On Convergence of Fuzzy Integrals over Complete Residuated Lattices

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Abstract

Recently we proposed a new type of fuzzy integrals defined over complete residuated lattices. These integrals are intended for the modeling of type $\langle 1, 1\rangle$ fuzzy quantifiers. An interesting theoretical question is, how to introduce various notions of convergence of this type of fuzzy integrals. In this contribution, we would like to present some results on strong and pointwise convergence of these fuzzy integrals, where the operation of the biresiduum is used to establish the measurement how close two elements of a residuated lattice are.

Keywords: Fuzzy integral, fuzzy quantifier, convergence, fuzzy measure

1. Introduction

In fuzzy integral theory, there are many results on various types of their convergence, e.g., variants of the Lebesgue's dominated convergence theorem or Fatou's lemma. Perhaps all results use some type of the continuity of fuzzy measure, moreover, the convergence is usually studied in some subset of real numbers which have many nice properties.

We recently introduced ([1], see also [2, 3]) a new type of fuzzy integral defined on fuzzy measure spaces over complete residuated lattices. We were motivated by our investigation in the field of fuzzy quantifiers [4]. When we studied various properties of these fuzzy integrals, we started to be interested whether we can state and prove results about their convergence in parallel to results proved for other types of fuzzy integrals.

However, in our case, when fuzzy measures are noncontinuous and a complete residuated lattice need not to be, for example, dense[1] or linearly ordered, in general, there is a problem how to propose some analogous theorems to the standard ones. If we concede the presumptions of continuity and null-additivity of fuzzy measures, density of complete residuated lattices (especially, of MV-algebras), measurability of mappings $f : M \to L$ to be integrated, then, for example, the Lebesgue's dominated convergence theorem or Fatou's lemma can be proved (using Theorem 4.5 in [1]). Note that the proof for our integral defined using minimum operation (i.e., $\otimes = \wedge$ in formula (8)) can be done analogously to the proof of Theorem 7.5 in [5]. The proof for general operation $\otimes$ turned out to be much more complicated, since the idempotency of the operation $\otimes$ cannot be used.[2] We will show it in this paper. A similar question has been investigated in [6], where $\otimes$ is a generalized t-norm defined on $[0, \infty]$.

In this paper, we first, after necessary preliminaries, in Section 4.1 define the notion of convergence of a sequence in a complete residuated lattice using the operation of biresiduum, which measures the closeness of two elements of the lattice. Then in Section 4.2 we define a global convergence of mappings and show that if a sequence of mappings f_n converges globally to f, then also their integrals converge to the integral of f.

Finally, because global convergence of mappings is quite strong, we will look in Section 4.3 at sequences of mappings which converge pointwise. We are able to prove convergence theorems in this case only when we suppose additional conditions on underlying complete residuated lattices, namely, we suppose complete dense linearly ordered MV-algebras. Moreover, we need fuzzy measures to be continuous. Under these conditions, we can prove convergence theorems for pointwise convergent sequences of mappings, too.

2. Preliminaries

2.1. Structures of truth values

In this paper, we suppose that the structure of truth values is a *complete linearly ordered residuated lattice*, i.e., an algebra $\mathbf{L} = \langle L, \wedge, \vee, \to, \otimes, \bot, \top\rangle$ with four binary operations and two constants such that $\langle L, \wedge, \vee, \bot, \top\rangle$ is a complete linearly ordered lattice, where $\bot$ is the least element and $\top$ is the greatest element of L, respectively, $\langle L, \otimes, \top\rangle$ is a commutative monoid (i.e., $\otimes$ is associative, commutative and the identity $a \otimes \top = a$ holds for any $a \in L$) and the adjointness property is satisfied, i.e.,

$$a \le b \to c \quad \text{iff} \quad a \otimes b \le c \tag{1}$$

[1] We say that a lattice is dense if for any $a < b$ in L there is $c \in L$ with $a < c < b$.

