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A NEW VARIANT OF GENERALIZED ASSIGNMENT PROBLEM WITH CONSIDERATION OF BALANCE OF AVERAGE ASSIGNMENT COST

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RÉSUMÉ : This paper studies a different but realistic variant of minimization version of generalized assignment problem (Min-GAP). The new Min-GAP is to assign *n* items (tasks) to *m* knapsacks (agents) subject to minimum demands constraints. The objective is to minimize the total assignment cost (primary objective). Furthermore, since knapsacks belong to different interest groups, their average assignment costs are required to be as close each other as possible (secondary objective). A Lagrangian/surrogate relaxation scheme is proposed for this biobjective optimization problem. Firstly, the primary objective is optimized. Secondly, by adding additional constraints, the primary objective is reoptimized while approaching the secondary objective indirectly. Experimental results demonstrate that the sample variance of the average assignment costs of knapsacks can be greatly reduced in most randomly generated instances with slight increasing in total costs.

MOTS-CLÉS : *Generalized assignment problem, Lagrangian/surrogate relaxation, Knapsack problems*

1. INTRODUCTION

The minimum version of generalized assignment problem (Min-GAP) is a widely studied optimization problem, which considers the minimum cost assignment of *n* items (tasks) to *m* knapsacks (agents) such that each item is assigned to one and only one knapsack subject to maximum capacity constraints. The Min-GAP has found applications in many real world problems, such as computer and communication networks, location problems, vehicle routing and machine scheduling. However, as we all know, Min-GAP is NP-hard (Martello and Toth, 1990)

A considerable body of literature exists in the search for effective enumeration algorithms to solve problems of a reasonable size to optimality, for example Ross and Soland (1975), Fisher, Jaikumar and VanWassenhove (1986). A detail introduction of generalized assignment problem can be found in Martello and Toth (1990).

In recent studies, the main works are focused on the various application of GAP under special conditions. For example, French and Wilson propose a heuristic algorithm for the multilevel generalized assignment problem (MGAP), in which task-agent assignment can be made at different levels, implying different costs and different amounts of resource used. Ceselli and Righini (2006) propose an exact method for the MGAP. Chen and Lu consider an assignment problem with multiple incommensurate inputs and outputs for each possible assignment (2007). Kogan et al (2005) consider dynamic generalized assignment problems with stochastic demands and multiple agent- task relationships.

In this paper, we study a different but realistic variant of minimization version of generalized assignment problem (NP-hard) encountered in an industrial project of Champagne Ardenne, France. In the green energy investment project, 4 plants belonging to 4 different interest groups need to be built to collect wheat straw from a large number of suppliers and then produce ethanol. In order to guarantee the profitability, each plant has a minimum demand that must be satisfied by assigning enough suppliers (from 1000-2000 suppliers in reality) to it. Each supplier can only be assigned to at most one plant. This is a cooperation-competition assignment problem which has two objectives. The primary one is to decide which suppliers should be assigned to which plants at the minimum transportation cost. That means the interest of the overall system is in the first place (cooperation). With this precondition, the secondary objective is to balance the average transportation cost of each plant which requires that they can be as close each other as possible for the reason that these plants belong to different interest groups (competition).

In fact, this transportation problem can be formulated as a biobjective Min-GAP by viewing plants as knapsacks, suppliers as items and the supply volumes of suppliers as the weights of items. However, comparing with the classical Min-GAP model, our model is a new variant. In the classical Min-GAP, each knapsack has a maximum capacity and each item must be assigned exactly to one knapsack, but in our model each knapsack has a mini-

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mum demand which has to be satisfied by assigning enough items to it. As the existence of a large number of items, only a part of them will be assigned to each knapsack. Further more, the secondary objective of balance average assignment cost which is rarely considered in literatures of Min-GAP is highly nonlinear.

Considering the intractability of the secondary objective and the large number of items, we focus on seeking an approximate solution to our model. A Lagrangian/surrogate relaxation scheme is proposed for this biobjective optimization problem. Firstly, the primary objective is optimized. Secondly, by adding additional constraints, the primary objective is optimized again while approaching the secondary objective indirectly. The rest of the paper is organized as follows. The new Min-GAP model is formulated in section 2. In section 3, a Lagrangian/surrogate relaxation approach is proposed for this Min-GAP. Numerical testing results are presented in section 4. Conclusions are given in section 5.

