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A Rational Krylov Iteration for Optimal $\mathcal{H}_2$ Model Reduction

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In this paper, we address the optimal $\mathcal{H}_2$ approximation of a stable, single-input single-output large-scale dynamical system. The problem we consider is as follows: Given an n^{th} order linear dynamical system $G(s) = C(sI - A)^{-1}B$ where $A \in \Re^{n\times n}$, and $B, C^T \in \Re^n$, find a stable r^{th} order reduced system $G_r(s) = C_r(sI_r - A_r)^{-1}B_r$ with $r < n$, such that $G_r(s)$ minimizes the $\mathcal{H}_2$ error, i.e.

$$G_r(s) = \arg \min_{\deg(\hat{G})=r} \left\| \ G(s) - \hat{G}(s) \ \right\|_{\mathcal{H}_2}. \qquad (1)$$

where $\|G\|_{\mathcal{H}_2} := \left(\int_{-\infty}^{+\infty} \mid G(\jmath w) \mid^2 dw\right)^{1/2}$.

In the sequel, we will construct the reduced order models $G_r(s)$ through *Krylov* projection methods. Toward this end, we construct matrices $V \in \mathbb{R}^{n\times r}$ and $Z \in \mathbb{R}^{n\times r}$ that span certain Krylov subspaces with the property that $Z^TV = I_r$. The reduced order model $G_r(s)$ will then be obtained as

$$A_r = Z^TAV, \quad B_r = Z^TB, \quad \text{and} \quad C_r = CV. \qquad (2)$$

The corresponding oblique projection is given by VZ^T.

Many researchers have worked on the problem (1); see [18], [16], [7], [5], [3], [17], [4] and references therein. Since obtaining a global minimum is a hard task, the goal is to generate a reduced-order model that satisfies the first-order conditions for (1). However, these methods require solving a (sequence) of large-scale Lyapunov equations and hence dense matrix operations including inversion, which rapidly become intractable as the dimension increases in large-scale settings. Indeed some of these methods are unsuitable even for medium scale problems. Here, we propose an iterative rational Krylov algorithm which efficiently seeks a minimizer to the problem (1). The method is based on the computationally proven approaches utilizing Krylov subspaces. The proposed method is suitable for large-scale settings where the order of the system, n, can grow to the order of many thousands of state variables.

Our starting point is the interpolation based first-order conditions for the optimal $\mathcal{H}_2$ approximation obtained by Meier and Luenberger [5]:

Theorem 1: Let $G_r(s)$ solve the optimal $\mathcal{H}_2$ problem and let $\hat{\lambda}_i$ denote the eigenvalues of A_r, i.e. $\hat{\lambda}_i$ are the Ritz values. For simplicity, assume that $\hat{\lambda}_i$ has multiplicity one. Then, the first-order necessary conditions for $\mathcal{H}_2$ optimality are

$$\left.\frac{d^k}{ds^k}G(s)\right|_{s=-\hat{\lambda}_i} = \left.\frac{d^k}{ds^k}G_r(s)\right|_{s=-\hat{\lambda}_i}, \quad k = 0, 1. \qquad (3)$$

Theorem 1 states that the reduced model has to *interpolate* $G(s)$ and its first derivative at the mirror images of the Ritz values. For equivalence between the interpolation-based framework of the optimal $\mathcal{H}_2$ problem [5] and Lyapunov-based framework [17], see Gugercin *et al.* [12].

The following result by Grimme [10] for interpolation-based (Krylov-based) model reduction will be the main tool for the proposed method.

Theorem 2: [10] Given $G(s) = C(sI - A)^{-1}B$ and r interpolation points $\{\sigma_i\}_{i=1}^r$, let $V \in \mathbb{R}^{n\times r}$ and $Z \in \mathbb{R}^{n\times r}$ be obtained as follows:

$$\text{Im}(V) = \text{Span}\{(\sigma_1 I - A)^{-1}B, \cdots, (\sigma_r I - A)^{-1}B\}$$
$$\text{Im}(Z) = \text{Span}\{(\sigma_1 I - A)^{-T}C^T, \cdots, (\sigma_r I - A)^{-T}C^T\}$$

with $Z^TV = I_r$. Then, the reduced model $G_r(s) = C_r(sI_r - A_r)^{-1}B_r$ obtained as in (2) interpolates $G(s)$ and its first derivative at $\{\sigma_i\}_{i=1}^r$.

