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# A multidimensional Schur-Cohn algorithm in tabular forms

Ioana Serban and Mohamed Najim, *fellow, IEEE*

***Abstract*— The paper proposes multidimensional extensions of the Bistritz and the Jury-Marden stability tests, based on a necessary and sufficient stability test recently introduced by the authors, in terms of functional Schur coefficients. This type of test has only one condition to be checked, instead of $N$ conditions in the classical Jury-Anderson test. The two proposed extensions are illustrated by a comparative 2-D example, and several 3-D examples of functional Schur coefficients are computed and graphically represented.**

***Index Terms*— Multidimensional Systems, Stability, Schur Cohn algorithm**

## I. INTRODUCTION

A problem of constant interest in system theory is to analyze the BIBO stability of multidimensional systems. Several methods of testing the BIBO stability in the multidimensional case were proposed in [9], [10], [12], [14], [25]. The 2-dimensional case was considered in [3], [11], [15], [16], [19]. Algebraic tests for multidimensional stability are treated in [18], [17].

There are two categories of algorithms used to locate the zeros of a one variable complex polynomial with respect to the unit circle: the first is scattering-type algorithms (Schur-Cohn, Marden-Jury) and the second is imittance type algorithms (Bistritz, Routh). For a unified approach of these algorithms see [13].

The purpose of this paper is to give a multidimensional generalization of the Bistriz test introduced in [6] and of the Jury-Marden test [21] using a new criterion of stability for multidimensional systems. This new criterion was obtained using the functional Schur coefficients, recently introduced by the authors [24]. The functional Schur parameters are a natural extension to the multidimensional case of the 1-D *Schur parameters* also known as the *reflection coefficients*. Using an algebraic extension of the reflection coefficients to the two-dimensional case [20], a sufficient but not necessary condition for stability of two-dimensional systems was already obtained in [1]. The use of the functional Schur coefficients leads to a necessary and sufficient condition of stability of multidimensional systems.

The paper is organized as follows: in Section II the slice technique method is presented, the functional Schur coefficients are introduced and the multidimensional stability criterion is formulated. Section III introduces a multidimensional form of the Bistritz stability algorithm. In section IV a multidimensional Jury-Table form procedure for testing the stability in the n-dimensional case is proposed. To illustrate the proposed procedures a general 2-D Schur-Cohn criterion is given in Section V, and a comparative 2-D example is provided. Also several examples for the 3-D case are given. In Section VI we provide conclusions and some further comments.

I. Serban is with the IMS-LAPS, Bordeaux 1 University / ENSEIRB, 351, cours de la Libération, Talence, France Ioana.Serban@laps.ims-bordeaux.fr

M. Najim is with the IMS-LAPS, Bordeaux 1 University / ENSEIRB, 351, cours de la Libération, Talence, France Mohamed.Najim@laps.ims-bordeaux.fr

## II. FUNCTIONAL SCHUR COEFFICIENTS AND MULTIDIMENSIONAL SCHUR-COHN CRITERION

In [24] the authors obtained a multidimensional extension of the analytic approach of the Schur-Cohn criterion (see [8] for more details). To each function $F$ a multidimensional analogous of the Schur parameters sequence is associated, and a multidimensional Schur-Cohn criterion is obtained. The multidimensional extension is based on the slice functions, first introduced by Rudin [23]. They where used in extending to the 2-D or n-D case several results well known in the 1-D case. We present here briefly the "slice" method.

Denote by $\mathbb{D}$ the set $\{z \in \mathbb{C} : |z| < 1\}$, and by $\mathbb{T}$ the set $\{z \in \mathbb{C} : |z| = 1\}$.

For each point $w = (w_1, ..., w_N)$ on the polytorus $\mathbb{T}^N$ let $\tilde{D}_w$ be the one-dimensional disk that "slices" $\mathbb{D}^N$ through the origin and through $w$:

$$\tilde{D}_w = \{\lambda w = (\lambda w_1, ..., \lambda w_N) : \lambda \in \mathbb{D}\}. \qquad (1)$$

It is obvious that if $u$ and $w$ are in $\mathbb{T}^N$ such that there is $z \in \mathbb{T}$ with $v = zu$ then $\tilde{D}_w = \tilde{D}_u$. To avoid considering redundant slices, one has to do a "normalization" on one coordinate, say for now the last one:

$$D_v = \{\lambda(v_1, ..., v_{N-1}, 1) : \lambda \in \mathbb{D}\} \qquad (v \in \mathbb{T}^{N-1}) \qquad (2)$$

In the following we introduce the definition of the slice of a multivariable polynomial.

