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A Note on the Fractional-Order Cellular Neural Networks

Ivo Petráš

***Abstract*— This paper deals on fractional - order cellular neural networks (CNNs). It is based on the well-known concept of the CNNs, where mathematical model of the CNN consists a fractional order derivative. Traditional first order cell is replaced by fractional order (non-integer) one. We present the method to derive such kind of the fractional order cell. An illustrative example has been used to demonstrate and investigate the existence of the fractional order CNN. We present the fractional three-cell CNN where the total order of system is less than three, namely 2.7, and the corresponding state space trajectories are shown as well.**

I. Introduction

A number of applications where fractional calculus has been used rapidly grows [2], [15], [16], [20], [26]. This mathematical phenomena allow to describe a real object more accurate that the classical "integer" methods. The real objects are generally fractional [7], however, for many of them the fractionality is very low. A typical example of a non-integer (fractional) order system is the voltage-current relation of a semi-infinite lossy RC transmission line [6], [18] or diffusion of the heat into a semi-infinite solid, where heat flow is equal to the half-derivative of the temperature [16].

The main reason for using the integer-order models was the absence of solution methods for fractional differential equations. We have to identify and describe the real object by the fractional order models. The first advantage is that we have more degrees of freedom in the model. The second advantage is that we have a "memory" in model. Fractional-order systems have an unlimited memory, being integer-order systems cases in which the memory is limited. Therefore we need a memory term (e.g. fractional integral or derivative) in the fractional order model. This memory term insure the history and its impact to present and future. This is a very important thing especially for fractional order model of the cell in the neural network.

Chua and Yang introduced the CNN in 1988 as a non-linear dynamical system composed by an array of elementary and locally interacting non-linear subsystems, so called cells [8]. Each cell is made of a linear capacitor, a non-linear voltage-controlled current source, and a few resistive linear circuit elements. This complex analog circuits were designed to real - time signal processing and image processing [9]. Practical realization and implementation of the CNN were described in several papers (see e.g.: [3], [8], [11], [12], [13]).

Arena $et\ al$ introduced a new class of the CNN with fractional (non-integer) order cells. They replaced first order cells with m-th order ones, where m being a non-integer quantity [1].

According to the general chaos theory, chaos may occur in non-linear autonomous system whose order is at least three. Hartley $et\ al$ introduced a fractional order Chua's system which is able to show chaotic dynamics even if the whole order of the differential equations is less than three [14]. From this consideration, the idea of developing a fractional order CNN, arose. The term of "system order" was introduced as well. The system order is not equal to the number of differential equations if we consider the fractional differential equations. The system order is equal to a highest derivative of the fractional differential equation of the mathematical model.

Arena and Hartley just simple replaced the fractional order derivative instead the integer order one. For numerical simulation they used an approximation method proposed by Charef $et\ al$ [25]. This approximation of fractional order operators is in the form of rational polynomial of high order in the frequency domain.

These considerations in mentioned Arena's and Hartley's works lead to several notes. We will discuss three of them. First note is on fractional order of the cell. We can not just replace the order of cell from integer to fractional one without real reason. The reason is described in this paper. Second note is on used approximation methods. If we use a high order approximation method then the total order of system is not equal to the highest derivative of the fractional differential equation but it is equal to the highest order of approximation polynomial. Third note is on system order. In both cases (Arena $et\ al$ and Hartley $et\ al$) is not clearly defined the term of system order, model order, number of initial conditions, number of state space variables and method for rewriting the state-space representation to fractional differential equation.

All mentioned notes will be explained in this paper, which is organized as follows. In Sec. 2, we briefly introduce the fractional calculus. Sec. 3 is on basic concept of CNN. Fractional order CNNs are introduced in Sec. 4. The illustrative example is described in Sec. 5. Sec. 6 concludes this paper with some additional remarks.

II. Fundamentals of Fractional Calculus

A. *Definitions of Fractional Derivatives*

The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with letter between Leibniz and L'Hospital in 1695.

Fractional calculus is a generalization of integration and differentiation to non-integer order fundamental operator

Ivo Petráš is with the Department of Applied Informatics and Process Control, Institute of Production Process Control, Faculty of B.E.R.G., Technical University, B. Němcovej 3, 042 00 Košice, Slovak Republic, (phone: +421-55-602-5194; fax: +421-55-602-5190; e-mail: ivo.petras@tuke.sk).

