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Stability Analysis of a Proportional with Intermittent Integral Control System

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Abstract— **The stability analysis of a proportional with intermittent integral (PII) control system is presented. This PII control system utilizes a proportional controller to attenuate unpredictable disturbances with an intermittently invoked integral controller to cancel constant disturbances. The integral control action is replaced by a constant offset set to its final value allowing perfect tracking and rejection of constant signals. By this design, the proportional controller can be made more aggressive while maintaining stability margins and control actions at similar levels. Due to its control fashion, this PII control system is modeled as a switched system composed of both linear time invariant and linear time-varying subsystems. The stability of this switched system is analyzed using a multiple Lyapunov functions approach. Additional sufficient conditions to ensure stability for the PII controller parameters are derived.**

I. INTRODUCTION

Disturbance attenuation is an important issue in many applications. If the disturbance has a predictable component, it can be identified or estimated, and can be compensated using open-loop control systems. However, it is undesirable to use purely open-loop controller for compensation of predictable disturbances, since slow changes, such as phase drift could eliminate the compensatory effect. In order to compensate these variations, conventional Internal Model Principle controllers, such as integral control, are used. But the conventional closed-loop controllers will limit the capabilities for compensating unpredictable disturbances. Brown *et al* [1][2] developed a control approach that combines open-loop and closed-loop control to perfectly cancel predictable while minimizing random disturbances which is referred to as *Intermittent Control*.

The simplest implementation of the intermittent control is the Proportional with Intermittent Integral (PII) control [1]. Motivated by a model of mammalian blood pressure regulation, this approach utilizes a proportional controller that intermittently invokes an integral controller. When the integral control loop is opened, the learned control action is maintained. Thus, the proportional controller can be made more aggressive while maintaining stability margins and/or control actions at similar levels. Simulations in [1][2] showed the performance improvement that can be achieved with this control approach, but a stability proof was left as an open issue. This paper provides the desired stability analysis.

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Due to its control fashion, a PII control system can be modeled as a switched system, which is a hybrid dynamical system that is composed of a family of continuous-time systems and a rule orchestrating the switching between the subsystems [3]. Therefore, stability analysis approaches for switched systems apply to the PII control system.

A notable approach for the stability analysis of switched systems is dwell time approach. This approach has been studied in [4][5]. For the stability under arbitrary switching problems, much of the work has been focused on the existence of a common Lyapunov function. It is well established [6] that if a common Lyapunov function exists for the subsystems of a switched linear system, then the system is uniformly exponentially stable for arbitrary switching signals. Moreover, the problem of determining the stability of a switched system in the case where the switching action is constrained in some manner arises in a number of important applications [6][7]. Branicky [8] proposed a multiple Lyapunov functions (MLF) approach to guarantee the stability of switched systems with a constraint on the rate of switching. His basic idea was to define a Lyapunov-like function for each subsystem. One then uses these functions to construct a stabilizing switching signal by only allowing the system to switch into a subsystem if the value of the corresponding Lyapunov-like function is less than it was when this subsystem last switched in. However, the challenge of this approach is to find the individual Lyapunov-like functions for each subsystem. The search for Lyapunov-like functions can be formulated as a Linear Matrix Inequality (LMI) problem and discussed in [9][10].

The stability theorem using MLF approach in [8] requires all subsystems to be stable, and only guarantees stability in the sense of Lyapunov. By relaxing the restrictions on the stabilities of all subsystems, we developed a less conservative theorem that guarantees both stability in the sense of Lyapunov and asymptotic stability in [11]. This stability theorem can be applied for the stability analysis of the PII control system. This paper is organized as follows: First, the PII control system is introduced in Section II. In Section III, the stability of the PII control system is analyzed, followed by the conclusion and future work in Section IV.

II. PROPORTIONAL WITH INTERMITTENT INTEGRAL CONTROL SYSTEM

The overall PII control system is shown in Fig. 1. The PII controller is indicated by the dash-line block. G_p represents the plant of interest. The controller has a time-varying integral gain with $\dot{K}_i(t) = -K_d K_i(t)$ with K_d and the opening and closing of switch S_1 defined in Table I. (Note $K_i(t)=0$ is equivalent to switch being open, and $K_i(t)\neq 0$ implies switch is closed.) The switching mechanisms of this PII