Let $P(z_1, z_2, \ldots, z_N)$ be a polynomial in $N$ variables of degree $n$:

$$P(\mathbf{z}) = \sum_{|\alpha| \le n} p_\alpha \mathbf{z}^\alpha$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_N) \in \mathbb{C}^N$, $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ and $\mathbf{z}^\alpha = z_1^{\alpha_1} \ldots z_N^{\alpha_N}$ whose degree is $|\alpha| = \alpha_1 + \cdots + \alpha_N$.

For each $v \in \mathbb{T}^{N-1}$ consider the restriction of the multivariable polynomial $P$ to the one-dimensional disk $D_v$, which can be regarded as a one variable polynomial:

$$P_v(\lambda) = P(\lambda v) \qquad (\lambda \in \mathbb{D}). \qquad (3)$$

$P_v$ is called *the slice of $P$ through $v$* [23].

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The authors proved recently [24] the following result: the $N$-variable polynomial $P$ has no zeros inside the unit polydisk is equivalent with the fact that the one variable polynomial $P_v(\lambda)$ has no zeros inside the unit polydisk, for each $v \in \mathbb{T}^{N-1}$. This was obtained by the means of the Schur coefficients sequence associated to the n variable polynomial $P$, which is an extension of the Schur parameters sequence in the multidimensional case. In the following we define the functional Schur coefficients sequence for an analytic function $F$ in several variables.

*Definition 2.1:* Let $F(\mathbf{z})$ be an analytic function in the open unit polydisk $\mathbb{D}^N$, and let $v = (v_1, ..., v_{N-1}) \in \mathbb{T}^{N-1}$. For each point $(v_1, ..., v_{N-1}, 1)$ on the polytorus $\mathbb{T}^N$ consider the restriction of $F$ to the one-dimensional disk $D_v$, which can be regarded as a one variable function (the slice of $F$ through $v$):

$$F_v(\lambda) = F(\lambda v_1, ..., \lambda v_{N-1}, \lambda) \qquad (\lambda \in \mathbb{D}), \tag{4}$$

For each $v$ on $\mathbb{T}^{N-1}$ define the sequence $(F_{v,k})_{k\geq 0}$:

$$F_{v,0} = F_v \qquad F_{v,k} = \Phi(F_{v,k-1}) \qquad (k \geq 1),$$

where the transform $\Phi$ that maps a complex function $f$ analytic around the origin to the function $\Phi(f)$ defined by $\Phi(f) = 0$ if $f$ is a constant unimodular function, and

$$\Phi(f)(z) = \begin{cases} \frac{f(z)-f(0)}{z(1-\overline{f(0)}f(z))} & z \neq 0 \\ f'(0)(1-|f(0)|^2)^{-1} & z = 0 \end{cases} \tag{5}$$

otherwise (see [8]).

The functions $\gamma_k : \mathbb{T}^{N-1} \to \mathbb{C}$ defined in [24] by

$$\gamma_k(v) = F_{v,k}(0) \qquad (v \in \mathbb{T}^{N-1}) \tag{6}$$

are called *the functional Schur coefficients of the function $F$*.

The following theorem is established in [24]

*Theorem 2.2:* (n-D Schur-Cohn criterion)

The following statements are equivalent:

A) $P$ has no zeros in the closed unit polydisk $\mathbb{D}^N$:

$$P(z_1, z_2, \ldots, z_N) \neq 0 \quad |z_i| \leq 1, \; i = 1 \ldots N \tag{7}$$

B) $P_v(\lambda) = P(\lambda v_1, ... \lambda v_{N-1}, \lambda)$ has no zeros in $\mathbb{D}$ for $v \in \mathbb{T}^{N-1}$;

C) $|\gamma_k(v)| < 1$ for all $v \in \mathbb{T}^{N-1}$ and $k = 0 \ldots n-1$, where $\gamma_k(v)$ are the functional Schur coefficients associated to $F_v = \frac{(P_v)^T}{P_v}$

Let us make somme comments on the continuity of the functional Schur coefficients. It is showed in [24] that the continuity of each $\gamma_k$, for $k = 0 \ldots n-1$ associated to $F_v = \frac{(P_v)^T}{P_v}$ is in fact equivalent with the stability of $P$. Furthermore, the $|\gamma_k|$ are lower semicontinuous, wich guarantees the existence of the maximum of the function $|\gamma_k(v)|$.

