DETECTION OF SIGNALS IN NONSTATIONARY RANDOM NOISE VIA STATIONARIZATION AND STATIONARITY TEST*

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ABSTRACT

In this paper, an effective method is proposed for detecting a signal which is corrupted by nonstationary random noise. First, the noisy observation data whose statistical property is nonstationary owing to the noise is modified such that they become stationary ones. Then, based on this stationarized observation process, the stationarity test is executed to detect the signal. The method is tested by simulations to show the efficacy.

1. INTRODUCTION

Recently, in the field of statistical signal processing there has been a renewal of research interest in the theory of nonstationary processes, and many of the new techniques to handle nonlinear and/or nonstationary processes have developed in the areas of statistics, engineering or more specified processes such as environmental science, psychophysics or mathematical finance [1].

Especially, the signal detection has been recognized as one of the most important problems in the signal processing area for several decades. However, the most conventional approaches to the signal detection is based on the assumption that the corrupting random noise is stationary with (time-invariant) power spectrum (e.g. [2,3]). As well-known, the short-time Fourier transform, wavelet and Wigner distribution are regarded as useful tools to detect signals in random noise; however, these may be effective only for the signals, leaving the nonstationarity property of the noise. In this paper, focussing our attention to the nonstationarity of the corrupting noise, we investigate a signal detection problem.

2. PROBLEM STATEMENT

Let $y(t)$ be a (scalar) observation data given in the form,

$$y(t) = s(t) + \eta(t), \qquad 0 \leq t, \tag{1}$$

where $s(t)$ is a (nonrandom and locally existing) signal to be detected; and $\eta(t)$ is the random noise. Up to the present time, in the most literature the noise $\eta(t)$ has been treated as a white noise, otherwise, treated as a stationary colored noise process. In the most real situation, it may have a nonstationarity nature. For example, under the sea the communication or sensing situation is greatly influenced by the conditions of tidal waves. If the intensity of the tidal wave increases, then the intensity (variance) of the random disturbance also becomes increasing. Consequently, in order to reflect the actual situation, the environmental noise should be modeled as a nonstationary process. Keeping this fact in mind, the random noise $\eta(t)$ is assumed to be generated by the process described by

$$d\eta(t) = -\beta(t)\eta(t)dt + \alpha(t)dw(t), \quad \eta(0) = \eta_0, \tag{2}$$

where $w(t)$ is a (scalar) standard Wiener process; $\alpha(t)$, $\beta(t)$ are slowly and smoothly varying positive but unknown functions; and the initial value η_0 is a Gaussian random variable with zero-mean and unit variance. Then, the noise $\eta(t)$ is the nonstationary process having time-varying power spectral density function. This spectral density function may be interpreted as the evolutionary spectral density in the sense of Priestley [4].

Hence, our purpose is to propose a method to detect the signal $s(t)$ from the nonstationary observation data $\{y(t)\}$. Our procedure for the signal detection is as follows:

(i) First, based on the noise model (2), the coefficient functions $\alpha(t)$ and $\beta(t)$ are estimated using observation data $\{y(t)\}$.

(ii) Then, using the estimates for $\alpha(t)$ and $\beta(t)$ obtained in (i), the observation data $y(t)$ is modified to become a stationary process. This procedure is called the *stationarization* of the nonstationary process in this paper.

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(iii) For the stationarized process, its stationarity is tested incorporated with the KM$_2$O-Langevin equation [5-7].

3. PROPOSED APPROACH

Recalling the basic assumption that the duration of the signal $s(t)$ is very local, consider the case of signal-free,

$$y(t) = \eta(t), \tag{3}$$

or equivalently, its stochastic differential,

$$\begin{aligned} dy(t) &= d\eta(t) \\ &= -\beta(t)y(t)dt + \alpha(t)dw(t), \end{aligned} \tag{4}$$

which shows that the observation process is clearly nonstationary.

A. Parameter Estimation:
We have assumed that the coefficient parameters $\alpha(t)$ and $\beta(t)$ change slowly and smoothly. More concretely speaking, in an interval I_t around the current time t, they are assumed to behave approximately like a constant, i.e.,

$$\alpha(t) = \alpha_t, \quad \beta(t) = \beta_t \quad \text{for } t \in I_t. \tag{5}$$

Recalling that the power spectral density of the process (2) with constant parameters α and β is given by $S(\lambda) = \alpha^2/(\lambda^2 + \beta^2)$, the power spectral density of the $\eta(t)$-process is approximately evaluated as

$$S_t(\lambda) = \frac{{\alpha_t}^2}{\lambda^2 + {\beta_t}^2} \tag{6}$$

under the local stationarity assumption (5). The suffix in the notation $S_t(\lambda)$ stands for the dependence on t. In this sense, $S_t(\lambda)$ may be interpreted as the time-varying spectral density (or, evolutionary spectral density in the sense of Priestley).