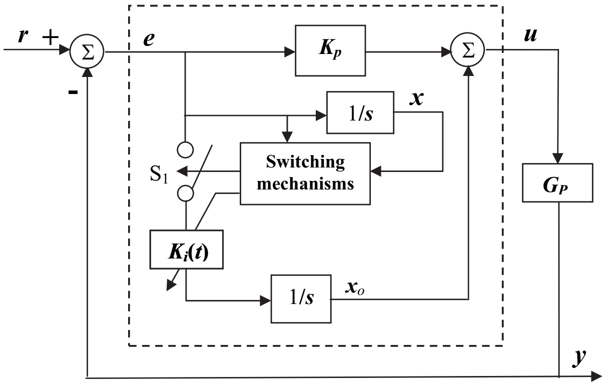

Fig. 1. Block diagram of a PII control system

TABLE I
DEFINITION OF PARAMETERS FOR PII CONTROL

If	Then
$K_i(t)=0$ and $\lvert x(t)\rvert > x_u$	$K_i(t^+)=K_i^*$, $x_o(t^+)=x_o(t)+\frac{K_i^* x(t)}{\max(1,K_s(t-t_l))}$
$K_i(t)=K_i^l$	$K_i(t^+)=0, x(t)=0, t_l=t$
$\lvert e(t)\rvert \geq e_u$	$K_d=0$
$\lvert e(t)\rvert < e_u$	$K_d=K_{decay}$

controller are as follows. Initially, the PII controller begins as a proportional controller (S_1 open, i.e., $K_i(t)=0$), and the integral controller is initialized with a nominal offset $x_o(t_0)=x_o$. The integrated error x is monitored. t_l is the time instant when error e starts to be integrated. When x exceeds a threshold x_u at time t, the integral controller is turned on by setting $K_i(t)=K_i^*$, and x_o is simultaneously augmented by x and scaled by the time spent reaching the threshold as given by

$$K_i(t^+)=K_i^* \tag{1}$$

$$x_o(t^+)=x_o(t)+\frac{K_i^* x(t)}{\max(1,K_s(t-t_l))} \tag{2}$$

where K_s is a scaling factor. The integral controller remains active as long as the error e is excessive. Once the error is not significant, *i.e.*, $|e|<e_u$, the integral control action is removed in a smooth manner. This is achieved by allowing the value of the integral gain, $K_i(t)$, to decay exponentially. When the integral gain decreases to a lower bound K_i^l, $K_i(t)$ is set to zero. Integral action is thus completely turned off (S_1 open). x is reset, t_l is set to t, and e starts to be integrated.

Table 2 in [2] gives reasonable methods of choosing those additional parameters listed in Table I when the design goals are fast response and maximum disturbance attenuation.

This PII control strategy has been shown [2] to be more effective than PI control on plants with infrequent step changes in set-point or load. By observing its switching mechanism, this PII control system can be modeled as a switched system. Therefore, stability analysis approaches for switched systems can be applied to this system.

III. STABILITY ANALYSIS OF THE PII CONTROL SYSTEM

A. Multiple Lyapunov Functions Stability Theorem

The MLF result presented in [8] requires that all the subsystems to be stale and have the same equilibrium point at the origin. With all these conditions on the subsystems, this result can not be applied to the PII control system. As an extension of this result, an MLF approach developed in [11] relaxes the constraints on the subsystems. It allows some subsystems to be unstable or not to share the common equilibrium point. In addition, the states of all subsystems are not necessarily the same, which is required by other approaches. A brief summary of the result in [11] is given below.

For a switched system with M subsystems, let set $\{t_j\}$ represent the switching times with $t_j \leq t_{j+1}$. Pairs of subsets of $\{t_j\}$ are defined as follows,

$$\{\bar{t}_{q,k}\}=\{t_j|\text{when subsystem } q \text{ is switched on}\};$$
$$\{\underline{t}_{q,k}\}=\{t_j|\text{when subsystem } q \text{ is switched off}\}$$

with $q\in\{1,2,\cdots,M\}$, and $\bar{t}_{q,k}<\bar{t}_{q,k+1}$, $\underline{t}_{q,k}<\underline{t}_{q,k+1}$. Let S be a switching sequence associated with the switched system. The interval completion $\mathscr{I}(S|q)$ is defined as,

$$\mathscr{I}(S|q)=\bigcup_k [\bar{t}_{q,k},\underline{t}_{q,k}]$$

For a switched system that has different state vectors for each of the individual subsystems, its dynamics can be described as

$$\dot{x}_q=f_q\big(x_q(t)\big), \qquad t\in\mathscr{I}(S|q) \tag{3}$$

where $x_q\in\mathbb{R}^{n_q}$. Its initial condition is given by

$$x_q(t_j)=H_{q,p}\big(x_p(t_j)\big) \tag{4}$$

where $q,p\in\{1,2,\cdots,M\},(q\neq p)$, and $\sigma(t_j)=q$, $\sigma(t_{j-1})=p$. Function H satisfies $\|H_{q,p}\big(x_p(t_j)\big)\|\leq K\|x_p(t_j)\|$, with K being a constant. For switched system (3), (4), Theorem 1 in [11] states that