In order to compare our criterion with others criteria, remind that the existing multidimensional tests of stability relies either on the DeCarlo Strintzis criterion or on the Jury-Anderson criterion:

*DeCarlo Strintzis criterion:* The condition (7) holds if and only if all the following $N$ conditions are satisfied:

$$P(z_1, 1, 1, \ldots, 1) \neq 0 \; |z_1| \leq 1, \tag{8}$$
$$P(1, z_2, 1, \ldots, 1) \neq 0 \; |z_2| \leq 1, \tag{9}$$
$$\ldots\ldots$$
$$P(1, 1, \ldots, 1, z_N) \neq 0 \; |z_N| \leq 1, \tag{10}$$
$$P(z_1, z_2, \ldots, z_N) \neq 0 \; |z_i| = 1, i = 1 \ldots, N. \tag{11}$$

*Jury-Anderson criterion:* The condition (7) holds if and only if all the following $N$ conditions are satisfied:

$$P(z_1, 0, 0, \ldots, 0) \neq 0 \; |z_1| \leq 1, \tag{12}$$
$$P(z_1, z_2, 0, \ldots, 0) \neq 0 \; |z_1| = 1, |z_2| \leq 1, \tag{13}$$
$$\ldots\ldots$$
$$P(z_1, \ldots, z_{N-1}, 0) \neq 0 \; |z_i| = 1, i = 1 \ldots N-2, |z_{N-1}| \leq 1, \tag{14}$$
$$P(z_1, z_2, \ldots, z_N) \neq 0 \; |z_i| = 1, i = 1 \ldots N-1, |z_N| \leq 1. \tag{15}$$

Both criteria involve checking a set of $N$ conditions. The new multidimensional stability criterion (2.2) proposed in Section V states that (7) holds if and only if the following unique condition is satisfied:

$$P(\lambda v_1, ... \lambda v_{N-1}, \lambda) \neq 0 \; |v_i| = 1, \; i = 1 \ldots N-1, |\lambda| \leq 1.$$

This condition is similar with condition (15), but is the only condition that one needs to check in order to verify if a polynomial $P$ is stable. Conditions (12) to (14) are no longer necessary.

In the following the equivalence of (A) with (B) in (2.2) is used in order to give a multidimensional Bistritz table. In section IV the equivalence of (A) with (C) in (2.2) gives a extension of the multidimensional Jury Table.

## III. MULTIDIMENSIONAL BISTRITZ TEST

This section combines the 1-D stability test proposed by Bistritz in [7] with the condition (B) of the (2.2), in order to obtain a new multidimensional stability test. The main advantage of the Bistritz procedure is that it associates to a given polynomial $p$ a sequence of symmetric polynomials, and therefore only half of the entries are needed for the table construction when testing the stability of $p$. Therefore it involves less overall computational cost then other known methods.

Let us first recall the division-free version of the Bistritz procedure as it is presented in [2].

Consider a 1-D polynomial defined by $p(z) = \sum_{i=0}^{n} p_i z^i$ such that $\Re \; p(1) \neq 0$, where $p_i$ are complex numbers and $\Re \; z$ denotes the real part of $z \in \mathbb{C}$. Denote by $\bar{z}$ the complex conjugate of $z \in \mathbb{C}$ and by $p^T(z) = z^n(\overline{p(1/\bar{z})})$ the transpose of the polynomial $p$. Then one can associate to $p$ a sequence $\{f_m\}_{m=-1,0,1,\ldots,n}$ of polynomials:

$$f_m(z) = \sum_{k=0}^{n-m} f_{m,k} z^k$$

and a sequence of scalars $\{\phi_m\}_{m=0,\ldots,n}$ defined as follows:

*The 1-D Bistritz test:*

*Initialization:* For $m = -1, 0$ define:

$$f_{-1} = (z-1)(p(z) - p^T(z)) \quad (16)$$

$$f_0(z) = p(z) + p^T(z) \quad (17)$$

$$\phi_0 = f_0(1); \quad (18)$$

*Recursion:* For $m = 0, \ldots, n-1$ define:

$$z f_{m+1}(z) = (f_{m-1,0}\overline{f}_{m,0} + \overline{f}_{m-1,0} f_{m,0} z) f_m(z) - f_{m,0}\overline{f}_{m,0} f_{m-1}(z) \quad (19)$$

$$\phi_{m+1} = f_{m+1}(1), \quad (20)$$

Bistritz showed [6] the following necessary and sufficient stability conditions for $p$.