With the help of Priestley's method of the estimation of evolutionary (time-varying) spectral density, the density $S_t(\lambda)$ can be estimated from the observation data $\{y(t)\}$ (provided that no signal exists in the data). Let it denote by $\hat{S}_t(\lambda)$.

From the relation (6), we have

$$\frac{1}{S_t(\lambda)} = \frac{1}{{\alpha_t}^2}\,\lambda^2 + \left(\frac{{\beta_t}^2}{{\alpha_t}^2}\right). \tag{7}$$

Then, the coefficients $1/{\alpha_t}^2$ and ${\beta_t}^2/{\alpha_t}^2$ can be determined as the least-squares estimates by minimizing the square-error, $|1/S_t(\lambda) - 1/\hat{S}_t(\lambda)|^2$.

Though both coefficients are obtained, we employ here only the first one to obtain $\hat{\alpha}_t = \sqrt{\hat{\alpha}_t^2}$. The use of the estimate obtained here for β_t is not recommended from the viewpoint of its accuracy. Instead, this is estimated from the standpoint of the energy of the noise process. It is calculated in the neighborhood of t as

$$v(t) = \int_{-\infty}^{\infty} S_t(\lambda)d\lambda = \frac{{\alpha_t}^2}{\beta_t}\,\pi. \tag{8}$$

From this, we have the estimate $\hat{\beta}_t$ by

$$\hat{\beta}_t = \frac{\hat{\alpha}_t^2}{\hat{v}_t}\,\pi, \tag{9}$$

where $\hat{v}_t = \int_{-\infty}^{\infty} \hat{S}_t(\lambda)\,d\lambda$.

B. Stationarization of $y(t)$-process:
Having obtained the parameter estimates $\hat{\alpha}_t$ and $\hat{\beta}_t$, the observation process (4) is expressed as follows:

$$dy(t) = -\hat{\beta}_t\,y(t)dt + \hat{\alpha}_t\,dw(t). \tag{10}$$

In order to relate this process to the (discrete-time) KM$_2$O-Langevin equation later, let us introduce the discretized version for (10). To do this, consider the stochastic differential (10) as the small increment, and write (10) as

$$\delta y_t + \hat{\beta}_t\,y_t\,\delta t = \hat{\alpha}_t\,\delta w_t. \tag{11}$$

Diving both sides by $\hat{\alpha}_t\,\delta t$ provided that $\hat{\alpha}_t \neq 0$, we have

$$\frac{\delta y_t + \hat{\beta}_t\,y_t\,\delta t}{\hat{\alpha}_t\,\delta t} = \frac{\delta w_t}{\delta t}. \tag{12}$$

Then, the right-hand side of (12) can be identified to a stationary white Gaussian noise sequence with zero-mean and unit power spectral density. Keeping this fact in mind, let us define for each t a sequence $\hat{y}_t$ by

$$\hat{y}_t = \frac{\delta y_t + \hat{\beta}_t\,y_t\,\delta t}{\hat{\alpha}_t}. \tag{13}$$

Then this process can be regarded as a (discrete-time) stationarized version of the observation process $y(t)$.

C. Representation by the KM$_2$O-Langevin Equations:
Broadly speaking, the KM$_2$O-Langevin equations, where KM$_2$O stands for the names of Kubo, Mori, Miyoshi and Okabe, can be regarded as (discrete-time) versions of AR models with time-dependent coefficients.

The forward and backward KM$_2$O-Langevin equations proposed by Okabe [5-7] are expressed as follows:

$$\begin{aligned} Y(k) = -\sum_{n=1}^{k-1} \gamma_+(k,n)Y(n) - \delta_+(k)Y(0) \\ + \nu_+(k) \end{aligned} \tag{14a}$$

$$\begin{aligned} Y(-k) = -\sum_{n=1}^{k-1} \gamma_-(k,n)Y(-n) - \delta_-(k)Y(0) \\ + \nu_-(-k) \quad (k = 1, 2, \cdots, K) \end{aligned} \tag{14b}$$

where $Y(\cdot)$ is a random sequence; $\nu_+(\cdot)$ and $\nu_-(\cdot)$ are the forward and backward innovation sequences, respectively; $\{\gamma_+(\cdot,\cdot), \delta_+(\cdot)\}$ and $\{\gamma_-(\cdot,\cdot), \delta_-(\cdot)\}$ are coefficients called the KM$_2$O-Langevin data and these can be determined uniquely from the observation data $Y(\cdot)$.