Proposed Algorithm: We propose a numerical algorithm which efficiently produces a reduced order model $G_r(s)$ satisfying the interpolation-based first-order necessary conditions (3). Our approach exploits the connection between the Krylov-based reduction and interpolation. Since the interpolation points in (3) depend on the final reduced model and are not known *a priori*, we use rational Krylov steps to iteratively correct the reduced-order model $G_r(s)$ so that the next (corrected) reduced-order model interpolates the full-order model at mirrored Ritz values $-\lambda_i(A_r)$ from the previous reduced-order model. This continues until Ritz values from consecutive reduced-order models stagnate. Below, we give a sketch of the algorithm:

Algorithm 1: **An Iterative Rational Krylov Algorithm (IRKA):**

1) Make an initial shift selection σ_i for $i = 1, \ldots, r$
2) $Z = \left[\ (\sigma_1 I - A^T)^{-1}C^T, \cdots, (\sigma_r I - A^T)^{-1}C^T\ \right]$.
3) $V = \left[\ (\sigma_1 I - A)^{-1}B, \cdots, (\sigma_r I - A)^{-1}B\ \right]$
4) $Z = Z(Z^TV)^{-T}$ (to make $Z^TV = I_r$)
5) while (not converged)
   a) $A_r = Z^TAV$,
   b) $\sigma_i \longleftarrow -\lambda_i(A_r)$ for $i = 1, \ldots, r$
   c) $Z = \left[(\sigma_1 I - A)^{-T}C^T, \cdots, (\sigma_r I - A)^{-T}C^T\right]$
   d) $V = \left[\ (\sigma_1 I - A)^{-1}B, \cdots, (\sigma_r I - A)^{-1}B\ \right]$
   e) $Z = Z(Z^TV)^{-T}$ (to make $Z^TV = I_r$)
6) $A_r = Z^TAV$, $B_r = Z^TB$, $C_r = CV$

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Upon convergence, desired interpolation conditions (3), i.e. the first-order conditions, will be satisfied. It should be noted that solution of the optimal $\mathcal{H}_2$ reduction problem is obtained via Krylov projection methods only and its computation is suitable in large-scale setting.

We have implemented the above algorithm for many different large-scale systems. In most of our numerical examples, the algorithm worked very efficiently and converged after a small number of steps, resulted in stable reduced systems. For some case problem where the global optimal is known, Algorithm 1 has converged to this global optimal. We have also developed a Newton formulation of this algorithm, see [12]. The Newton formulation seemed to result in faster convergence and prevent rare convergence failures of Algorithm 1.

Corollary 1: Let $G_r(s)$ be the reduced model resulting from Algorithm 1. Then, $G_r(s)$ is the optimal approximation of $G(s)$ with respect to the $\mathcal{H}_2$ norm among all reduced order systems having the same reduced system poles as $G_r(s)$. Therefore, Algorithm 1 generates a reduced model $G_r(s)$ which is the optimal solution for a restricted $\mathcal{H}_2$ problem.

CD Player Example: The original model describes the dynamics between a lens actuator and the radial arm position in a portable CD player. The model has 120 states, i.e., n=120, with a single input and a single output. As illustrated in [6], even though the Krylov-based methods resulted in good local behavior, they are observed to yield large $\mathcal{H}_\infty$ and $\mathcal{H}_2$ error compared to balanced truncation [13], [14] which is well known to yield small $\mathcal{H}_\infty$ and $\mathcal{H}_2$ error norms, see [6], [11].

In this example, we compare the performance of the proposed method **IRKA**, Algorithm 1, with that of balanced truncation. We reduce the order to r as r varies from 2 to 40; and for each r value, we compare the $\mathcal{H}_2$ error norms due to balanced truncation and due to Algorithm 1. For the proposed algorithm, the initial shifts are *randomly* selected with real parts in the interval $[10^{-1}, 10^3]$ and the imaginary parts in the interval $[1, 10^5]$. The results showing the relative $\mathcal{H}_2$ error for each r are depicted in Figure 1. The figure reveals that the proposed method with initial random shift selection outperforms balanced truncation for almost all the r values except $r = 2, 24, 36$. However, even for these three r values, the resulting $\mathcal{H}_2$ error is not far away from the one due to balanced truncation. For the range $r = [12, 22]$, **IRKA** clearly outperforms the balanced truncation. We would like to emphasize that there results were obtained by a *random* shift selection and staying in the numerically efficient Krylov projection framework *without* requiring any solutions to large-scale Lyapunov equations. This is the main difference of our algorithm from the existing methods and this makes the proposed algorithm numerically efficient in large-scale settings.