2. The new Min-GAP formulation

The new variant of Min-GAP considered in this paper is based on the assumption that the sum of all items' weight is greater than the sum of the total minimum demands of all knapsacks. That means it is not necessary for each item to be assigned to one knapsack. But, if an item is assigned, it can only be assigned to exactly one knapsack. Our primary objective is to minimize the total assignment cost under the constraints that all the knapsacks' minimum demands have to be satisfied. To simplify, we still assume that the weight of each item is identical for each knapsack and each knapsack has the same minimum demand. The notations required for modeling the problem are given as follows:

Parameters

N : The set of items; $N = \{1, 2, ..., n\}$;

M : The set of knapsacks; $M = \{1, 2, ..., m\}$;

w_j : The weight of item j ;

C : Minimum demand of knapsack;

d_{ij} : The unit cost of assigning item j to knapsack i ;

Variables

$$x_{ij} = \begin{cases} 1 & \text{item } j \text{ is assigned to knapsack } i \\ 0 & \text{otherwise} \end{cases}$$

Omitting the secondary objective, the new Min-GAP model which has minimum demand constraints can be formulated as the following binary integer-programming problem (P):

$$\text{Model P:} \quad \text{Z=M in } \sum_{i=1}^{m}\sum_{j=1}^{n} w_j d_{ij} x_{ij} \tag{1}$$

$$\text{s.t.} \quad \sum_{i=1}^{m} x_{ij} \le 1, \qquad \forall j \in N \tag{2}$$

$$\sum_{j=1}^{n} x_{ij} \cdot w_j \ge C\,, \qquad \forall i \in M \tag{3}$$

$$x_{ij} \in \{0,1\} \quad \forall i \in M,\ j \in N \tag{4}$$

Constraints (2) mean that each item is assigned to at most one knapsack. Constraints (3) ensure that the minimum demand of each knapsack must be satisfied. So the average assignment cost of knapsack *i* is

$$c_{\text{ave}}^{i} = \sum_{j=1}^{n} w_j d_{ij} x_{ij} \Bigg/ \sum_{i=j}^{n} w_j x_{ij} \qquad \forall i \in M \tag{5}$$

The secondary objective can be formulated by minimizing the sample variance of c_{ave}^{i} as Eq.(6) shown.

$$z = \text{Min } \frac{1}{m-1}\sum_{i=1}^{m}\left(c_{\text{ave}}^{i} - \bar{c}_{\text{ave}}\right)^2 \tag{6}$$

Where $\bar{c}_{\text{ave}} = \sum_{i=1}^{m} c_{\text{ave}}^{i} \Big/ m$ is the mean value of c_{ave}^{i}, $\forall i \in M$. No matter as a constraint or as an objective function, Eq.(6) is very complicated to be tackled .

In this paper, the secondary objective is considered indirectly and approached by adding additional constraints. For any given parameter $\underline{c}$, we require that the average assignment cost c_{ave}^{i} of each knapsack *i* must greater than or equal to $\underline{c}$. Violating this constraint will lead to, for example at knapsack *i*, $\sum_{j=1}^{n}\left(d_{ij} - \underline{c}\right) w_j x_{ij} \le 0$. This means more items with unit assignment cost greater than $\underline{c}$ should be assigned to knapsack *i*. However, our objective is still to minimize the total assignment cost, thus it will force each c_{ave}^{i} close to $\underline{c}$ as much as possible. So that our secondary objective, the balance of average assignment costs among knapsacks can be finally achieved. By testing a number of possible value of $\underline{c}$, hopefully we can find a solution that has small sample variance of c_{ave}^{i} at the expense of slight increase of the total assignment cost. Thus the following constraints are added to Model P.

$$\sum_{j=1}^{n}\left(d_{ij}-\underline{c}\right)w_j x_{ij}\geq 0, \qquad i\in M \tag{7}$$

The new Min-GAP is defined by Eq.(1)-(4) and (7).

3. Lagrangian/surrogate relaxation approach for new Min-GAP

A two stages relaxation scheme is proposed in this section. Firstly, a Lagrangian relaxation approach is presented to decompose our Min-GAP into *m* independent two dimensional knapsack problems (2KPs). Secondly, instead of seeking for an optimal solution, a surrogate relaxation approach is proposed to find a valid lower bound for the 2KPs.