${}_aD_t^\alpha$, where a and t are the limits of the operation. The continuous integro-differential operator is defined as

$$ {}_aD_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha} & \Re(\alpha) > 0, \\ 1 & \Re(\alpha) = 0, \\ \int_a^t (d\tau)^{-\alpha} & \Re(\alpha) < 0. \end{cases} $$

The two definitions used for the general fractional differintegral are the Grunwald-Letnikov (GL) definition and the Riemann-Liouville (RL) definition [16], [15]. The GL is given here

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{\alpha}{j} f(t-jh), \quad (1) $$

where $[.]$ means the integer part. The RL definition is given as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (2) $$

for $(n-1 < \alpha < n)$ and where $\Gamma(.)$ is the *Gamma* function.

The Laplace transform method is used for solving engineering problems. The formula for the Laplace transform of the RL fractional derivative (2) has the form [16]:

$$ \int_0^\infty e^{-st}\, {}_0D_t^\alpha f(t)\, dt = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k \; {}_0D_t^{\alpha-k-1} f(t)\big|_{t=0}, $$

for $(n-1 < \alpha \leq n)$, where $s \equiv j\omega$ denotes the Laplace operator.

Some others important properties of the fractional derivatives and integrals we can find out in several works (e.g.: [15], [16], etc.).

B. Numerical Methods for Calculation of Fractional Derivatives

For numerical calculation of fractional-order derivation we can use the relation (3) derived from the Grunwald-Letnikov definition (1). This approach is based on the fact that for a wide class of functions, two definitions - GL (1) and RL (2) - are equivalent. The relation for the explicit numerical approximation of α-th derivative at the points kT, $(k = 1, 2, \dots)$ has the following form [16], [21], [22]:

$$ {}_{(k-L/T)}D_{kT}^\alpha f(t) \approx T^{-\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f_{k-j}, \quad (3) $$

where L is the "memory length", T is the time step size of the calculation (sampling period) and $(-1)^j\binom{\alpha}{j}$ are binomial coefficients $c_j^{(\alpha)}$, $(j = 0, 1, \dots)$. For their calculation we can use the following expression [24]:

$$ c_0^{(\alpha)} = 1, \qquad c_j^{(\alpha)} = \left(1 - \frac{1+\alpha}{j}\right) c_{j-1}^{(\alpha)}. \quad (4) $$

Described numerical method is so called Power Series Expansion (PSE) of a generating function. It is important to note that PSE leads to approximation in the form of polynomials, that is, the discretized fractional operator is in the form of FIR filter, which has only zeros. Other approach can be realized by continued fraction expansion (CFE) of the generating function and then the discretized fractional operator is in the form of IIR filter, which has poles and zeros. In other words, for evaluation purposes, the rational approximations obtained by CFE frequently converge much more rapidly than the PSE and have a wider domain of convergence in the complex plane. On the other hand, the approximation by PSE and the short memory principle is convenient for the dynamical properties consideration [21].

A detailed review of the other approximation methods and techniques (Carlson's, Chareff's, CFE, Matsuda's, Oustaloup's, etc.) for continuous and discrete fractional-order models in form of IIR and FIR filters was done in work [26]. Described forms of approximation were also compared with PSE and the others as for example Muir recursion [22]. Some other approaches were described in works [5], [17], [19]. Last but not least we should mention the approach proposed by Hwang, which is based on B-splines function [23].

III. Basic concept of CNN

The basic circuit unit of the CNN is a cell. It contains linear and non-linear circuit elements, which typically are: linear capacitor, linear resistors, linear and non-linear controlled sources, and independent sources. Any cell in the CNN is connected only to its neighbor cells. Theoretically we can define the CNN of any dimension, e.g.: two-dimensional array of $M \times N$, having M rows and N columns. We call the cell on the i-th row and the j-the column cell - $C(i,j)$. Observe from Fig. 1 that each cell $C(i,j)$ contains one independent voltage source $E_{i,j}$, one independent current source I, one linear capacitor C, two linear resistors R_x and R_y, controlling input voltage u_{ij}, state voltage of the cell x_{ij}, feedback from the output voltage y_{ij} of each neighbor cell $C(k,l)$. In fact each cell $C(i,j)$ mutually interacts with all the cells belonging to its neighbors $N_r(i,j)$ by means of the voltage controlled current source $I_{xy}(i,j;k,l) = A(i,j;k,l)y_{kl}$, $I_{xu}(i,j;k,l) = B(i,j;k,l)u_{kl}$ and $I_{xx}(i,j;k,l) = C(i,j;k,l)x_{kl}$. The constant coefficients $A(i,j;k,l)$, $B(i,j;k,l)$ and $C(i,j;k,l)$ are known as the cloning templates. If they are equal for each cell, they are called space-invariant. The CNN is described by the following state equations of all its cells [8]:

$$ \begin{aligned} C\frac{dx_{ij}(t)}{dt} = & - \frac{1}{R_x} x_{ij}(t) \\ & + \sum_{C(k,l)\in N_r(i,j)} A(i,j;k,l) y_{kl}(t) \\ & + \sum_{C(k,l)\in N_r(i,j)} B(i,j;k,l) u_{kl}(t) \\ & + C(i,j;k,l) x_{kl}(t) + I \end{aligned} \quad (5) $$

with $x_{ij}(0) = x_{ij0}$, $C > 0$, $R_x > 0$, $1 \leq i \leq M$, and $1 \leq j \leq N$, where

$$ N_r(i,j) = \{C(k,l) : \max(|k-i| - |l-j|) \leq r\} $$

is the r-neighborhood. Input equation is: $u_{ij}(t) = E_{ij}$. Output equation is:

$$y_{ij}(t) = \frac{1}{2}(|x_{ij}(t)+1| - |x_{ij}(t)-1|).$$

This model with direct dependence of state variable on the state of the neighboring cells is known as a state controlled CNN. Such kind of the CNN are also able to show chaotic behaviors [9], [10].

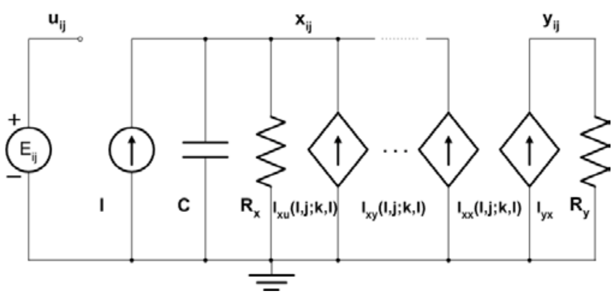

Fig. 1. The CNN cell

The only non-linear element in each cell is a piecewise-linear voltage-controlled current source: $I_{yx} = (1/R_y)f(x_{ij})$.

IV. Concept of fractional order CNN

In this section we will disscus the first note mentioned in introduction. We will derive the fractional order model of the CNN described in the previous section.

Westerlund in 1994 proposed a new linear capacitor model [4]. It is based on Curie's empirical law of 1889 which states that the current through a capacitor is

$$i(t) = \frac{U_0}{h_1 t^m},$$

where h_1 and m are constant, U_0 is the dc voltage applied at $t = 0$, and $0 < m < 1$.

For a general input voltage $u(t)$ the current is

$$i(t) = C\frac{d^m u(t)}{dt^m}, \qquad (6)$$

where C is capacitance of the capacitor. It is related to the kind of dielectric. Another constant m (order) is related to the losses of the capacitor. Westerlund provided in his work the table of various capacitor dielectric with appropriated constant m which has been obtained experimentally by measurements.

Carlson was also studying the fractional capacitor and approximation technique in 1963 [5]. He used a regular Newton process to approximate the fractional capacitor impedance $Z(s) = \frac{C}{s^m},\ \ 0 < m < 1,\ \ m \in \Re$.

Applying the Kirchhoff law and relation (6) to standard model of the CNN which is described by the equation (5), we obtain a fractional order model of the CNN in the following form:

$$\begin{aligned} C\frac{d^m x_{ij}(t)}{dt^m} = & - \frac{1}{R_x}x_{ij}(t) \\ & + \sum_{C(k,l)\in N_r(i,j)} A(i,j;k,l)y_{kl}(t) \\ & + \sum_{C(k,l)\in N_r(i,j)} B(i,j;k,l)u_{kl}(t) \\ & + C(i,j;k,l)x_{kl}(t) + I \end{aligned} \qquad (7)$$

with $x_{ij}(0) = x_{ij0}, C > 0, 0 < m < 1, R_x > 0, 1 \le i \le M$, and $1 \le j \le N$.

In Arena $et\ al$ work the parameter m is $1 < m < 1.5$ and two-cells CNN was studied. Taking into account the consideration that $0 < m < 1$ and the fact that we would like to study behavior of system with the total order less than three, we have to consider three-cells fractional order CNN. Referring to the general definition of CNN given by (7) and choosing the opposite-sign template we obtain the following three-cells CNN ($M = 3$, $N = 1$, $C = 1$, $R = 1$, and $u_{kl} = 0$):

$$\begin{aligned} {}_0D_t^m x_1 &= -x_1 + p_1 f(x_1) - s f(x_2) - r f(x_3) \\ {}_0D_t^m x_2 &= -x_2 + s f(x_1) - p_2 f(x_2) - r f(x_3) \\ {}_0D_t^m x_3 &= -x_3 - r f(x_1) - r f(x_2) + p_3 f(x_3) \end{aligned} \qquad (8)$$

V. Illustrative example

Let us assume that we have the three-identical-cells CNN described by equations (8), with fractional order $m = 0.9$ (order of real analog capacitor). We will use the relations (3) and (4) for time step $T = 0.001$. Simulation results are depicted on Fig. 2 and Fig. 3.