It is proved that the following statements are equivalent:

(S1) The random sequence $Y(\cdot)$ is locally and weakly stationary;

(S2) The processes $\{\nu_+(\cdot)\}$ and $\{\nu_-(\cdot)\}$ are sequences of mutually independent random variables.

In this paper, the stationarity test for an observed random data will be carried out based on these statements.

D. Stationarity Test:
Let I_t be a time interval around the current time t, $I_t = [t-M, t+M]$ $(M > 0)$. Partition I_t into $2K$ equi-intervals. For a set of given data $\{\hat{y}_{t\pm i\Delta}\}_{i=0,1,\cdots,K}$ $(\Delta = M/K)$ during the time interval I_t, normalize them such that they are distributed as $N[0,1]$, and let these normalized data be denoted $\{\hat{Y}_{t\pm i\Delta}\}_{i=0,1,\cdots,K}$. Then, identify these data with $\{Y(\pm k)\}_{n=0,1,\cdots,K}$ in (14).

The stationarity test is executed as follows:

(i) First, compute sample covariances $R(\ell)$ for $\ell = 0, 1, 2, \cdots, L$ $(< K)$, where L is an integer determined from the confidence level.

(ii) Then, with $\gamma_+(k,0) = \delta_+(k)$, $\gamma_-(k,0) = \delta_-(k)$ and $V_+(0) = V_-(0) = R(0)$ as initial conditions, two sets of parameters $\{V_+(k), \gamma_+(k,\cdot), \delta_+(k)\}$ and $\{V_-(k), \gamma_-(k,\cdot), \delta_-(k)\}$ are computed by the recursive algorithm, where $\{V_\pm(\cdot)\}$ are variances of $\nu_\pm(\cdot)$.

(iii) After determined these parameter sets, the fluctuation $\nu_+(k)$ is computed from the forward KM$_2$O-Langevin equation (14a) as follows:

$$\left.\begin{aligned} &\nu_+(0) = Y_t(0) \\ &\nu_+(k) = Y_t(k) + \sum_{n=0}^{k-1} \gamma_+(k,n) Y_t(n) \\ &\qquad\qquad\qquad (k = 1, 2, \cdots, L). \end{aligned}\right\} \tag{15}$$

Using the realizations $\{\hat{Y}_{t\pm i\Delta}\}$ in (15) as mentioned above, we obtain the data $\{\nu_+(k)\}$.

(iv) In order to facilitate the stationarity test, the data $\{\nu_+(k)\}$ are normalized using $V_+(\cdot)$. Let the normalized random data be denoted by $\xi(k)$. Then, the claim is to test whether the data $\{\xi(k)\}$ are zero-mean white noise with unit variance or not. In order to test the whiteness of the data, the portmanteau test [8] is used.

4. SIMULATION EXPERIMENTS

Figures 1-3 show a typical one set of simulation results. These are obtained for noise process (2) with

$$\beta(t) = \frac{C}{\sqrt{2\pi}D} \exp\left\{-\frac{t^2}{2D^2}\right\}, \ \alpha(t) = \sqrt{2\beta(t)}. \tag{16}$$

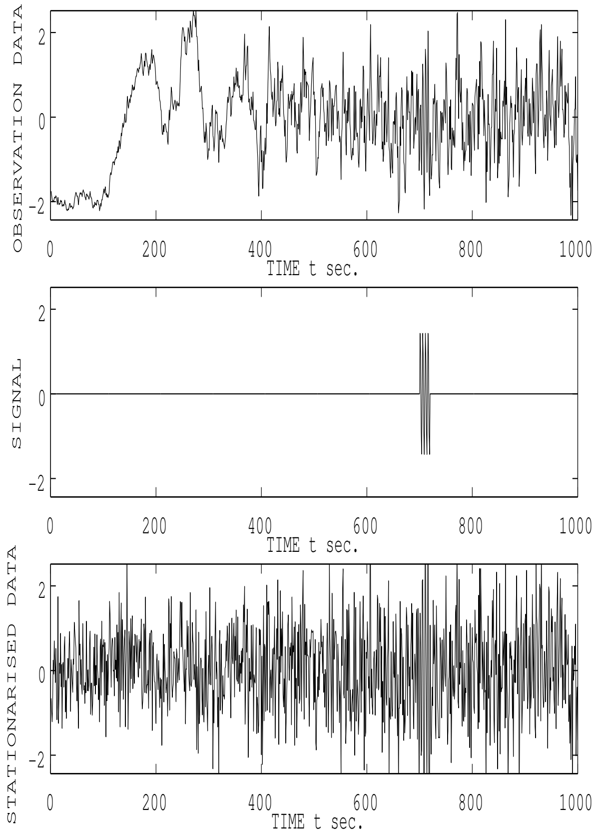

Fig. 1. A sample path of the observation data $y(t)$ (top), the embedded signal $s(t)$ (middle) and the stationarized data $\hat{y}(t)$ (bottom).