[2] It means that $a \wedge b = b$ for $a \ge b$, but $a \otimes b \le b$ for $a \ge b$ in general.

holds for each $a, b, c \in L$, where $\leq$ denotes the corresponding lattice ordering. A residuated lattice is *divisible*, if $a \otimes (a \to b) = a \wedge b$ holds for arbitrary $a, b \in L$, and satisfies the *law of double negation*, if $(a \to \bot) \to \bot = a$ holds for any $a \in L$. A divisible residuated lattice satisfying the law of double negation is called an *MV-algebra*. For other information about residuated lattices we refer to [7, 8].

Example 2.1. It is easy to prove (see e.g. [9]) that the algebra

$$\mathbf{L}_{\mathrm{T}} = \langle [0,1], \min, \max, T, \to_T, 0, 1 \rangle,$$

where T is a left continuous t-norm [10] and $a \to_T b = \bigvee\{c \in [0,1] \mid T(a,c) \leq b\}$, defines the residuum, is a complete residuated lattice. In this paper, we will refer to the complete residuated lattice determined by the Łukasiewicz t-norm , i.e.,

$$T_{Ł}(a,b) = \max(a + b - 1, 0).$$

Its residuum is as follows:

$$a \to_{Ł} b = \min(1, 1 - a + b).$$

This complete residuated lattice will be denoted by $\mathbf{L}_{Ł}$. Note that $\mathbf{L}_{Ł}$ is a complete MV-algebra called an *Łukasiewicz algebra* (*on* $[0,1]$), where, for example, the distributivity of $\otimes$ over $\wedge$ is satisfied.[3]

Let us define two additional operations for all $a, b \in L$:

$$a \leftrightarrow b = (a \to b) \wedge (b \to a) \quad (2)$$

$$\neg a = a \to \bot \quad (3)$$

which are called the biresiduum and the negation, respectively.

2.2. Fuzzy sets

Let $\mathbf{L} = \langle L, \wedge, \vee, \to, \otimes, \bot, \top \rangle$ be a complete residuated lattice and M be a universe of discourse (possibly empty). A mapping $A : M \to L$ is called a *fuzzy set on* M.[4] A value $A(m)$ is called a *membership degree of* m *in the fuzzy set* A. The set of all fuzzy sets on M is denoted by $\mathcal{F}(M)$. Obviously, if $M = \emptyset$, then the empty mapping $\emptyset$ is the unique fuzzy set on $\emptyset$ and thus $\mathcal{F}(\emptyset) = \{\emptyset\}$. A fuzzy set A on M is called *crisp*, if there is a subset Z of M such that $A = 1_Z$, where 1_Z denotes the characteristic function of Z.[5] Particularly, $1_\emptyset$ denotes the empty fuzzy set on M, i.e., $1_\emptyset(m) = \bot$ for any $m \in M$. The set of all crisp fuzzy sets on M is denoted by $\mathcal{P}(M)$. A fuzzy set A is *constant*, if there is $c \in L$ such that $A(m) = c$ for any $m \in M$. For simplicity, a constant fuzzy set is denoted by the corresponding element of L, e.g., a, b, c.[6]

Let us denote $\mathrm{Supp}(A) = \{m \mid m \in M \ \&\ A(m) > \bot\}$ and $\mathrm{Core}(A) = \{m \mid m \in M \ \&\ A(m) = \top\}$ the *support* and the *core* of a fuzzy set A, respectively. Obviously, Supp and Core are natural mappings from $\mathcal{F}(M)$ to $\mathcal{P}(M)$ with $\mathrm{Supp}(1_Z) = \mathrm{Core}(1_Z) = Z$ for any crisp fuzzy set. A fuzzy set A is called *normal*, if $\mathrm{Core}(A) \neq \emptyset$. Let $A \in \mathcal{F}(M)$ and Z be a subset of M. Then $A \upharpoonright Z : M \to L$ denotes the common restriction of $A : M \to L$ to the set Z. Let $A, B \in \mathcal{F}(M)$. We say that a fuzzy set A is a *fuzzy subset of* a fuzzy set B and denote it by $A \subseteq B$, if $A(m) \leq B(m)$ for any $m \in M$. The set of all fuzzy subsets of A on M is denoted by $\mathcal{F}(A)$. Let $\{A_i \mid i \in I\}$ be a non-empty family of fuzzy sets on M. Then the *union of* A_i is defined by