*Stability conditions* (Theorem 1 in [2]): The polynomial $p$ is stable if and only if:

$$\frac{\phi_m}{\phi_0} > 0, m = 1, \ldots, n. \quad (21)$$

Without restraining the generality one can suppose that $\phi_0 > 0$ (if not, consider $-p$ instead of $p$). Then the stability conditions are:

$$\phi_m > 0, m = 0, \ldots, n. \quad (22)$$

Remark that only the case of the normal recursion is addressed in this paper. Bistritz showed several techniques of treatment of the singular cases [4] that can be extended for multidimensional polynomials.

In the following we descride how the 1D Bistritz test can be extended to the multidimensional case by using the multidimensional criteria given in Section II. Consider $P$ be a $N$-variable polynomial:

$$P(\mathbf{z}) = \sum_{|\alpha| \le n} p_\alpha \mathbf{z}^\alpha \quad (\mathbf{z} = (z_1, ..., z_N) \in \mathbb{T}^N) \quad (23)$$

Let $v = (v_1, ..., v_{N-1}) \in \mathbb{T}^{N-1}$. Then the slice pf $P$ through $v$ is:

$$P_v(\lambda) = \sum_{k=1}^{n} c_k(v)\lambda^k \qquad (\lambda \in \mathbb{D}), \quad (24)$$

where the coefficients $c_k$ are polynomials in $v$ given by:

$$c_i(v) = \sum_{|\alpha| = i} p_\alpha v_1^{\alpha_1} v_2^{\alpha_1} ... v_{N-1}^{\alpha_{N-1}} \qquad (0 \le i \le n). \quad (25)$$

In fact the polynomial $P_v(\lambda)$ can be seen as a one variable polynomial with coefficients that are parameterized over $v = (v_1, ..., v_{N-1}) \in \mathbb{T}^{N-1}$. Therefore one can apply the procedure proposed by Bistriz to $P_v(\lambda)$, as indicated in the following.

Firstly we have to check if $\Re\, P_v(1) \neq 0$. In the contrary case consider the polynomial $P(z) = \overline{P_v(1)} P_v(\lambda)$. Secondly we associate to $P_v(\lambda)$ a sequence $\{f_{m[v]}\}_{m=-1,0,1,\ldots,n}$ of polynomials:

$$f_{m,[v]}(\lambda) = \sum_{k=0}^{n-m} f_{m,k,[v]}\lambda^k$$

and a sequence of $\{\phi_m(v)\}_{m=0,\ldots,n}$ defined as follows:

*Multidimensional Bistritz test:*

*Initialization:* For $m = -1, 0$ define:

$$f_{-1,[v]}(\lambda) = (\lambda - 1)(P_v(\lambda) - P_v^T(\lambda)) \quad (26)$$

$$f_{0,[v]}(\lambda) = P_v(\lambda) - P_v^T(\lambda) \quad (27)$$

$$\phi_0(v) = f_{0,[v]}(1); \quad (28)$$

*Recursion:* For $m = 0, \ldots, n-1$ define:

$$\lambda f_{m+1,[v]}(\lambda) = (f_{m-1,0,[v]}\overline{f}_{m,0,[v]} + \overline{f}_{m-1,0,[v]} f_{m,0,[v]}\lambda) f_{m,[v]}(\lambda) - f_{m,0,[v]}\overline{f}_{m,0,[v]} f_{m-1,[v]}(\lambda) \quad (29)$$

$$\phi_{m+1}(v) = f_{m+1,[v]}(1), \quad (30)$$

From condition (B) of (2.2) we have the equivalence between stability of the multivariable polynomial $P(z)$ and stability of $P_v(\lambda)$, for all $v \in \mathbb{T}^{N-1}$. Therefore we can state the following stability conditions for $P(z)$.