> Suppose we have candidate Lyapunov-like functions V_q for each of the individual subsystems. Let Π be the set of all switching sequences associated with the system, and Ξ be an arbitrary subset of Π. If for each $S\in\Xi$, the following conditions are satisfied,
>
> 1) There exists at least one V_i, $i\in\{1,2,\cdots,M\}$, such that
>   a) $\dot{V}_i(x_i(t))\leq 0$, for all $t\in\mathscr{I}(S|i)$;
>   b) $V_i(x_i(\bar{t}_{i,k+1}))\leq V_i(x_i(\bar{t}_{i,k}))$, $\forall k$;
> 2) For all other V_q's, $(q\neq i)$,
>   a) there exists a positive constant m, such that $|V_q(x_q(t))|\leq m|V_i(x_i(\bar{t}_i^*))|$, for $\bar{t}_{q,j}\leq$

$t < \underline{t}_{q,j}$, where interval $[\bar{t}_{q,j},\ \underline{t}_{q,j}]$ is a subset of $\mathscr{I}(S|q)$ for any j, and $\bar{t}_i^* = \max_k\{\bar{t}_{i,k}|\bar{t}_{i,k} < \bar{t}_{q,j}\}$;

b) x_q does not have finite escape time and is guaranteed to enter subsystem i or the switched system can be guaranteed to enter subsystem i prior to entering subsystem q,

the switched system (3), (4) is stable in the sense of Lyapunov for all switchings in Ξ. Furthermore, if $\dot{V}_i(x_i(t)) < 0$, and one of the following two conditions is satisfied,

- the sequence $\{V_i(x_i(\bar{t}_{i,k}))\}$ converges to zero as $k \to \infty$;
- $\{\bar{t}_{i,k}\}$ is a finite sequence and the switched system stays in subsystem i after the last switching,

the switched system is asymptotically stable.

Compared with existing MLF approaches in the literature, this theorem allows some subsystems to be unstable as long as their corresponding candidate Lyapunov-like functions have converging upper bounds which are functions of stable subsystems' Lyapunov functions.

B. Switched System Model of the PII Control System and Stability Theorem

The plant G_p in Fig. 1 can be represented by state space equations as

$$\begin{aligned} \dot{x}_p &= A_p\ x_p + B_p\ u \\ y &= C_p\ x_p \end{aligned} \tag{5}$$

where $x_p \in \mathbb{R}^n$. System control signal is

$$u = K_p\ e + x_o \tag{6}$$

and the error signal is

$$e = r - y = r - C_p x_p \tag{7}$$

so

$$\dot{x}_p = (A_p - K_p B_p C_p)x_p + B_p x_o + K_p B_p r \tag{8}$$

From Fig. 1, we have

$$\dot{x}_o = K_i(t)\ e = -K_i(t)C_p x_p + K_i(t) r \tag{9}$$

Let

$$z = \begin{bmatrix} x_o \\ x_p \end{bmatrix}, \quad z \in \mathbb{R}^{n+1} \tag{10}$$

The state equation of the PII control system can be expressed as

$$\begin{aligned} \dot{z} &= \begin{bmatrix} \dot{x}_o \\ \dot{x}_p \end{bmatrix} = \begin{bmatrix} 0 & -K_i(t)\ C_p \\ B_p & A_p - K_p B_p C_p \end{bmatrix} \begin{bmatrix} x_o \\ x_p \end{bmatrix} + \begin{bmatrix} K_i(t) \\ K_p B_p \end{bmatrix} r \\ &:= A_\sigma(t) z + B_\sigma(t) r \end{aligned} \tag{11}$$

$$y = \begin{bmatrix} 0 & C_p \end{bmatrix} \begin{bmatrix} x_o \\ x_p \end{bmatrix} = Cz \tag{12}$$

where $A_\sigma(t) \in \mathbb{R}^{(n+1)\times(n+1)}$, $B_\sigma(t) \in \mathbb{R}^{(n+1)\times 1}$, and $z(t_0) = z_0$.

