C}}_{yy} = N^{\frac{1}{2}} \mathbf{X} + \mathbf{o}_P(1) \tag{20}$$

where $\mathbf{o}_P(1)$ is a $(2h+1) \times (2g+1)$ matrix where every entry decays to zero in probability. Define the $(h+1) \times 1$ vector

$$\Delta\tilde{\mathbf{C}}_{s_i s_i} = \left[\Delta\tilde{C}_{s_i s_i}(0), \Delta\tilde{C}_{s_i s_i}(1), \dots, \Delta\tilde{C}_{s_i s_i}(h) \right]^T \tag{21}$$

and define the vectors $\Delta\tilde{\mathbf{C}}_{t_i t_i}$, $\tilde{\mathbf{C}}_{s_i t_i}$ and $\tilde{\mathbf{C}}_{t_i s_i}$ in the similar manner. Define the $(2g+1) \times 1$ vector

$$\mathbf{COS}_i = [\ \cos(-g\nu_i), \quad \dots \quad \dots, \quad \cos(g\nu_i)\ ]^T \tag{22}$$

Similarly we will define the vector $\mathbf{SIN}_i$. Finally, define the *even mirror duplication* $(2h+1) \times h$ matrix $\mathbf{D}_+$ by

$$\mathbf{D}_+ = \begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} \tag{23}$$

and the *odd mirror duplication* $(2h+1) \times h$ matrix $\mathbf{D}_-$ by

$$\mathbf{D}_- = \begin{bmatrix} 0 & \dots & 0 & -1 \\ 0 & \dots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & -1 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix} \tag{24}$$

Since $\Delta\tilde{C}_{s_i s_i}(\tau)$ and $\Delta\tilde{C}_{t_i t_i}(\tau)$ are even functions of τ and $[\tilde{C}_{t_i s_i}(\tau) - \tilde{C}_{s_i t_i}(\tau)]$ is an odd function of τ, we can write $\mathbf{X}$ as:

$$\mathbf{X} = \sum_{i=1}^{I} \Bigg[\delta_{\{\nu_i \ne 0\}} \frac{1}{2} \Big\{ \mathbf{D}_+ (\Delta\tilde{\mathbf{C}}_{s_i s_i} + \Delta\tilde{\mathbf{C}}_{t_i t_i}) \mathbf{COS}_i^T$$
$$+ \mathbf{D}_- (\tilde{\mathbf{C}}_{t_i s_i} - \tilde{\mathbf{C}}_{s_i t_i}) \mathbf{SIN}_i^T \Big\} + \delta_{\{\nu_i = 0\}} \mathbf{D}_+ \Delta\tilde{\mathbf{C}}_{s_i s_i} \mathbf{1}^T \Bigg] \tag{25}$$

where $\mathbf{1}$ is the $(2g+1)\times 1$ vector of ones. Using simple matrix manipulations (see [5], p. 254) we have

$$\begin{aligned}&\mathrm{vec}(\mathbf{X})\\&=\sum_{i=1}^{I}\Bigg[\delta_{\{\nu_i\neq 0\}}\frac{1}{2}\Big\{(\mathbf{COS}_i\otimes\mathbf{D}_+)(\Delta\tilde{\mathbf{C}}_{s_is_i}+\Delta\tilde{\mathbf{C}}_{t_it_i})\\&\qquad+(\mathbf{SIN}_i\otimes\mathbf{D}_-)(\tilde{\mathbf{C}}_{t_is_i}-\tilde{\mathbf{C}}_{s_it_i})\Big\}\\&\qquad+\delta_{\{\nu_i=0\}}(\mathbf{1}\otimes\mathbf{D}_+)\Delta\tilde{\mathbf{C}}_{s_is_i}\Bigg]\end{aligned}\tag{26}$$

Based on the classical results of Bartlett for 1-D continuous-spectrum time series ([1], Proposition 7.3.1-7.3.4), we know that the vector $N^{\frac{1}{2}}\Delta\tilde{\mathbf{C}}_{s_is_i}$ is asymptotically normal with an asymptotic mean equal to zero, and an asymptotic covariance matrix $\mathbf{V}_i=[v_{\tau\kappa}]$, $(\tau,\kappa=0,\ldots,h)$, where

$$\begin{aligned}v_{\tau\kappa}=(\eta-3)C_i(\tau)C_i(\kappa)+\sum_{z=-\infty}^{\infty}\Big[&C_i(z)C_i(z-\tau+\kappa)\\&+C_i(z+\tau)C_i(z-\kappa)\Big]\end{aligned}\tag{27}$$

The vector $N^{\frac{1}{2}}\Delta\tilde{\mathbf{C}}_{t_it_i}$ has an identical asymptotic distribution.