$$\left(\bigcup_{i \in I} A_i\right)(m) = \bigvee_{i \in I} A_i(m) \quad (4)$$

for any $m \in M$ and the *intersection of* A_i is defined by

$$\left(\bigcap_{i \in I} A_i\right)(m) = \bigwedge_{i \in I} A_i(m) \quad (5)$$

for any $m \in M$. Let A, B be fuzzy sets on M. The *difference* of A and B is a fuzzy set $A \setminus B$ on M defined by

$$(A \setminus B)(m) = A(m) \otimes (B(m) \to \bot) \quad (6)$$

for any $m \in M$ and the *complement* of A is a fuzzy set $\overline{A} = 1_M \setminus A$.

3. Fuzzy measures and integrals

Let us consider algebras of fuzzy sets as a base for defining fuzzy measures of fuzzy sets. Contrary to the classical definition of algebra of sets or fuzzy sets (see e.g. [11, 12, 5, 13, 14, 15]) , we consider a σ-algebra of fuzzy sets that are subsets of a given fuzzy set.

Definition 3.1. Let A be a non-empty fuzzy set on M. A subset $\mathcal{F}$ of $\mathcal{F}(A)$ is a *σ-algebra of fuzzy sets on* A, if the following conditions are satisfied

1. $1_\emptyset, A \in \mathcal{F}$,
2. if $X \in \mathcal{F}$, then $A \setminus X \in \mathcal{F}$,
3. if $X_i \in \mathcal{F}$, $i = 1, 2, \ldots$, then $\bigcup_{i=1}^{n} X_i \in \mathcal{F}$.

A pair $(A, \mathcal{F})$ is called a *fuzzy measurable space* (on A), if $\mathcal{F}$ is a σ-algebra of fuzzy sets on A and we say that X is *$\mathcal{F}$-measurable*, if $X \in \mathcal{F}$.

[3] That means $\bigwedge_{i \in I}(a \otimes b_i) = a \otimes \bigwedge_{i \in I} b_i$ holds.

[4] In many papers (see e.g. [7]), a mapping $A : M \to L$ is called **L**-fuzzy set or **L**-fuzzy subset on M. Since we will always deal with a fixed complete residuated lattice in the following text, we suppose that the denotation "fuzzy set" without a reference to the considered residuated lattice is sufficient.

[5] Sometimes, if it would help to the better readability of text, we will write X, Y, Z instead of $1_X, 1_Y, 1_Z$.

[6] We suppose that the meaning of this symbol will be unmistakable from the context, that is, it should be clear when an element of L is considered and when a constant fuzzy set is assumed.

Theorem 3.1. *Let* $\mathbf{L}$ *be a complete MV-algebra and* $\mathcal{F}$ *be a* σ*-algebra of fuzzy sets on* A*. Then* $\mathcal{F}$ *is closed under countable intersection, i.e., for any* $Y_i \in \mathcal{F}$*,* $i = 1, 2, \dots$*, we have*

$$\bigcap_{i=1}^{\infty} Y_i \in \mathcal{F}. \quad (7)$$

Proof. We may write

$$(\bigcup_{i=1}^{\infty}(A \setminus Y_n)(m) = \bigvee_{i=1}^{\infty}(A(m) \otimes (Y_i(m) \to \bot)) =$$

$$A(m) \otimes \bigvee_{i=1}^{\infty}(Y_i(m) \to \bot) =$$

$$A(m) \otimes (\bigwedge_{i=1}^{\infty} Y_i(m) \to \bot) = (A \setminus \bigcap_{i=1}^{\infty} Y_n)(m).$$

As a simple consequence of the double negation holding in MV-algebras, one can prove that $A \setminus (A \setminus \bigcap_{i=1}^{\infty} Y_i) = \bigcap_{i=1}^{\infty} Y_i$ and, hence, $\bigcap_{i=1}^{\infty} Y_i \in \mathcal{F}$. □

Let $f : M \to L$ be a mapping (fuzzy set). We define

$$F_a = \{m \mid m \in M \text{ and } f(m) \geq a\}$$
$$F_{a^+} = \{m \mid m \in M \text{ and } f(m) > a\}$$

for any $a \in L$.