3.1. Decomposition

There are several ways to relax Min-GAP. We can dualize constraints (2), constraints (3) or constraints (7) or any combination of them. However, constrains (2) are generalized upper bound (GUB) constraints, the relaxation of them might provide better bounds (Fisher 2004). So constrains (2) are relaxed and incorporated into the objective function by introducing a set of non-negative Lagrange multipliers $\boldsymbol{\lambda}\in\mathbf{R}^n$. Then the Lagrangian relaxed problem $\mathrm{P}_{LR}(\boldsymbol{\lambda})$ is

$$\mathrm{Z}_{\mathrm{LR}}(\boldsymbol{\lambda})=\mathrm{Min}\ \sum_{i=1}^{m}\sum_{j=1}^{n}w_j d_{ij}x_{ij}+\sum_{j=1}^{n}\lambda_j\left(\sum_{i=1}^{m}x_{ij}-1\right)$$

Subject to (3), (4), (7)

The corresponding dual problem is

$$Z_{\mathrm{D}}=\underset{\boldsymbol{\lambda}}{\mathrm{Max}}\ \mathrm{Z}_{\mathrm{LR}}(\boldsymbol{\lambda}) \tag{D}$$

It is well known that for any $\boldsymbol{\lambda}\geq\mathbf{0}$, $\mathrm{Z}_{\mathrm{LR}}(\boldsymbol{\lambda})$ provides a lower bound of the Min-GAP so as to $\mathrm{Z}_{\mathrm{D}}\leq Z$ (Fisher 2004). The $\mathrm{P}_{LR}(\boldsymbol{\lambda})$ can be decomposed into *m* independent subproblems which can be formulated as follows:

$$\mathrm{Z}_{\mathrm{2KP}}^{i}(\boldsymbol{\lambda})=\mathrm{Min}\ \sum_{j=1}^{n}\left(w_j d_{ij}+\lambda_j\right)x_{ij}$$

Subject to (3), (4), (7)

This subproblem is a two-dimensional 0-1 knapsack problem 2KP and NP-hard (Martello and Toth 2003). In the 2KP or the *m*KP (Freville 2004), all of the weights are positive. However, in our subproblem the weight may be negative as the factor $d_{ij}-\underline{c}$ in constraints (7) indicated. Considering the computational time for finding an optimal solution of 2KP, we prefer to find a valid lower bound. A surrogate relaxation approach is performed. By associating two nonnegative surrogate multipliers σ,τ with constraints (3) and (7), respectively, we can write the surrogate relaxation of 2KP as

$$S(\boldsymbol{\lambda},\sigma,\tau,i)=\mathrm{Min}\ \sum_{j=1}^{n}\left(w_j d_{ij}+\lambda_j\right)x_{ij} \tag{8}$$

$$\text{s.t.}\quad \sum_{j=1}^{n}\left(\sigma+\tau d_{ij}-\tau\underline{c}\right)x_{ij}\cdot w_j\geq\sigma C \tag{9}$$

and (4)

It is obvious that, if $\sigma+\tau d_{ij}-\tau\underline{c}\leq 0$, then $x_{ij}=0$. This is a 0-1 knapsack problem which can be solved efficiently by Martello and Toth's algorithm MT2. It is well known that, for any positive pair (σ,τ) , $S(\boldsymbol{\lambda},\sigma,\tau,i)$ provides a lower bound for 2KP. If the solution of $S(\boldsymbol{\lambda},\sigma,\tau,i)$ is feasible for 2KP, then it is also optimal. The optimal surrogate multiplier pair (σ,τ) produces the surrogate dual bound

$$S(\boldsymbol{\lambda},\sigma,\tau,i)=\underset{\sigma,\tau\geq 0}{\mathrm{Max}}\quad S(\boldsymbol{\lambda},\sigma,\tau,i) \tag{S}$$

Both the lagrangian and the surrogate dual problem (D) and (S) can be solved by applying subgradient optimization. The lower bound for the Min-GAP z_{low} is obtained as,

$$z_{\mathrm{low}}=\sum_{i=1}^{m}S(\boldsymbol{\lambda},\sigma,\tau,i)-\sum_{j=1}^{n}\lambda_j \tag{S}$$

3.2. Subgradient optimization

In this section, a subgradient optimization is performed to solve the lagrangian dual problem (D). As similar to that for (D), the subgradient optimization for surrogate dual problem (S) is omitted here.

