AUTOCORRELATION-BASED ALGORITHM FOR ARMA MODEL ORDER SELECTION IN COLORED GAUSSIAN NOISE

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ABSTRACT

In this paper, we have addressed the ARMA model order selection problem for the case of colored Gaussian noise using autocorrelation. The most well known solutions for the ARMA model order problem are the Akiake information criterion (AIC), the minimum description length (MDL), and the minimum eigenvalue (MEV) criterion. In the MEV method, observation and/or modeling error is assumed to be zero-mean white Gaussian. This paper presents a generalization of the original results in the MEV method to the colored Gaussian noise for the second order statistics. Simulations show the performance of the generalization results.

1. INTRODUCTION

Time series analysis provides a parametric model for stationary stochastic processes such as autoregressive (AR), moving average (MA), and autoregressive moving average (ARMA) processes. In system identification and signal modeling, estimating the parameters of a model, such as ARMA model is a major goal. Many techniques have been developed to solve this problem. However, most of them assume that the model order is known. In most realistic situations, the model order is not known and has to be estimated before solving the parameter estimation problem. The problem of ARMA model order estimation plays a significant role in many areas such as spectral estimation, seismology, speech processing, system identification, biomedical signal processing, and adaptive control. The most well known solutions for the ARMA model order problem are the Akiake information criterion (AIC) [1], the minimum description length (MDL) [2, 3], and the minimum eigenvalue (MEV) criterion [4].

The MEV criterion of Liang *et al*. [4] has been investigated in [5, 6, 7, 8] and is shown to have accuracy never before achieved. Liang assumed that the observation and/or modeling error to be zero mean white Gaussian. Al-Smadi and Wilkes [6] extended the results of Liang to third order cumulants (TOCs). Their paper also extended the MEV method to the colored Gaussian noise in the third order cumulants case. They used the property that higher order cumulants are blind to any kind of Gaussian process. The paper in [5] extended the results of Liang to the two dimensional model order estimation. The paper in [7] proposed a method that uses the determinant of sub-matrices of the MEV criterion to search for the corner that contains the estimates of the true order. The search was based on determining the singularity of principal sub-matrices through determinants.

This paper presents a generalization of the original results in [4] to the colored Gaussian noise for the second order statistics (i. e., autocorrelation). Gaussian noise is important in science and engineering. There are many reasons for that. For example, it is simple, tractable, and fairly realistic model [9]. That is, the Gaussian noise has many properties that make analytic results possible. It also describes several types of physical phenomena that are usually confirmed by experiments. Furthermore, the central limit theorem provides the mathematical justification for using the Gaussian distribution as a model for a large number of different physical phenomena in which the observed random variable is the result of a large number of individual random processes. These reasons make the Gaussian process very fundamental and important model in engineering and science problems.

2. STATEMENT OF THE PROBLEM

A general model for ARMA process can be represented as follows:

$$y(n) = \sum_{i=1}^{p} a_i y(n-i) + \sum_{i=0}^{q} b_i x(n-i) \qquad (1)$$

where $y(n)$ is the observed time series data, $x(n)$ is the excitation input signal. The a_i (i= 1, 2, …, p) and b_i (i= 0, 1, …, q) are the coefficients of the ARMA model, while (p, q) is the order of the ARMA model.

It is assumed that the model in (1) is stable, invertible, and has no pole-zero cancellations. Clearly, the problem is twofold: one has both to select the order of the predictor and then to compute the predictor coefficients. That is, the ARMA model identification problem is as follows: given a set of time series observations $y(n)$, where $0 \le n \le N$ (N denotes the number of samples), from an unknown ARMA(p,q) process, we have to determine the unknown parameters a_i and b_i, in addition to the model order (p, q).

The signal $y(n)$ is observed in additive noise $w(n)$

$$y(n) = x(n) + w(n) \tag{2}$$

where $w(n)$ is additive Gaussian noise. The system in (1) can be written in a compact form as

$$D_{pq}\, \underline{\theta} = \underline{w} \tag{3}$$

where D_{pq} is an $N\times(p+q+2)$ composite data matrix, $\underline{\theta}$ is a $(p+q+1)$ parameter vector, and $\underline{w}$ is a $N\times1$ observation noise/modeling error vector which is assumed to be zero-mean white Gaussian noise.