The concept $\mathcal{F}$-measurability of a mapping f is defined standardly as follows (cf. [5]).

Definition 3.2. Let $(A, \mathcal{F})$ be a fuzzy measurable space and $X \in \mathcal{F}$. We say that a mapping $f : M \to L$ is $\mathcal{F}$-measurable on X, if $X \upharpoonright F_a \in \mathcal{F}$ for any $a \in L$.

The following lemma will be used for proving convergence theorems.

Lemma 3.2. *Let* $\mathbf{L}$ *be a complete dense MV-algebra and* $X \in \mathcal{F}$*. A mapping* $f : M \to L$ *is* $\mathcal{F}$*-measurable on* X *if and only if* $X \upharpoonright F_{a^+} \in \mathcal{F}$ *for any* $a \in L \setminus \{\top\}$*.*

Proof. Let f be $\mathcal{F}$-measurable, $a \in L \setminus \{\top\}$ and $a_1 > a_2 > \cdots$ be such that $\bigwedge_{n=1}^{\infty} a_n = a$. Clearly, $F_{a_n} \subseteq F_{a^+}$ for any $n = 1, 2, \dots$ and thus $\bigcup_{n=1}^{\infty} F_{a_n} \subseteq F_{a^+}$. If $f(m) > a$, then there exists n_0 such that $f(m) \geq a_{n_0} > a$ which implies $\bigcup_{n=1}^{\infty} F_{a_n} \supseteq F_{a^+}$ and thus $\bigcup_{n=1}^{\infty} F_{a_n} = F_{a^+}$. Since $X \upharpoonright F_{a_n} \in \mathcal{F}$ for any $n = 1, 2, \dots$, then also $\bigcup_{n=1}^{\infty} X \upharpoonright F_{a_n} = X \upharpoonright \bigcup_{n=1}^{\infty} F_{a_n} = X \upharpoonright F_{a^+} \in \mathcal{F}$.

Let $X \upharpoonright F_{a^+} \in \mathcal{F}$ for any $a \in L \setminus \{\top\}$ and $b \in L$ be an arbitrary element. If $b = 0$, then $X \upharpoonright F_0 = X \in \mathcal{F}$. Let us suppose that $b > 0$ and consider $b_1 < b_2 < \cdots$ such that $\bigvee_{n=1}^{\infty} b_n = b$. Clearly, $F_{b_n^+} \supseteq F_b$ and thus $\bigcap_{n=1}^{\infty} F_{b_n^+} \supseteq F_b$. If $m \in \bigcap_{n=1}^{\infty} F_{b_n^+}$, then $f(m) \geq b_n$ for any $n = 1, 2, \dots$ which implies $f(m) \geq \bigvee_{n=1}^{\infty} b_n = b$. Hence, $m \in F_b$ and $\bigcap_{n=1}^{\infty} F_{b_n^+} \subseteq F_b$ and $\bigcap_{n=1}^{\infty} F_{b_n^+} = F_b$. Since $X \upharpoonright F_{b_n^+} \in \mathcal{F}$ for any $n = 1, 2, \dots$, then, according to Theorem 3.1, we have $\bigcap_{n=1}^{\infty} X \upharpoonright F_{b_n^+} = X \upharpoonright \bigcap_{n=1}^{\infty} F_{b_n^+} = X \upharpoonright F_b \in \mathcal{F}$. □

In [1] (see also [2]), we simply defined fuzzy measure as a non-decreasing mapping μ of $\mathcal{F}$ to L such that $\mu(1_\emptyset) = \bot$ and $\mu(A) = \top$. A triplet $(A, \mathcal{F}, \mu)$ is called a *fuzzy measure space*, if $(A, \mathcal{F})$ is a fuzzy measurable space and μ is a fuzzy measure on $(A, \mathcal{F})$. Later, when the convergence theorems of fuzzy integrals sequences will be investigated, we need to suppose the continuity of fuzzy measures.