The global structure of subgradient optimization for solving dual problem (D) can be summarized as follows:

Step 0. Initiate $\boldsymbol{\lambda}^0:=\mathbf{0}$, $\theta_0:=1$, $k:=0$ and Z^* (a upper bound found by using a heuristic to construct an initial feasible solution of the Min-GAP);

Step 1. Solve the relaxed problem $\mathrm{P}_{LR}(\boldsymbol{\lambda}^k)$;

Step 2. Set step size t_k in iteration k by $t_k := \theta_k \left[Z^* - Z_{LR}\left(\boldsymbol{\lambda}^k\right)\right] / \gamma^k$, where $Z_{LR}\left(\boldsymbol{\lambda}^k\right)$ is the current dual value. $\gamma^k := \sum_{j=1}^{n}\left(\sum_{i=1}^{m} x_{ij}^k - 1\right)^2$;

Step 3. Update the Lagrange multipliers in iteration $k+1, \lambda_j^{k+1} := \max\left\{0, \lambda_j^k + t_k\left(\sum_{i=1}^{m} x_{ij}^k - 1\right)\right\}$;

Step 4. Construct a feasible solution and update the upper bound Z^*; If the dual objective value is not improved in a given number of iterations, set $\theta := \theta/2$;

Step 5. Check the stopping criterions:

(1) The dual value $Z_{LR}\left(\boldsymbol{\lambda}^k\right)$ is not improved for a given number of iterations, or

(2) The total number of iterations is greater than a given integer.

If one of the criterions is met, stop and output the best feasible solution and the duality gap; else, $k := k+1$, return to Step 1;

3.3. Construction of feasible solution

A good upper bound can effectively accelerate the convergence of subgradient optimization. Based on the solution of relaxed problem x_{ij}^*, a simple heuristic is proposed to construct a feasible solution as following shown.

Step 0: For any $j \in N$, if $\sum_{i=1}^{m} x_{ij}^* - 1 > 1$ (that means item *j* has been assigned to several knapsacks), then assigns item *j* to the knapsack *i* whose $w_j d_{ij}$ value is the minimal one in these knapsacks and set $x_{ij}^* = 1$, the others $x_{ik}^* = 0,\ k \neq j$;

Step 1: For any knapsack *i*, calculate the supplied volume $\sum_{j=1}^{n} x_{ij}^* \cdot w_j$. If $\sum_{j=1}^{n} x_{ij}^* \cdot w_j < C$, then select enough items that haven't been assigned to satisfy constrain (3) and (7) by solving a two-dimensional 0-1 Knapsack Problem.

3.4. Selection of $\underline{c}$

Omitting constraints (7) and only considering the primary objective, the Min-GAP can be solved by using the above described lagrangian/surrogate relaxation method. The 2KP turns into 0-1 knapsack problem which can be solved efficiently by Martello and Toth's subroutine MT2. We denote $\underline{c}_{\min}$ as the minimal average assignment cost and $\underline{c}_{\max}$ as the maximal one. $\underline{c}$ takes value $\underline{c}_{\min} + i \cdot \left(\underline{c}_{\max} - \underline{c}_{\min}\right)/D$ sequentially, where D is a given positive integer number and $i = 1..D-1$. The Min-GAP solution that has minimum sample variance of average assignment costs and with an acceptable increasing rate of global cost is selected as the output solution of our algorithm.

4. Numerical results

We evaluated our algorithm's performance by using randomly generated problem instances. The parameters of the instances are generated in this way: Firstly, we generate coordinates for items and knapsacks in a square plane calculate the distances between items and knapsacks and the distances are rounded to integer and used as the unit assignment costs. In addition, the weight of each item is randomly and uniformly generated from the intervals [10, 50], it is also rounded to integer. Our algorithm was coded by using Visual Fortran 6.5. The numerical tests were performed on a Pentium (R) 1.73GHz PC with 1 GB RAM. The notations used for presenting the results of the instances are first given in Table 1.