By observing the switching mechanisms, this PII control system can be modeled as a switched linear system consisting of two subsystems. The switching signal $\sigma(t) \in \{1,2\}$ and matrices $A_\sigma(t),\ B_\sigma(t)$ can be written as

$$A_1(t) = \begin{bmatrix} 0 & -K_i^* \exp(\int_{t_0}^t -K_d \mathrm{d}\tau)\ C_p \\ B_p & A_p - K_p B_p C_p \end{bmatrix} \tag{13}$$

$$A_2(t) = \begin{bmatrix} 0 & 0 \\ B_p & A_p - K_p B_p C_p \end{bmatrix} := \begin{bmatrix} 0 & 0 \\ B_p & A_{22} \end{bmatrix} \tag{14}$$

$$B_1(t) = \begin{bmatrix} K_i^* \exp(\int_{t_0}^t -K_d \mathrm{d}\tau) \\ K_p B_p \end{bmatrix} \tag{15}$$

$$B_2(t) = \begin{bmatrix} 0 \\ K_p B_p \end{bmatrix} \tag{16}$$

For the switched linear system (11), (12), we have the following theorem.

Theorem 1: If the following assumptions are satisfied,

1) $K_p,\ K_i^*,\ K_d$ are chosen such that there exist finite positive constants $\mu_1,\ \mu_2$, and for all $K_i \in [0, K_i^*]$, every pointwise eigenvalue of $A_1(t)$ satisfies $\mathrm{Re}[\lambda(A_1(t))] \le -\mu_1$, and $\mathrm{Re}[\lambda(A_{22})] \le -\mu_2$.
2) K_i^* and K_d are chosen such that the upper bound β on $\|\dot{A}_1(t)\|$ satisfies $\beta < \min\{\mu_1^4/(3\alpha_1^2),\ 2\mu_1^2\}$, where α_1 is an upper bound on $\|A_1(t)\|$.
3) $K_i^*,\ K_d$, and K_i^l are chosen such that the minimum dwell time of the PI subsystem, τ_{1m}, given by $\tau_{1m} = \ln(K_i^*/K_i^l)/K_d$, satisfies
$$\tau_{1m} \ge \frac{\ln(a_1^2 a_3)}{\lambda_1}$$
where $a_1,\ a_3$, and λ_1 are parameters associated with $A_\sigma(t)$ and will be defined later.

the PII control system (11), (12) is stable in the sense of Lyapunov. ■

As described in Section II, the switchings of the PII control system only happen when certain conditions are satisfied. Thus MLF results for analyzing stability of switched systems with constrained switchings can be applied to prove this theorem. In order to apply Theorem 1 from [11], we will analyze the stability of each subsystem so that candidate Lyapunov-like functions for each subsystem can be constructed. In this paper, the norm of a vector x is referring to the Euclidean norm, $\|x\| = \sqrt{x^T x}$, and the norm of a matrix A is referring to the induced norm, $\|A\| = \max_{\|x\|=1} \|Ax\|$.

C. Stability analysis of the PI subsystem

It can be seen from (13) to (16) that subsystem 1 is linear time-varying (LTV), while subsystem 2 is linear time invariant (LTI). Both subsystems are forced systems with reference signal r as their input. However, this input signal does not affect the internal stability of the two subsystems. Thus the stability theory for homogeneous systems can be applied to both subsystems.

The stability analysis of subsystem 1 will be based on Theorem 8.7 in [12]:

> Suppose for the linear state equation $\dot{x} = A(t)x$, $x(t_0) = x_0$ with $A(t)$ continuously differentiable there exist finite positive constants α, μ such that, for all t, $\|A(t)\| \leq \alpha$ and every pointwise eigenvalue of $A(t)$ satisfies $\mathrm{Re}[\lambda(A(t))] \leq -\mu$. Then there exists a positive constant β such that if the time derivative of $A(t)$ satisfies $\|\dot{A}(t)\| \leq \beta$ for all t, the state equation is uniformly exponentially stable.