The data covariance matrix is obtained as

$$R_{pq} = D_{pq}^{T} D_{pq} \tag{4}$$

Liang *et al*. [4] proposed a method to estimate the ARMA model order p and q under the assumption that the modeling error $w(n)$ is white Gaussian noise. The method is derived from the minimum description length (MDL) criterion [2] and is based on the minimum eigenvalue of the covariance matrix R_{pq}. Liang's method leads to the criterion

$$J_{MEV}(p,q) = \lambda_{min} (N)^{(\frac{p+q}{N})} \tag{5}$$

where p is an estimate of the number of poles, q is an estimate of the number of zeros, N is the number of data points, and λ_{min} is the minimum eigenvalue of R_{pq}.

The MEV criterion calculates a table of $J_{MEV}(p, q)$ for all values of p and q. The table is organized so that p increases from left to right while q increases from top to bottom down the table. The search method utilizes row-ratio and column-ratio tables. The tables are formed by dividing each row (or column) of the $J_{MEV}(p, q)$ by the previous row (or column). An estimate of the number of poles, p, is set equal to the column number that contains the minimum value of column ratio table. Similarly, the number of zeros, q, is set equal to the number of the row having the minimum value of the row ratio table.

3. GENERALIZATION OF THE MEV RESULTS TO COLORED NOISE

Now, we will generalize the results of Liang to the colored Gaussian noise case where the Gaussian noise is assumed to have a short correlation length. For a Gaussian distribution with parameters μ and σ^2, the probability density function (pdf) is given by

$$f(w) = \frac{1}{\left(2\pi\sigma^2\right)^{\frac{1}{2}}}\, e^{-\frac{1}{2}\left(\frac{w-\mu}{\sigma}\right)^2} \tag{6}$$

The pdf for an N-dimensional Gaussian distribution with N-vector parameter $\underline{\mu}$ and an $N\times N$ matrix parameter R_{ww} is given by [10]

$$f(\underline{w}) = f(\underline{y}/\underline{\theta}) = \frac{1}{(2\pi)^{\frac{N}{2}}\left(|R_{ww}|\right)^{\frac{1}{2}}}\, e^{-\frac{1}{2}[(\underline{w}-\underline{\mu})^T R_{ww}^{-1} (\underline{w}-\underline{\mu})]} \tag{7}$$

where the notation $|R_{ww}|$ is used for the determinant of R_{ww}. If the mean is considered to be zero, the pdf becomes

$$f(\underline{w}) = f(\underline{y}/\underline{\theta}) = \frac{1}{(2\pi)^{\frac{N}{2}}\left(|R_{ww}|\right)^{\frac{1}{2}}}\, e^{-\frac{1}{2}[\underline{w}^T R_{ww}^{-1} \underline{w}]} \tag{8}$$

Substituting Equation (3) into Equation (8), we obtain

$$f(\underline{w}) = f(\underline{y}/\underline{\theta}) = \frac{1}{(2\pi)^{\frac{N}{2}}\left(|R_{ww}|\right)^{\frac{1}{2}}}\, e^{-\frac{1}{2}[\underline{\theta}^T D_{pq}^T R_{ww}^{-1} D_{pq}\, \underline{\theta}]} \tag{9}$$

Using the MDL criterion [2, 3]

$$J_{MDL} = -log[f(\underline{y}/\underline{\theta})] + \frac{1}{2} d_k log(N) \tag{10}$$

where the first term is the log-likelihood function and the second term provides the penalty for over parameterization and d_k is the number of independently adjusted parameters within the model.

$$d_k = p + q + 1 \tag{11}$$

Substituting Equations (9) and (11) into Equation (10), we obtain

$$J_{MDL} = \frac{N}{2} log(2\pi) + \frac{1}{2} log|R_{ww}| + \frac{1}{2}\, \underline{\theta}^T D_{pq}^T R_{ww}^{-1} D_{pq}\, \underline{\theta} + \frac{1}{2}(p+q+1) log(N) \tag{12}$$