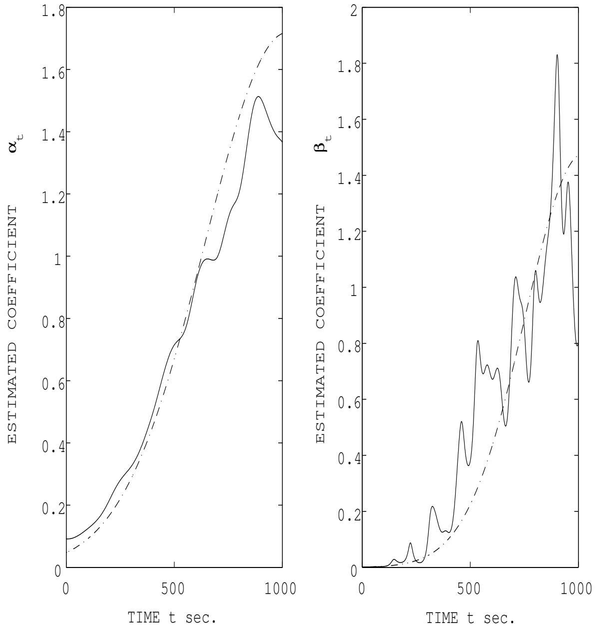

Fig. 2. Estimation of $\alpha(t)$ (left) and $\beta(t)$ (right) (where the dash-dot line indicates their true values.)

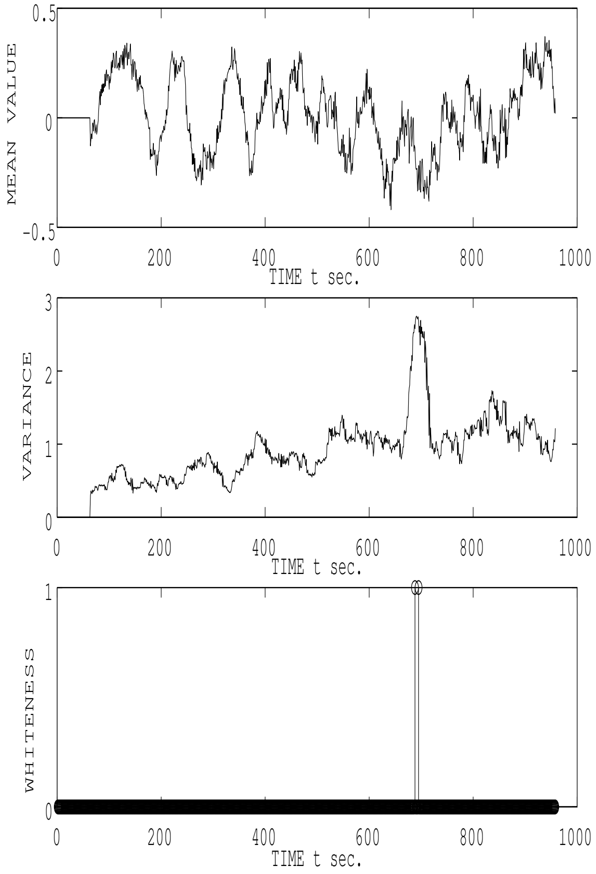

Fig. 3. The mean and the variance of $\{\xi(k)\}$ (top and middle), and the whiteness test.

($C = 1000$, $D = 270$). The top figure in Fig. 1 depicts the sample path of the observation $y(t)$ in which a locally distributed four-cycle sinusoidal wave shown in the middle is embedded during the time interval [701,720] s. The sample path exhibits a typical nonstationary process. The stationarized process is shown in the bottom.

The estimated coefficients $\hat{\alpha}_t$ and $\hat{\beta}_t$ are illustrated in Fig. 2. Comparing them with their true values, we may fairly say that the estimates are well performed.

Figure 3 shows the results of the test of stationarity for signal detection. The mean values of the normalized data $\{\xi(k)\}$ is depicted in the top, showing that its process seems to be zero in the sense of time average. The variance is shown in the middle figure. Clearly, a salient feature can be observed around $t = 700$ s where the localized signal exists. Whiteness test of the process $\{\xi(k)\}$ is shown in the bottom. The two levels, 0 and 1, indicate that the hypotheses for the whiteness will be accepted or not, respectively. The figure shows that the whiteness of $\{\xi(k)\}$ is rejected around $t = 700$ s, and hence the decision is made that the signal exists there.

5. REFERENCES

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