Since all entries in $A_1(t)$ are bounded for all $t > 0$, there exists $\alpha_1 > 0$ such that

$$\|A_1(t)\| \leq \alpha_1 \tag{17}$$

The upper bound β on $\|\dot{A}_1(t)\|$ can be derived as

$$\|\dot{A}_1(t)\| = \begin{bmatrix} 0 & K_i^* K_d e^{-K_d t} C_p \\ 0 & 0 \end{bmatrix} \leq K_i^* K_d \max_{i=1}^{n} |C_{pi}| := \beta \tag{18}$$

where C_{pi}'s are entries of matrix C_p. An explicit bound on β was derived by Desoer [13] as

$$\beta \leq \frac{\mu_1^4}{3\alpha_1^2} \tag{19}$$

This condition is satisfied by assumption 2 of theorem 1. With assumption 1, it can be concluded that subsystem 1 of the switched system (11) is uniformly exponentially stable, *i.e.*, the transition matrix of $A_1(t)$ satisfies

$$\|\Phi_1(t,\tau)\| \leq a_1 \mathrm{e}^{-\lambda_1(t-\tau)} \tag{20}$$

where explicit expressions of constants a_1, λ_1 are derived as follows.

For each t, let $(n+1) \times (n+1)$ matrix $Q_1(t)$ be the solution of equation

$$A_1^T(t) Q_1(t) + Q_1(t) A_1(t) = -I \tag{21}$$

which is

$$Q_1(t) = \int_0^\infty \mathrm{e}^{A_1^T(t)\sigma} \mathrm{e}^{A_1(t)\sigma} \mathrm{d}\sigma \tag{22}$$

thus

$$\begin{aligned} \|Q_1(t)\| &\leq \int_0^\infty \|\mathrm{e}^{A_1^T(t)\sigma} \mathrm{e}^{A_1(t)\sigma}\| \mathrm{d}\sigma \\ &\leq \int_0^\infty \mathrm{e}^{-2\mu_1\sigma} \mathrm{d}\sigma = \frac{1}{2\mu_1} \end{aligned} \tag{23}$$

From the proof of Theorem 8.7 in [12], the lower bound on $\|Q_1(t)\|$ is

$$\|Q_1(t)\| \geq \frac{1}{2\alpha_1} \tag{24}$$

The bound on $\|\dot{Q}_1(t)\|$ is derived using the boundedness of $\|Q_1(t)\|$,

$$\begin{aligned} \|\dot{Q}_1(t)\| &\leq 2\|\dot{A}_1(t)\| \; \|Q_1(t)\|^2 \\ &\leq \frac{\beta}{2\mu_1^2} := 1 - \nu \end{aligned} \tag{25}$$

where the positive $\nu = 1 - \frac{\beta}{2\mu_1^2}$ exists under assumption 2 of theorem 1.

From the proof of Theorem 7.4 in [12], we have

$$a_1 = \sqrt{\frac{\alpha_1}{\mu_1}} \tag{26}$$

$$\lambda_1 = \nu\mu_1 = \mu_1 - \frac{\beta}{2\mu_1} \tag{27}$$

where α_1, β are given by (17) and (18).

D. Stability analysis of the proportional subsystem

The state-space equation of subsystem 2 is

$$\begin{bmatrix} \dot{x}_o(t) \\ \dot{x}_p(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B_p & A_{22} \end{bmatrix} \begin{bmatrix} x_o(t) \\ x_p(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K_p B_p \end{bmatrix} r \tag{28}$$

or

$$\dot{x}_o = 0 \tag{29}$$

$$\dot{x}_p = A_{22} x_p + B_p x_o + K_p B_p r \tag{30}$$

Since x_o remains constant when subsystem 2 is active, subsystem 2 can be considered as a forced linear system with reduced number of states. As expressed in (30), the state x_o in subsystem 1 can be seen as a constant input to subsystem 2. The number of states for subsystem 2 is n, while subsystem 1 has $(n+1)$ states.

Since $\mathrm{Re}[\lambda(A_{22})] \leq -\mu_2$ from assumption 1, the state transition matrix of A_{22} satisfies

$$\|\mathrm{e}^{A_{22}(t-\tau)}\| \leq a_2 \mathrm{e}^{-\mu_2(t-\tau)} \tag{31}$$

The explicit expression of a_2 is derived as follows.