Repeating the same arguments as in [1] , Proposition 7.3.1-7.3.3, it is not difficult to show that the vector $N^{\frac{1}{2}}(\tilde{\mathbf{C}}_{t_is_i}-\tilde{\mathbf{C}}_{s_it_i})$ is asymptotically normal with a zero asymptotic mean and an asymptotic covariance matrix $\mathbf{U}_i=[u_{\tau\kappa}]$, $(\tau,\kappa=0,\ldots,h)$, where

$$\begin{aligned}u_{\tau\kappa}=\sum_{z=-\infty}^{\infty}\Big[&2C_i(z)C_i(z+\tau-\kappa)-C_i(z+\tau)C_i(z-\kappa)\\&-C_i(z-\tau)C_i(z+\kappa)\Big]\end{aligned}\tag{28}$$

Since $s_i(n)$ and $t_i(n)$ are independent and zero-mean, it can be easily verified that the vector $(\tilde{\mathbf{C}}_{t_is_i}-\tilde{\mathbf{C}}_{s_it_i})$ is uncorrelated with the vectors $\Delta\tilde{\mathbf{C}}_{s_is_i}$ and $\Delta\tilde{\mathbf{C}}_{t_it_i}$. Clearly, $\Delta\tilde{\mathbf{C}}_{s_is_i}$ and $\Delta\tilde{\mathbf{C}}_{t_it_i}$ are independent. Moreover, the vectors $N^{\frac{1}{2}}\Delta\tilde{\mathbf{C}}_{s_is_i}$, $N^{\frac{1}{2}}\Delta\tilde{\mathbf{C}}_{t_it_i}$ and $N^{\frac{1}{2}}(\tilde{\mathbf{C}}_{t_is_i}-\tilde{\mathbf{C}}_{s_it_i})$ are asymptotically jointly normal (see Appendix for the proof). Hence, from [1], Proposition 6.3.1 (Cramer-Wold device) and Proposition 6.4.2, $N^{\frac{1}{2}}\mathrm{vec}(\mathbf{X})$ is asymptotically normal with a zero asymptotic mean and an asymptotic covariance matrix

$$\begin{aligned}\mathbf{\Gamma}=\sum_{i=1}^{I}\Bigg[&\delta_{\{\nu_i\neq 0\}}\frac{1}{4}\big\{(\mathbf{COS}_i\otimes\mathbf{D}_+)2\mathbf{V}_i(\mathbf{COS}_i\otimes\mathbf{D}_+)^T\\&+(\mathbf{SIN}_i\otimes\mathbf{D}_-)\mathbf{U}_i(\mathbf{SIN}_i\otimes\mathbf{D}_-)^T\big\}\\&+\delta_{\{\nu_i=0\}}(\mathbf{1}\otimes\mathbf{D}_+)\mathbf{V}_i(\mathbf{1}\otimes\mathbf{D}_+)^T\Bigg]\end{aligned}\tag{29}$$

Therefore, by Proposition 1 above, and by [1], Proposition 6.3.3, $N^{\frac{1}{2}}\mathrm{vec}(\Delta\tilde{\mathbf{C}}_{yy})$ has the same limiting distribution as $N^{\frac{1}{2}}\mathrm{vec}(\mathbf{X})$. Thus, (29), combined with the 1-D Bartlett formulas (27) and (28), provides a Bartlett-type formula for the asymptotic covariance matrix of the sample autocovariances. We have just proved the following theorem:

Theorem 1 *Let $\{y(n,m)\}$ be given by (2) and (3), such that Assumption 1 and Assumption 2 are satisfied. Then, as $\Psi(N,M)\to\infty$, the vector $N^{\frac{1}{2}}$vec$(\Delta\tilde{\mathbf{C}}_{yy})$ is asymptotically normal with zero asymptotic mean and asymptotic covariance matrix $\mathbf{\Gamma}$, given by (29) where $\mathbf{V}_i$ and $\mathbf{U}_i$ are given by (27), (28), respectively.*

4. DISCUSSION

In the last section we have established the limiting distribution of the sample covariances of a horizontal evanescent field observed in the presence of a Gaussian purely-indeterministic field. Using a "naive" analogy with the case of a 1-D mixed-spectrum regular process, one could expect an $(NM)^{\frac{1}{2}}$ convergence rate to the limiting distribution. However, it turns out that the rate is $N^{\frac{1}{2}}$. This fact could be explained by observing that the stochastic properties of the evanescent field are determined by those of its modulating 1-D processes. Since the rate of convergence of the sample covariances of the modulating processes is $N^{\frac{1}{2}}$, so is the rate for the evanescent field itself. As a result of this slow convergence rate, one can expect that in estimating the covariances of a 2-D homogeneous random field, the dominant estimation error would be due to the evanescent components.

5. CONCLUSIONS

In the paper we have considered the asymptotic properties of the sample covariance sequence of a field composed of a sum of horizontal evanescent components and a Gaussian purely-indeterministic component. We have proved that the sample covariance function of the field is asymptotically normal and derived a Bartlett-type formula for the asymptotic covariances of the sample autocovariances. The proven results are straightforward to extend to the more general case where the purely-indeterministic component has an absolutely summable sequence of fourth-order cummulants. (Thus, the case where the noise field is Gaussian, becomes a special case, as its fourth-order cummulants are zero).

6. APPENDIX

We would like to show that the vectors $N^{\frac{1}{2}}\Delta\tilde{\mathbf{C}}_{s_is_i}$, $N^{\frac{1}{2}}\Delta\tilde{\mathbf{C}}_{t_it_i}$ and $N^{\frac{1}{2}}(\tilde{\mathbf{C}}_{t_is_i}-\tilde{\mathbf{C}}_{s_it_i})$ are asymptotically jointly normal. Since the proof follows the technique of [1], Propositions

7.3.2 and 7.3.3, we will only present the outlines of the proof.

First, we assume that the modulating 1-D processes $\{s_i(n)\}$ and $\{t_i(n)\}$ of each evanescent field are **finite** order MA processes, *i.e.*,

$$s_i(n) = \sum_{j=-m}^{m} a_i(j)\xi_i(n-j)\ , \qquad (30)$$

$$t_i(n) = \sum_{j=-m}^{m} a_i(j)\zeta_i(n-j)\ , \qquad (31)$$

where $\xi_i(n)$, $\zeta_i(n)$ are independent 1-D zero mean white innovation processes of identical variance σ^2 and

$$\mathrm{E}\Big[\xi_i^4(n)\Big] = \mathrm{E}\Big[\zeta_i^4(n)\Big] = \eta\sigma^4 \qquad (32)$$

For every i, the sequence $a_i(j)$ is an absolutely summable deterministic sequence, *i.e.*, $\sum_{j=-m}^{m} |a_i(j)| < \infty$.

Define a sequence of random $3(h+1)\times 1$ vectors $R(n)$ by

$$R(n) = \begin{bmatrix} s_i(n)s_i(n) - C_i(0) \\ \vdots \\ s_i(n)s_i(n+h) - C_i(h) \\ t_i(n)t_i(n) - C_i(0) \\ \vdots \\ t_i(n)t_i(n+h) - C_i(h) \\ t_i(n)s_i(n) - s_i(n)t_i(n) \\ \vdots \\ t_i(n)s_i(n+h) - s_i(n)t_i(n+h) \end{bmatrix} \qquad (33)$$

Thus, $\{R(n)\}$ is a strictly stationary $(2m+h)$-dependent sequence (for a definition of m-dependence, see [1], Definition 6.4.3) and

$$N^{-1}\sum_{n=0}^{N-1} R(n) = \begin{bmatrix} \Delta\tilde{\mathbf{C}}_{s_i s_i} \\ \Delta\tilde{\mathbf{C}}_{t_i t_i} \\ \tilde{\mathbf{C}}_{s_i s_i} - \tilde{\mathbf{C}}_{t_i t_i} \end{bmatrix} \qquad (34)$$