*Stability conditions.* The polynomial $P(z)$, and equivalently $P_v(\lambda)$, is stable for all $v \in \mathbb{T}^{N-1}$ if and only if:

$$\frac{\phi_m(v)}{\phi_0(v)} > 0, (\forall)\ v \in \mathbb{T}^{N-1}, \text{ and } m = 1, \ldots, n. \quad (31)$$

As in the 1-D case, one can suppose that $\phi_0(v) > 0$ for all $v \in \mathbb{T}^{N-1}$ is satisfied (if not, the polynomial is not stable). Thus the stability conditions became

$$\phi_m(v) > 0, (\forall)\ v \in \mathbb{T}^{N-1}, \text{ and } m = 0, \ldots, n. \quad (32)$$

## IV. MULTIDIMENSIONAL JURY TABLE

Another classical method to test if a one variable polynomial $p$ has all the roots inside the unit circle is the Jury Table (see [16]). The advantage of the Jury Table is that it gives explicitly the principal leading minors in the Schur-Cohn matrix associated to $p$. It is well known that the positivity of the principal leading minors of the Schur-Cohn matrix is equivalent with the positivity of $1 - |\gamma_k|^2$, for $k = 0, \ldots, n-1$, where $\gamma_k$ are the Schur parameters associated to $p$. Therefore we can use the Jury Table with condition (C) in the (2.2) in order to obtain the multidimensional form of the jury table.

First we recall briefly the construction of the 1-D Jury Table. Consider the one variable polynomial $p(z) = \sum_{i=0}^{n} p_i z^i$.

In order to test if the roots of $P$ are inside the unit circle we consider on the first row of the table the coefficients of the polynomial transpose $p^T$:

*Jury Table*

1. For $i = 0, \ldots, n$ let $b_i^0 = \overline{p}_{n-i}$.

2. For $k = 1, \ldots n$ let $m$ be equal to 0 if $k = 1, 2$ and $m = 1$ if $k > 2$. Then construct the $k^{th}$ row of the table with the entries $b_i^k$ for $i = 0, \ldots, n-k-1$ defined by:

$$b_i^k = \left(\frac{1}{b_0^{k-1}}\right)^m \begin{vmatrix} b_0^{k-1} & b_{n-k+1-i}^{k-1} \\ \bar{b}_{n-k+1}^{k-1} & \bar{b}_i^{k-1} \end{vmatrix} \tag{33}$$

3. $p(z) \neq 0$ for all $|z| \geq 1$ if and only if $b_0^k > 0$ for all $k = 1, \ldots, n$.

Consider now a multivariable polynomial $P$ as in (23) and let $P_v(\lambda)$ be the slice of $P$ through $v$ as in (24) with the coefficients (25). Then one can construct the following multidimensional jury table:

*Multidimensional Jury Table*

1. For $i = 0, \ldots, n$ let $b_i^0(v) = c_i(v)$.
2. For $k = 1, \ldots n$ let $m$ be equal to 0 if $k = 1, 2$ and $m = 1$ if $k > 2$. Then construct the $k^{th}$ row of the table with the entries $b_i^k$ for $i = 0, \ldots, n-k-1$ defined by:

$$b_i^k(v) = \left(\frac{1}{b_0^{k-1}(v)}\right)^m \begin{vmatrix} b_0^{k-1}(v) & b_{n-k+1-i}^{k-1}(v) \\ \overline{b_{n-k+1}^{k-1}(v)} & \overline{b_i^{k-1}}(v) \end{vmatrix} \tag{34}$$

3. $P(z) \neq 0$ for all $|z| \leq 1$ if and only if $b_0^k(v) > 0$ for all $k = 1, \ldots, n$ and $v \in \mathbb{T}^{N-1}$.

In the next section a general 2D Schur Cohn criterion is given, and we show how the proposed procedures can be applied for the two-dimensional systems.

## V. IMPLEMENTATION AND EXAMPLES

To illustrate this new stability criterion we provide with a 2-dimensional example.