The solution of Lyapunov equation

$$A_{22}^T Q_2 + Q_2 A_{22} = -I \tag{32}$$

is given by

$$Q_2 = \int_0^\infty \mathrm{e}^{A_{22}^T\sigma} \mathrm{e}^{A_{22}\sigma} \mathrm{d}\sigma \tag{33}$$

The upper and lower bounds on $\|Q_2\|$ can be derived as

$$\frac{1}{2\alpha_2} \leq \|Q_2\| \leq \frac{1}{2\mu_2} \tag{34}$$

where α_2 is an upper bound on $\|A_{22}\|$. Then

$$a_2 = \sqrt{\frac{\alpha_2}{\mu_2}} \tag{35}$$

Since $\mathrm{e}^{A_{22}(t-\tau)}$ is the transition matrix of A_{22},

$$x_p(t) = \mathrm{e}^{A_{22}(t-\tau)} x_p(\tau) \tag{36}$$

The complete solution of (30) when $r = 0$ is

$$x_p(t) = \mathrm{e}^{A_{22}(t-t_0)} x_p(t_0) + \int_{t_0}^t \mathrm{e}^{A_{22}(t,\sigma)} B_p x_o \mathrm{d}\sigma \tag{37}$$

with initial time at t_0. Therefore, by augmenting x_p with x_o, we have

$$\begin{aligned} \begin{bmatrix} x_o(t) \\ x_p(t) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ \int_{t_0}^t \mathrm{e}^{A_{22}(t,\sigma)} B_p \mathrm{d}\sigma & \mathrm{e}^{A_{22}(t,\tau)} \end{bmatrix} \begin{bmatrix} x_o(\tau) \\ x_p(\tau) \end{bmatrix} \\ &:= \mathrm{e}^{A_2(t,\tau)} \begin{bmatrix} x_o(\tau) \\ x_p(\tau) \end{bmatrix} \end{aligned} \tag{38}$$

Based on the structure of $e^{A_2(t,t_0)}$, its induced norm can be written as

$$\|e^{A_2(t,t_0)}\| = \max_{\|x\|=1} \|e^{A_2(t,t_0)}x\| \tag{39}$$

Let vector x be partitioned as $[x_s, x_v^T]^T$, where x_s is a scalar, and x_v is an $n \times 1$ vector. We have

$$e^{A_2(t,t_0)}x = \begin{bmatrix} x_s \\ \int_{t_0}^{t} e^{A_{22}(t,\sigma)} B_p d\sigma x_s + e^{A_{22}(t,t_0)} x_v \end{bmatrix} \tag{40}$$

Thus

$$\begin{aligned} \|e^{A_2(t,t_0)}x\| &= \sqrt{x_s^2 + \|\int_{t_0}^{t} e^{A_{22}(t,\sigma)} B_p d\sigma x_s + e^{A_{22}(t,t_0)} x_v\|^2} \\ &\leq \sqrt{x_s^2 + 2\|\int_{t_0}^{t} e^{A_{22}(t,\sigma)} B_p d\sigma\|^2 x_s^2 + 2\|e^{A_{22}(t,t_0)} x_v\|^2} \end{aligned} \tag{41}$$

Let x_{vm} be the vector that maximizes $\|e^{A_{22}(t,t_0)}x_v\|$, *i.e.*,

$$\|e^{A_{22}(t,t_0)}\| = \max_{\|x_v\|=1} \|e^{A_{22}(t,t_0)}x_v\| = \|e^{A_{22}(t,t_0)}x_{vm}\| \tag{42}$$

Clearly, vector x that maximizes (41) will be of the form

$$x = \begin{bmatrix} \alpha \\ \sqrt{1-\alpha^2} x_{vm} \end{bmatrix}, \qquad (0 \leq \alpha \leq 1) \tag{43}$$

Since B_p is known, and from (31), we have

$$\begin{aligned} \|\int_{t_0}^{t} e^{A_{22}(t,\sigma)} B_p d\sigma\| &\leq \int_{t_0}^{t} \|e^{A_{22}(t,\sigma)}\| d\sigma \|B_p\| \\ &\leq \int_{t_0}^{t} a_2 e^{-\mu_2(t-\sigma)} d\sigma \|B_p\| \\ &= \frac{a_2}{\mu_2}(1 - e^{-\mu_2(t-t_0)})\|B_p\| \\ &\leq \frac{a_2}{\mu_2}\|B_p\| \end{aligned} \tag{44}$$

Inequality (41) can be derived as

$$\|e^{A_2(t,t_0)}x\| \leq \sqrt{\left(1 + \frac{2a_2^2}{\mu_2^2}\|B_p\|^2\right)\alpha^2 + 2(1-\alpha^2)a_2^2} \tag{45}$$

$\|e^{A_2(t,t_0)}x\|$ takes maximum value when $\alpha = 0$ or 1. Thus

$$\|e^{A_2(t,t_0)}\| \leq \max\left\{\sqrt{1 + \frac{2a_2^2}{\mu_2^2}\|B_p\|^2},\ \sqrt{2}a_2\right\} := a_3 \tag{46}$$