For any vector $\lambda \in \mathcal{R}^{3(h+1)}$, the sequence $\lambda^T R(n)$ is an $(2m+h)$-dependent process. Since $s_i(n)$ and $t_i(n)$ are independent with zero mean, it can be easily verified that the vector $(\tilde{\mathbf{C}}_{t_i s_i} - \tilde{\mathbf{C}}_{s_i t_i})$ is uncorrelated with the vectors $\Delta\tilde{\mathbf{C}}_{s_i s_i}$ and $\Delta\tilde{\mathbf{C}}_{t_i t_i}$. Moreover, the latter two are independent. Hence,

$$\lim_{N\to\infty} N var\{\sum_{n=0}^{N-1} \lambda^T R(n)\} = \lambda^T \mathbf{\Lambda}\lambda > 0 \qquad (35)$$

where

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{V}_i & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{U}_i \end{bmatrix} \qquad (36)$$

is a positive definite matrix, where $\mathbf{V}_i$ and $\mathbf{U}_i$ are given by (27), (28), respectively. Then by the central limit theorem for strictly stationary m-dependent sequences ([1], Theorem 6.4.2) the sequence $N^{-1}\sum_{n=0}^{N-1} \lambda^T R(n)$ is asymptotically normal with zero mean and covariance $N^{-1}\lambda^T\mathbf{\Lambda}\lambda$ for any $\lambda \in \mathcal{R}^{3(h+1)}$. Applying the Cramer-Wold device ([1], Proposition 6.3.1) we obtain that $N^{\frac{1}{2}}(\tilde{\mathbf{C}}_{t_i s_i} - \tilde{\mathbf{C}}_{s_i t_i})$, $N^{\frac{1}{2}}\Delta\tilde{\mathbf{C}}_{s_i s_i}$ and $N^{\frac{1}{2}}\Delta\tilde{\mathbf{C}}_{t_i t_i}$ jointly asymptotically normal. Extension to the case where the modulating processes $\{s_i(n)\}$ and $\{t_i(n)\}$ of each evanescent field is the **infinite** order MA model (by $m \to \infty$) follows from *exactly* the same reasons as in [1], Proposition 7.3.3.

7. REFERENCES

[1] P.J. Brockwell and R.A. Davis, *Time Series: Theory and Methods*, Springer, 1991.

[2] J. M. Francos, A. Z. Meiri, and B. Porat, “A Wold-like decomposition of 2-D discrete homogenous random fields,” *Ann. Appl. Prob.*, vol. 5, pp. 248-260, 1995.

[3] G. Cohen and J. M. Francos, “Spectral Representation and Asymptotic Properties of Certain Deterministic Fields with Innovation Components,” *Probab. Theory Relat. Fields*, vol. 127, pp. 277-304, August 2003.

[4] E. J. Hannan, *Multiple Time Series*, John Wiley & sons, 1970.

[5] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.

[6] S. M. Kay, *Fundamentals of Statistical Signal Processing: Estimation Theory*, Prentice Hall, 1993.

[7] M. Kliger and J. M. Francos, “Asymptotic normality of the sample covariances of evanescent fields in noise,” *in preparation*.

[8] T.-H. Li, B. Kedem and S. Yakowitz, “Asymptotic normality of sample autocovariance with an application in frequency estimation,” *Stochastic Processes appl.*, vol. 52, pp. 329-349, 1994.

[9] T.-H. Li, “Bartlett-type formulas for complex multivariate time series of mixed spectra,” *Stat. Prob. Let.*, vol. 28, pp. 259-268, 1996.

[10] M.S. Maskisack and D.S. Poskitt, “Some properties of autoregressive estimates for processes with mixed spectra,” *J. Time Series Anal.*, vol. 11, pp. 325-337, 1990.

[11] P. Stoica, Y. Soderstrom and F.-N. Ti, “Overdetermined Yule-Walker estimation of the parameters of multiple sinusoids: accuracy aspects, *Signal Process.*, vol. 16, pp. 155-174, 1989.