In this section we shall consider $P(z_1, z_2)$ a two-variable polynomial of degree $n$. Denote by $P_v(\lambda) = P(\lambda v, \lambda)$ the slice of $P$ through $v \in \mathbb{T}$ ($\lambda \in \mathbb{D}$). Denote by $\gamma_k(v)$ the functional Schur coefficients associated to $F = \frac{(P_v)^T}{P_v}$, denote by $\phi_m$ the sequence of scalars associated to $P_v$ by the Bistritz table and finally, denote by $b_0^k(v)$ be the first entries on each row in the multidimensional Jury Table (34). We have the following two-dimensional criterion:

*2-D Schur-Cohn criterion:*

The following assertions are equivalent:

A) $P(z_1, z_2) \neq 0 \quad |z_1| \leq 1, \quad |z_2| \leq 1$
B) $P_v(\lambda) := P(\lambda v, \lambda) \neq 0, \quad |v| = 1, |\lambda| \leq 1$
C) $|\gamma_k(v)| < 1$ for all $v \in \mathbb{T}$ and $k = 0 \ldots n-1$
D) $b_0^k(v) > 0$ for all $v \in \mathbb{T}$ and $k = 1, \ldots, n$.
and, if the restrictions needed for the Bistritz procedure are satisfied, i.e. $\Re P(1) > 0$, they are furthermore equivalent with:
E) $\phi_m(v) > 0$ for all $v \in \mathbb{T}$ and $m = 1 \ldots n-1$

One can easily see that the sequence of $\{\phi_m(v)\}_m$ and the sequence of $\{b_0^k(v)\}_k$ are trigonometric polynomials in $v$ real valued on $\mathbb{T}^2$. In fact they are Tchebîshev polynomials, and thus their positivity is reduced to the problem of stability is finally reduced to the problem positivity of a real polynomial on a given interval. Several techniques can be used to check the positivity of a real polynomial, as for instance the classical Sturm algorithm or the 1-D real Bistritz table [5].

In the following a classic example of polynomial will be tested for stability. First the functional Schur coefficients will be computed and graphically represented. The stability will be checked in two ways: using the proposed multidimensional Bistritz procedure and using the multidimensional Jury table.

*Example 1.* Let $P_1$ be

$$P_1(z_1, z_2) = 1 + .5z_1 + .5z_2 + .25z_1z_2 + .25z_1^2 + .25z_2^2. \tag{35}$$

Then the slice of $p$ through $v \in \mathbb{T}$ is

$$P_v(\lambda) = 1 + .5(1+v)\lambda + .25(1 + v + v^2)\lambda^2 \tag{36}$$

In order to illustrate the functional Schur coefficients associated to $P$, we shall compute and graphically represent them. Consider the function $P_v^T/P_v$. By (6) we obtain:

$$\gamma_1(v) = .25 + .25\bar{v} + .25\bar{v}^2 \tag{37}$$

$$\gamma_2(v) = \frac{.25 - .125v + .25\bar{v} - .25\bar{v}^2}{.8125 - .125(v + \bar{v}) - .625(v^2 + \bar{v}^2)} \tag{38}$$

According to (2.2), if $|\gamma_i(v)|^2 < 1$, for all $v \in \mathbb{T}$ and for $i = 1, 2$, then the polynomial $P(z_1, z_2)$ is stable. This can be easily observed by considering a graphical representation of the absolute value of $|\gamma_i|^2$, for $v = e^{it}$ ($t \in R$), as in Fig (1).

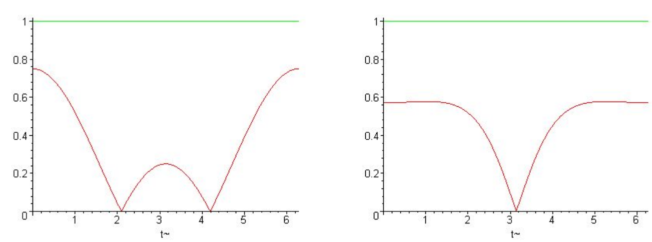

Fig. 1. Functional Schur coefficients of $P_1$

We shall now apply the Bistritz procedure to test the stability of $P$. First we need to check if $\Re P_v(1)$ is not zero, for any $v \in \mathbb{T}$. Let $x$ be the real part of $v \in \mathbb{T}$, i.e. $x = \Re v = cost$ ($t \in R$). We have $\Re P_v(1) = 1.5 + .75x + .5x^2$, which is tested to be positive with a standard 1D algorithm. Next, the following sequence of polynomials is obtained through the Bistritz recursion (26) - (30) :

$$\phi_i(x) = \sum_{k=0}^{n} a_{ik}x^k \quad (x \in [-1, 1])$$

where the coefficients $a_{ik}$ are given by the entries of $A_i$, for $i = 0, \ldots, 2$, given by:

$$A_0 := \begin{pmatrix} 3 \\ 1.5 \\ 1 \end{pmatrix} \; A_1 := \begin{pmatrix} -5.625 \\ -1.3125 \\ .375 \\ 1.25 \\ .5 \end{pmatrix} \; A_2 := \begin{pmatrix} 7.6201 \\ 4.4077 \\ 6.4365 \\ -5.0332 \\ -3.8633 \\ -2.5547 \\ .7031 \\ .6562 \\ .3125 \end{pmatrix}$$