Let us assume that

$$R_{ww} = \sigma^2 \widetilde{R}_{ww} \tag{13}$$

where $\widetilde{R}_{ww}$ is scaled so that the diagonal is 1. Then

$$R_{ww}^{-1} = \frac{1}{\sigma^2} \widetilde{R}_{ww}^{-1} \tag{14}$$

where σ^2 is the variance of $\underline{w}$. Hence,

$$J_{MDL} = \frac{N}{2} log(2\pi) + \frac{1}{2} log|\sigma^2 \widetilde{R}_{ww}| + \frac{1}{2\sigma^2}\, \underline{\theta}^T D_{pq}^T \widetilde{R}_{ww}^{-1} D_{pq}\, \underline{\theta} + \frac{1}{2}(p+q+1) log(N) \tag{15}$$

Using the property that $\left|kA\right| = k^n\left|A\right|$ for a matrix A with order $n \times n$ [11], then

$$\frac{1}{2} log|\sigma^2 \widetilde{R}_{ww}| = \frac{1}{2} log[\sigma^{2N} |\widetilde{R}_{ww}|]$$

$$= \frac{N}{2} log\,\sigma^2 + \frac{1}{2} log|\widetilde{R}_{ww}| \tag{16}$$

If p and q are fixed and $\underline{\theta}$ has a unit Euclidean norm (i.e., has unit length), then the $\underline{\theta}$ that minimizes Equation (15) is the eigenvector associated with the minimum eigenvalue, λ_{min}, of $D_{pq}^T \widetilde{R}_{ww}^{-1} D_{pq}$. With $\underline{\theta} = \underline{\theta}_{min}$, the following expression is obtained

$$\underline{\theta}_{\min} D_{pq}^T \widetilde{R}_{ww}^{-1} D_{pq} \underline{\theta}_{\min}^T = \underline{w}^T \widetilde{R}_{ww}^{-1} \underline{w} = \lambda_{\min} \tag{17}$$

Substituting Equations (16) and (17) into Equation (15), we obtain

$$J_{MDL} = \frac{N}{2} log(2\pi) + \frac{N}{2} log\,\sigma^2 + \frac{1}{2} log|\widetilde{R}_{ww}| + \frac{1}{2\sigma^2}\lambda_{min} + \frac{1}{2}(p+q+1) log(N) \tag{18}$$

Now, dropping the terms that do not depend on p, q, or $\underline{\theta}$, we have

$$J_{MDL} = \frac{N}{2} log\,\sigma^2 + \frac{1}{2} log|\widetilde{R}_{ww}| + \frac{1}{2\sigma^2}\lambda_{min} + \frac{1}{2}(p+q) log(N) \tag{19}$$

Corollary: The expected value of the term $\underline{w}^T R_{ww}^{-1} \underline{w}$ is N, i.e.

$$E[\underline{w}^T R_{ww}^{-1} \underline{w}] = N$$

Proof: Let us denote the trace of the matrix by tr, then

$$E[\underline{w}^T R_{ww}^{-1} \underline{w}] = E[tr\{\underline{w}^T R_{ww}^{-1} \underline{w}\}] =$$

$$tr\{E[\underline{w}\underline{w}^T] R_{ww}^{-1}\} = tr\{R_{ww} R_{ww}^{-1}\} = tr\{I\} = N \tag{20}$$

Now, using Equations (13) and (17), we obtain

$$\underline{w}^T \widetilde{R}_{ww}^{-1} \underline{w} = \lambda_{min} \approx N\sigma^2 \tag{21}$$

Thus, Equation (19) becomes

$$J_{MDL} = \frac{N}{2} log \frac{\lambda_{\min}}{N} + \frac{1}{2} log|\widetilde{R}_{ww}| + \frac{N}{2} + \frac{1}{2}(p+q) log(N) \tag{22}$$

As a simplifying assumption, we assume that $\widetilde{R}_{ww}$ is fixed to the value obtained when $\underline{\theta}$ contains the true parameters. Thus, $|\widetilde{R}_{ww}|$ is independent of p and q. Hence, $log|\widetilde{R}_{ww}|$ can be ignored. In addition, $\frac{N}{2}$ is independent of p and q. Therefore, the criterion in Equation (22) becomes

$$J_{MDL} = \frac{N}{2} log \frac{\lambda_{\min}}{N} + \frac{1}{2}(p+q) log(N) \tag{23}$$