E. Stability analysis of the overall PII control system

For switched system (11) with initial time $t_0 \geq 0$ and initial state z_0, its switching signal $\sigma(t) \in \{1,2\}$. $\{A_1, A_2\}$ in (13)-(14) constitute a family of matrices describing the subsystems. As stated in Section II, initially, the switched system begins with the pure proportional subsystem, subsystem 2. Assume there are N switchings over the interval (t_0, t) with set $\{t_1, t_2, \cdots, t_N\}$ representing the switching times. At time $t > t_N$, subsystem 1 is assumed to be active. The switching on and off time sets for the two subsystems are

$$\{\bar{t}_{1,k}\} = \{t_1, t_3, \cdots, t_N\}; \quad \{\underline{t}_{1,k}\} = \{t_2, t_4, \cdots, t_{N-1}\}; \tag{47}$$

$$\{\bar{t}_{2,k}\} = \{t_2, t_4, \cdots, t_{N-1}\}; \quad \{\underline{t}_{2,k}\} = \{t_3, t_5, \cdots, t_N\}. \tag{48}$$

The interval completions for the two subsystems are

$$\mathscr{I}(S|1) = [t_1, t_2] \bigcup [t_3, t_4] \bigcup \cdots \bigcup [t_N, t] \tag{49}$$

$$\mathscr{I}(S|2) = [t_0, t_1] \bigcup [t_2, t_3] \bigcup \cdots \bigcup [t_{N-1}, t_N] \tag{50}$$

Note that the initial time t_0 and final time t are not considered as switching times, but they are included in the interval completions.

In Subsections III-C and III-D, the solutions to the Lyapunov equations (21), (32) are given by (22) and (33). Using these two solutions, two candidate Lyapunov-like functions can be constructed corresponding to the two subsystems,

$$V_1 = z^T Q_1(t) z = z^T \int_0^{\infty} e^{A_1^T(t)\sigma} e^{A_1(t)\sigma} d\sigma z \tag{51}$$

$$V_2 = x_p^T Q_2 x_p = x_p^T \int_0^{\infty} e^{A_{22}^T\sigma} e^{A_{22}\sigma} d\sigma x_p \tag{52}$$

Verifying condition (1.a) of Theorem 1 in [11],

$$\dot{V}_1 = z^T (A_1^T Q_1 + Q_1 A_1 + \dot{Q}_1) z \leq -\nu\|z\|^2 < 0 \tag{53}$$

with positive ν defined in (25). Since there are only two subsystems and the switching sequence is minimal, for any four consecutive switching times, $t_{k-1},\ t_k,\ t_{k+1},\ t_{k+2}$,

$$\begin{aligned} z(t_{k+2}) &= e^{A_2(t_{k+2}, t_{k+1})} z(t_{k+1}) \\ &= e^{A_2(t_{k+2}, t_{k+1})} \Phi_1(t_{k+1}, t_k) z(t_k) \end{aligned} \tag{54}$$

where it is assumed that subsystem 1 is switched on at t_k and t_{k+2}, and subsystem 2 is switched on at t_{k-1} and t_{k+1}.

From (20) and (46), we have

$$\begin{aligned} \|z(t_{k+2})\| &\leq \|e^{A_2(t_{k+2}, t_{k+1})}\| \cdot \|\Phi_1(t_{k+1}, t_k)\| \cdot \|z(t_k)\| \\ &\leq a_3 a_1 e^{-\lambda_1 \tau_k} \|z(t_k)\| \end{aligned} \tag{55}$$

where $\tau_k = t_{k+1} - t_k$ is the activation time (or called *dwell time* in many literature) of subsystem 1 between the two consecutive switchings.

Since the PI subsystem (subsystem 1) switches to the proportional subsystem (subsystem 2) when the time-varying integral gain $K_i(t)$ decays to the lower bound K_i^l, the minimum dwell time for the PI subsystem can be obtained as,

$$\tau_{1m} = \frac{1}{K_d} \ln \frac{K_i^*}{K_i^l} \tag{56}$$

Considering assumption 3 of Theorem 1, which can also be written as

$$a_1^2 a_3 e^{-\lambda_1 \tau_{1m}} \leq 1 \tag{57}$$

where $a_1,\ a_3,\ \lambda_1$ are derived in previous sections, inequality (55) is equivalent to

$$a_1 \|z(t_{k+2})\| \leq a_1^2 a_3 e^{-\lambda_1 \tau_{1m}} \|z(t_k)\| \leq \|z(t_k)\| \tag{58}$$

or

$$\frac{1}{2\mu_1}\|z(t_{k+2})\|^2 \leq \frac{1}{2\alpha_1}\|z(t_k)\|^2 \tag{59}$$