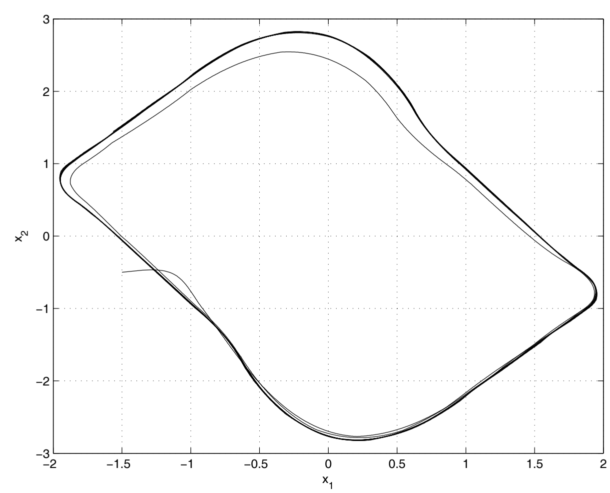

Fig. 2. State space trajectory (limit cycle) of the CNN (8) for the parameters: $p_1 = 1.6$, $p_2 = 1.6$, $p_3 = 0.9$, $s = 6.3$, $r = 3.75$ projected onto $x_1 - x_2$ plane. Initial conditions were: $\vec{x} = [-1.5, -0.5, 1.0]$.

Fig. 2 and Fig. 3 show the state space trajectories of the fractional order CNN with total order 2.7 for various parameters of the equations (8).

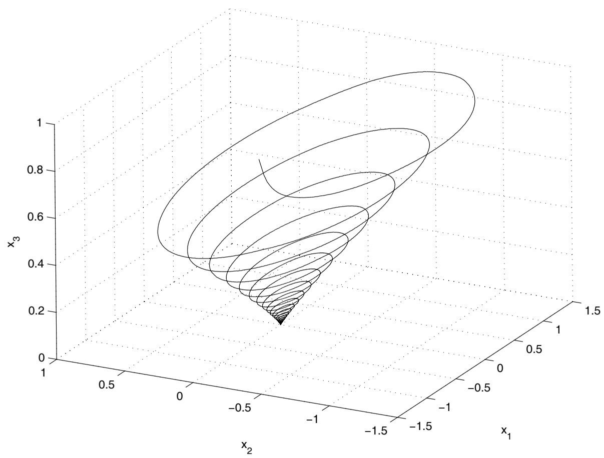

Fig. 3. State space trajectory (stable spiral) of the CNN (8) for the parameters: $p_1 = 1.2$, $p_2 = 2.1$, $p_3 = 1.1$, $s = 6.98$, $r = 2.4$ projected into $x_1 - x_2 - x_3$ space. Initial conditions were: $\vec{x} = [-1.5, -0.5, 1.0]$.

VI. Conclusions

In this paper we have presented approach to derive the fractional order cellular neural network and method for its numerical simulation. By illustrative examples we shown behavior of fractional CNN for various system parameters.

In this section we will also discuss the second and the third notes mentioned in introduction subsection. Till present time is no analytical solution for chaotic systems. Some authors consider the numerical solutions (attractors) as a numerical error. In fact, the deterministic chaos exists but computation of strange attractors is very important thing and therefore we have to find and appropriate approximation methods. Utilization of methods in the form of rational polynomial leads to high order system. In this case we must consider different initial conditions and large numerical error which is amplify by system constant and approximation polynomial constants. We recommend to use a method in the form of FIR filter with a large number of coefficients. It is also high order system but numerical error is much smaller than in methods in the form of IIR filter [21], [22]. However, the time of computation is longer because of number of the coefficients.

System order in such case is equal to sum of particular fractional orders of differential equations. The conclusion of this work confirms the conclusions of the works [14], [3], [16] that there is a need to refine the notion of the order of a system which can not be considered only by the total number of differentiation.

As has been demonstrated, the idea of fractional calculus requires to reconsider dynamical system concepts. Some of them have been noted in this article.

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