Rearranging and using logarithmic properties, we have

$$\frac{2}{N} J_{MDL} = log\left[\lambda_{\min}\left(N^{\left(\frac{p+q}{N}\right)}\right)\right] \tag{24}$$

Since the log function is monotonically increasing, then the following criterion contains the same information as Equation (5), that is

$$J(p,q) = \lambda_{min}(N)^{(\frac{p+q}{N})} \tag{25}$$

where p is an estimate of the number of poles, q is an estimate of the number of zeros, N is the number of data points, and λ_{min} is the minimum eigenvalue of $R_{pq} = D_{pq}^T \widetilde{R}_{ww}^{-1} D_{pq}$. Since $w(n)$ is assumed to have a short correlation length, then $\widetilde{R}_{ww}$ is nearly diagonal. Therefore, to obtain λ_{min}, $\widetilde{R}_{ww}$ is approximated by identity matrix. This assumption is also necessary for computational reasons, i.e., in computing $\widetilde{R}_{ww}^{-1}$. Therefore, if the modeling error $w(n)$ is colored Gaussian process with a short correlation length instead of being assumed white, then the basic MEV procedure can be used.

The only available data is the output observation. However, the excitation input signal is necessary to compute the cross-cumulants, which is an intermediate step in the identification process. Therefore, the modeling technique in [4] was used to form a large AR model to fit the observed output sequence, and the input sequence was estimated using inverse filtering.

4. SIMULATIONS

In this section, several examples were considered to confirm the theoretical developments. A finite length sequence was used as the input to simulate the following examples. The length of the data was taken to be N = 2000 points for each experiment. All computations were performed using MATLAB. Simulations were carried out using white and colored Gaussian additive noise. It should be noted that we did not use any noise reduction techniques as was done in [4].

Example 1**:** Data where generated according to the model [4]

y(n) -1.2798*y(n-1)* + 0.781*y(n-1)* -.1635*y(n-1)* + .7566*y(n-1)* -1.062*y(n-1)* .7821*y(n-1)* = *x(n)* -.2998*x(n-1)* + .415*x(n-1)* -.28*x(n-1)* +.497*x(n-1)* (26)

This is an ARMA model with six poles and four zeros. The poles are located at $0.98\,e^{\pm j30^0}$, $0.96\,e^{\pm j135^0}$, and $0.94\,e^{\pm j60^0}$. The zeros are located at $0.86\,e^{\pm j120^0}$ and $0.82\,e^{\pm j45^0}$. The observed time series is *y(n) = x(n) + w(n)*. The data was generated by passing a Gaussian input through this filter. The output was then corrupted with additive colored Gaussian noise at different signal-to-noise ratio (SNR) on the output sequence. The colored Gaussian noise *w(n)* was obtained by passing a zero-mean random Gaussian distribution through the following *sinc* function.

$$h(k) = 0.3\, sinc(0.01k) \qquad -5 \le k \le 5 \qquad (27)$$

After that Equation (25) was calculated and the MEV method was used to estimate the model order. The MEV criterion was also calculated for the white Gaussian noise. The simulation with noise with different seeds was performed 100 trials each time. The results are shown in Table 1.

Table (1) Model order estimation results for Example (1).

	Number of Correct Estimate	
dB	Colored Gaussian Noise	White Gaussian Noise
0	0	0
5	56	54
10	90	91
15	100	100

The estimate of the order (*p, q*) is considered correct only if both *p* and *q* orders are identified correctly. It is worth noting that the MEV method does not require prior estimation of the model parameters. This means that the MEV method has fewer computations than other MDL-based model order estimation techniques. It should be mentioned that the original paper of Liang *et al.* [4] showed that the MDL did not work well at low SNR. In addition, the MDL technique is more computationally expensive.

5. CONCLUSION

In this paper, we have addressed the order selection problem for the case of colored Gaussian noise in the second order statistics. The proposed solution is a generalization to the original results of Liang *et al.* in the MEV criterion. Simulation results are presented with colored Gaussian additive noise to show the performance of the generalized method.

6. REFERENCES

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