From the upper and lower bounds on Q_1 derived in (23) and (24), the bounds for the Lyapunov function $V_1(z(t)) = z^T(t)Q_1 z(t)$ are

$$\frac{1}{2\alpha_1}\|z\|^2 \le V_1(z(t)) \le \frac{1}{2\mu_1}\|z\|^2 \tag{60}$$

The last two inequalities yield

$$V_1(z(t_{k+2})) \le \frac{1}{2\mu_1}\|z(t_{k+2})\|^2 \le \frac{1}{2\alpha_1}\|z(t_k)\|^2 \le V_1(z(t_k)) \tag{61}$$

which means that at t_{k+2}, V_1 is no greater than its value last time when subsystem 2 is switched on at t_k. Thus condition (1.b) of Theorem 1 in [11] is verified.

At t_{k+1},

$$\|z(t_{k+1})\|^2 = \|x_o(t_{k+1})\|^2 + \|x_p(t_{k+1})\|^2 \ge \|x_p(t_{k+1})\|^2 \tag{62}$$

During interval $[t_{k+1}, t_{k+2}]$, subsystem 2 is active, so for all $t \in [t_{k+1}, t_{k+2}]$

$$\|x_o(t)\| = \|x_o(t_{k+1})\|$$
$$\|x_p(t)\| \le a_2 e^{-\mu_2(t-t_{k+1})}\|x_p(t_{k+1})\|$$

and

$$\begin{aligned} V_2(x_p(t)) &\le \|Q_2\|\|x_p(t)\|^2 \\ &\le \frac{a_2^2 e^{-2\mu_2(t-t_{k+1})}}{2\mu_2}\|x_p(t_{k+1})\|^2 \\ &\le \frac{\alpha_2}{2\mu_2^2}\|x_p(t_{k+1})\|^2 \end{aligned} \tag{63}$$

From (62) and (63), we have

$$\begin{aligned} V_1(z(t_{k+1})) &\ge \frac{1}{2\alpha_1}\|z(t_{k+1})\|^2 \\ &\ge \frac{1}{2\alpha_1}\|x_p(t_{k+1})\|^2 \\ &\ge \frac{\mu_2^2}{\alpha_1\alpha_2}V_2(x_p(t)) \end{aligned} \tag{64}$$

Since during $[t_k, t_{k+1}]$, $\dot{V}_1 < 0$, we have $V_1(z(t_k)) \ge V_1(z(t_{k+1}))$, which gives

$$\frac{\alpha_1\alpha_2}{\mu_2^2}V_1(z(t_k)) \ge V_2(x_p(t)) \tag{65}$$

for all $t \in [t_{k+1}, t_{k+2}]$.

All conditions of Theorem 1 in [11] are thus satisfied. Therefore, the switched system (11) is stable in the sense of Lyapunov, which proves Theorem 1.

IV. Conclusions

A proportional with intermittent integral (PII) controller is briefly introduced. Since the integral action is intermittently engaged with certain rules, a feedback system with the PII controller can be modeled as a switched system with two constituent subsystems. An LTI subsystem and a slowly time-vary subsystem's stabilities are both analyzed to construct corresponding candidate Lyapunov-like functions. Using the multiple Lyapunov functions approach, the overall PII control system is proved to be stable in the sense of Lyapunov.

This PII control approach is capable of canceling constant disturbances and attenuating unpredictable disturbances. But for systems subject to narrow-band disturbances and unpredictable disturbances, this approach has its limitations. In [2], the concept of integral control in PII control was extended to deal with any predictable disturbances using Internal Model Principle (IMP) controller. This control strategy is known as Intermittent Cancellation Control (ICC). However, ICC is restricted to plants that have infrequent changes in set point or disturbance characteristics, and the period of the predictable disturbance is required *a priori*. As of our future work, we will extend the IMP controller in the ICC approach to an adaptive IMP control algorithm, developed by Brown and Zhang [14]. The adaptive IMP feedback loop is invoked to identify and cancel the predictable disturbances. Once the predictable disturbances are canceled, the feedback loop is opened creating an open-loop controller. The main challenge will be finding a switching sequence to stabilize the switched system.

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