A standard 1D algorithm shows that $\phi_1(x) > 0$ and $\phi_2(x) > 0$ when $x \in [-1, 1]$, which leads to the conclusion that $P(z_1, z_2)$ is stable. The assertion $\phi_0(x) > 0$ for $x \in [-1, 1]$ was already verified, as $\phi_0(v) = 2\Re P_v(1)$

Consider again the polynomial given by (35), and its associated slice (36). We shall use now the multidimensional Jury Table in order to test the stability of $P$.

We have denoted again by $x$ the real part of $v$. When computing the principal leading minors of the Schur Cohn matrix associated to $P_v(\lambda)$ by the recursion in (34) the following sequence of polynomials is obtained:

$$b_0^i(x) = \sum_{k=0}^{n} b_{ik} x^k \quad (x \in [-1, 1]), \quad i = 1, 2$$

where the coefficients $b_{ik}$ are given by the entries of $B_i$, for $i = 1, 2$:

$$B_1 := \begin{pmatrix} .8125 \\ -.125 \\ -0.0625 \end{pmatrix} \; B_2 := 256 \begin{pmatrix} 139 \\ -48 \\ -6 \\ 0 \\ 1 \end{pmatrix}$$

It is easy to see that they are strictly positive for $x \in [-1, 1]$, for instance using [22], [26].

Remark again that $b_0^1$ and $b_0^2$ are positive if and only if the functional Schur coefficients associated to $P_1$ are not greater than 1 in absolute value.

We would like to make some concluding remarks concerning the 2D case. First of all, note that the proposed algorithms are algebraic finite algorithms. The number of conditions to be tested is reduced in comparison with the 2D Jury table or the 2D Bistritz test. For instance, the examination of stability of a 2-D polynomial of degree $n$ using the Jury Table is performed in $n$ "steps", each step corresponding to the positivity condition for the $k$-th functional Schur coefficient ($k = 1 \ldots n$) and at each step a polynomial of degree $2^k$ is tested for positivity. The multidimensional Bistritz procedure is similar with the Jury procedure in terms of number of steps and polynomial order involved in the procedure. However, the symmetry of the polynomials involved in the recursion should be exploited when efficiently implementing this procedure, and furthermore, Bistritz showed [2] that some redundant factors can be eliminated in order to reduce even more the computational complexity. The multidimensional Jury procedure has the advantage of explicitly giving the leading Schur Cohn minors, which are intimately related with the functional Schur coefficients.

Let us consider now the 3-D case.

*Example 2.* Consider the polynomial

$$P_2(z_1, z_2, z_3) := 5 + z_2 + z_1 z_3^2 - z_1 z_2 z_3 + z_1 z_2.$$

The slice of $P_2$ is

$$\begin{aligned} P_{v_1,v_2}(\lambda) &= P_2(\lambda, \lambda v_1, \lambda v_2) \\ &= 5 + \lambda v_1 + \lambda^2 v_1 + \lambda^3 (v_2^2 - v_1 v_2) \end{aligned}$$

One can compute the Schur-Cohn minors exactly as in the 2-D case, using the 3-D Jury table, and check them for positivity. One can use [22] to relax such a condition, by testing if a 2-D polynomial can be written in sum of squares. However, to the authors knoledge, there are no finite algebraic algorithms for testing the positivity condition of a 2-D real polynomial.

We compute in the following the functional Schur coefficients of $P_2$ and represent their absolute values, see Fig. 2. As all of them are not greater than one, the polynomial $P_2$ is stable.