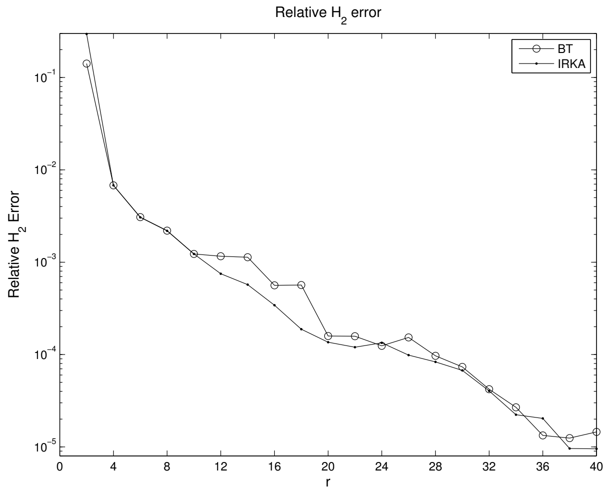

Fig. 1. Relative $\mathcal{H}_2$ norm of the error system vs r

To examine the convergence behavior, we reduce the order to $r = 8$ and $r = 10$ using Algorithm 1 and at each step of the iteration, we compute the $\mathcal{H}_2$ error due to the current estimate and plot this error vs iteration index. The results are shown in Figure 2. The figure illustrates at each step of the iteration, the $\mathcal{H}_2$ norm of the error is reduced and the algorithm converges after 4 steps.

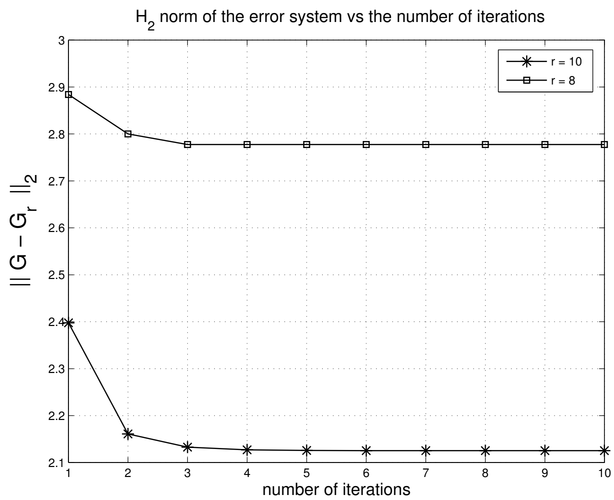

Fig. 2. $\mathcal{H}_2$ norm of the error system vs the number of iterations

A Semi-discretized Heat Transfer Problem: This problem arises during a cooling process in a rolling mill when different steps in the production process require different temperatures of the raw material. The problem is modeled as boundary control of a two dimensional heat equation. A finite element discretization results in a system of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t).$$

with state-dimension $n = 20209$, i.e., $A, E \in \mathbb{R}^{20209\times 20209}$, $B \in \mathbb{R}^{20209\times 7}$, $C \in \mathbb{R}^{6\times 20209}$. Note that in this case $E \neq I_n$, but the algorithm works fine with the obvious modifications. For details regarding the modeling, discretization, optimal control design, and model reduction for this example, see [15], [1], [2]. We consider the full-order SISO system that associates the sixth input of this system with the second output. We apply our algorithm and reduced the order to $r = 6$. Amplitude Bode plots of $G(s)$ and $G_r(s)$ are shown in Figure 3.

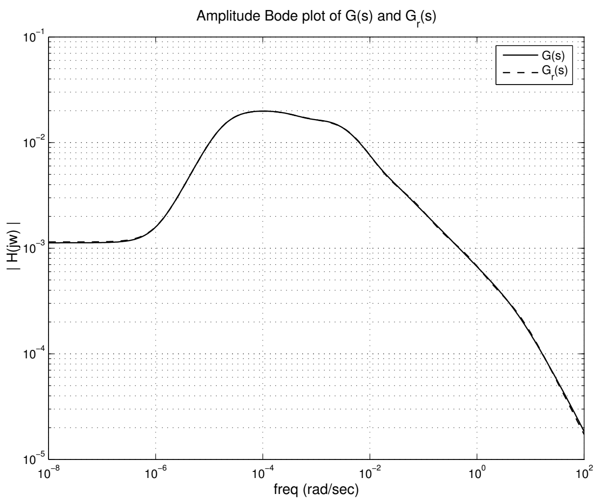

Fig. 3. Amplitude Bode plots of $G(s)$ and $G_r(s)$

The output response of $G_r(s)$ is virtually indistinguishable from $G(s)$ in the frequency range considered. **IRKA** converged in 7 iteration steps, although some interpolation points converged in the first 2-3 steps. The relative $\mathcal{H}_\infty$ error obtained with our sixth order system was 7.85×10^{-3}. Note that in order to apply Lyapunov-based methods, e.g. [3], [16], for this example, one would need to solve 2 *generalized* Lyapunov equations (since $E \neq I_n$) of order 20209 *at each step of the iteration*.

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