AVE_{MAX}	Maximum average assignment cost of all knapsacks
AVE_{MIN}	Minimum average assignment cost of all knapsacks
VAR	Sample variance of average assignment costs
Gap(%)	Duality gap of Min-GAP
TIME	Computational time (seconds)
Cost Increase(%)	The percentage increment of objective function: (Upper Bound of Min-GAP - Upper Bound of without balancing)/ Upper Bound of Min-GAP × 100%

Table 1: Notations used in numerical results

The duality gap is defined as: (Upper Bound- Lower bound)/ Lower Bound × 100%.

The interval $\left[\underline{c}_{\min}, \underline{c}_{\max}\right]$ is divided 7 fractions. $\underline{c}$ sequentially takes the value $\underline{c}_{\min} + i \cdot \left(\underline{c}_{\max} - \underline{c}_{\min}\right)/7$, $i = 1..6$. The Min-GAP solution, which has the minimum sample variance of average assignment costs and the duality gap smaller than 6%, is chosen as the output solution for our biobjective optimization problem. Because of space limitation, we don't present all the testing data here. Testing results of 10 randomly generated instances are given as Table 2-3 shown.

	Before balancing				After balancing				Cost Increase(%)	TIME
	AVE_{MAX}	AVE_{MIN}	VAR	Gap(%)	AVE_{MAX}	AVE_{MIN}	VAR	Gap(%)		
1	2.025	1.140	8.134	1.522	1.923	1.583	1.763	2.548	4.704	1433.55
2	**1.961**	**1.105**	**6.604**	**0.394**	**1.828**	**1.535**	**0.848**	**5.403**	**8.327**	**820.42**
3	2.615	1.285	18.484	1.572	2.474	1.733	2.823	2.258	4.971	1052.32
4	1.812	1.250	4.157	1.675	1.771	1.438	0.805	2.507	3.408	1152.53
5	2.186	1.096	10.600	0.545	2.223	1.465	5.565	3.080	4.203	839.68

Table 2: Numerical results for n=1000, m=10, $C = \sum_{i=1}^{n} w_i \Big/ 11$

	Before balancing				After balancing				Cost Increase(%)	TIME
	AVE_{MAX}	AVE_{MIN}	VAR	Gap(%)	AVE_{MAX}	AVE_{MIN}	VAR	Gap(%)		
1	2.619	1.714	16.587	0.200	2.549	2.167	4.092	1.144	4.533	1905.73
2	**4.234**	**2.222**	**84.711**	**0.343**	**4.217**	**2.894**	**47.266**	**4.589**	**4.990**	**1817.81**
3	2.235	1.866	31.248	0.835	2.184	2.175	2.78×10^{-3}	1.100	3.141	1503.64
4	2.655	2.258	14.055	0.464	2.536	2.457	0.216	0.811	1.244	2535.34
5	2.068	1.704	7.129	0.145	2.041	1.947	0.315	0.603	4.952	2534.41

Table 3: Numerical results for n=2000, m=4, $C = \sum_{i=1}^{n} w_i \Big/ 5$

In Tables 2-3, columns 2-5 report respectively the maximum, minimum average assignment cost of knapsacks, the sample variance of average assignment costs and the duality gap of the Min-GAP without considering the balance of average assignment costs. Columns 6-9 present the results that after performing balancing scheme. We find that the sample variance has been reduced significantly for all tested instances. In most cases, the variance has been reduced to over 60% or more. However, there are some instances which have little improvement on the variance as shown in boldface. That is because the two objectives: global interests and individual interests (average assignment costs are required to be as close each other as possible) are incompatible under certain circumstances. A good balance of individual interests is always at the price of sacrificing the overall interests.

5. Conclusions

This paper describes a real-life assignment problem, a new variant of biobjective Min-GAP with minimum capacity constraints, which received little attention but are encountered frequently in reality. The problem is not only to minimize the total assignment cost (primary objective), but also to balance the average assignment cost in knapsacks (secondary objective). A Lagrangian/surrogate relaxation scheme is proposed. Firstly, the primary objective is optimized. Secondly, by adding additional constraints, the secondary objective is approached indirectly. Numerical results on the randomly generated instances demonstrate that the sample variance of average transportation costs has been reduced significantly in all of the tested instances. In most of cases, the variance has been reduced to over 60% or more with the increase of the global cost in an acceptable range.

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