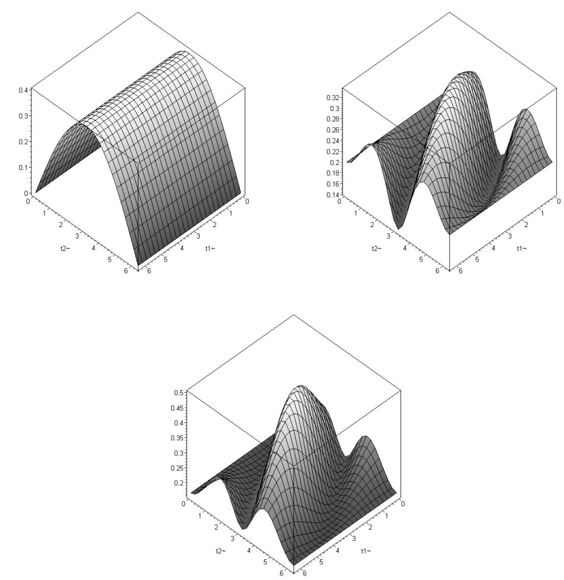

Fig. 2. Functional Schur coefficients of $P_2(z_1, z_2, z_3) := 5 + z_2 + z_1 z_3^2 - z_1 z_2 z_3 + z_1 z_2$

In the same way, in Fig. 3 and 4 the functional Schur coefficients of

$$P_3(z_1, z_2, z_3) = 0.85i + 0.25z_1^2 + 0.25z_2^2 + 0.25z_3^2 + 0.1z_2^3$$

and

$$P_4(z_1, z_2, z_3) = 0.82i + 0.25z_1^2 + 0.25z_2^2 + 0.25z_3^2 + 0.1z_2^3$$

are represented. One can see that a small perturbation of the first coefficients changes the stability into instability, as $P_3$ is stable but $P_4$ is unstable (the last functional Schur coefficient exceeds 1).

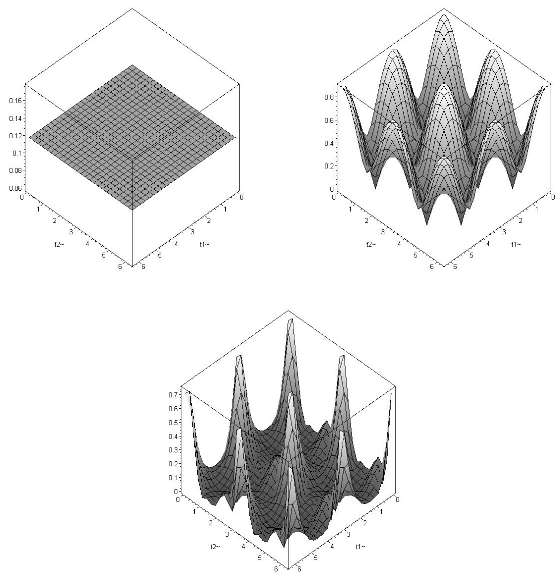

Fig. 3. Functional Schur coefficients of $P_3(z_1, z_2, z_3) = 0.85i + 0.25z_1^2 + 0.25z_2^2 + 0.25z_3^2 + 0.1z_2^3$

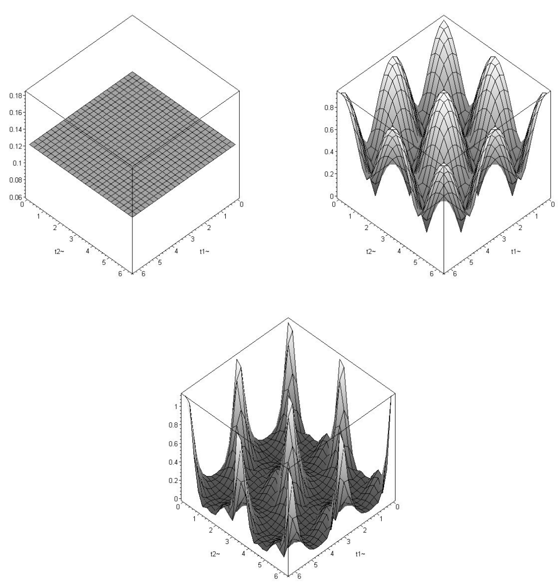

Fig. 4. Functional Schur coefficients of $P_4(z_1, z_2, z_3) = 0.82i + 0.25z_1^2 + 0.25z_2^2 + 0.25z_3^2 + 0.1z_2^3$

## VI. CONCLUSIONS AND FUTURE WORKS

In this contribution a new multidimensional Schur-Cohn type stability criterion was introduced, and a procedure for testing this criterion was given. The implementation technique in the two-dimensional case was detailed, and illustrated by means of an example. Application in design of filters with guaranteed stability is straightforward.

This criterion is based on the use of functional Schur coefficients and future research should be directed to investigate their properties.

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