Definition 3.3. We say that a fuzzy measure μ on $(A, \mathcal{F})$ is *continuous*, if

1. $\{Y_n\} \subseteq \mathcal{F}$, $Y_1 \subset Y_2 \subset \cdots$, $Y = \bigcup_{i=n}^{\infty} Y_n \in \mathcal{F}$, then $\lim_{n\to\infty} \mu(Y_n) = \mu(Y)$.
2. $\{Y_n\} \subseteq \mathcal{F}$, $Y_1 \supset Y_2 \supset \cdots$, $Y = \bigcap_{i=n}^{\infty} Y_n \in \mathcal{F}$ and there exists n_0 such that $\mu(Y_{n_0}) < \top$, then $\lim_{n\to\infty} \mu(Y_n) = \mu(Y)$.

Remark 3.1. One can see that $\lim_{n\to\infty} \mu(Y_n) = \bigvee_{n=1}^{\infty} \mu(Y_n)$ for item 1, and $\lim_{n\to\infty} \mu(Y_n) = \bigwedge_{n=1}^{\infty} \mu(Y_n)$ for item 2 of the previous definition. If μ satisfies 1. (2.), then we say that μ is *continuous from below* (*above*), respectively.

Fuzzy integrals defined over algebras of fuzzy sets which have the membership degrees in a complete residuated lattice have been proposed in [1].

Definition 3.4. Let $(A, \mathcal{F}, \mu)$ be a fuzzy measure space with $M = \mathrm{Dom}(A)$, $f : M \to L$ be a mapping and X be an $\mathcal{F}$-measurable fuzzy set. Then the $\otimes$-fuzzy integral of f on X is given by

$$\int_X^{\otimes} f \, d\mu = \bigvee_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \otimes \mu(Y)), \quad (8)$$

where $\mathcal{F}_X^- = \{Y \mid Y \in \mathcal{F} \text{ and } 1_\emptyset \neq Y \subseteq X\}$. If $X = A$, then we write $\int^{\otimes} f \, d\mu$.

One may see that no measurability of the mapping f is supposed in the definition, contrary to the standard definition of fuzzy integral (cf. [5],[16]). Supposing the $\mathcal{F}$-measurability of the mapping f and restricting ourselves to the complete MV-algebras as structures for the membership values of our mappings (fuzzy sets), we may state the following very important theorem characterizing the proposed fuzzy integrals. In the rest of this paper, we will write *fuzzy integral* instead of $\otimes$*-fuzzy integral*.

Theorem 3.3. *Let* $\mathbf{L}$ *be a complete MV-algebra,* $(A, \mathcal{F}, \mu)$ *be a fuzzy measure space with* $M = \mathrm{Dom}(A)$ *and* $f : M \to L$ *be* $\mathcal{F}$*-measurable on* X*. Then*

$$\int_X^{\otimes} f \, d\mu = \bigvee_{a \in L} (a \otimes \mu(X \upharpoonright F_a)). \quad (9)$$

Proof. See [1]. □

Under some presumptions on the complete MV-algebras (namely, the density), the fuzzy integrals can be described using fuzzy sets $X \upharpoonright F_{a^+}$.

Corollary 3.4. *Let* **L** *be a complete dense MV-algebra,* $(A, \mathcal{F}, \mu)$ *be a fuzzy measure space with a continuous fuzzy measure* μ *and* $M = \mathrm{Dom}(A)$*,* $f : M \to L$ *be* $\mathcal{F}$*-measurable on* X*. Then*

$$\int_X^{\otimes} f \; d\mu = \bigvee_{a \in L \setminus \{\top\}} (a \otimes \mu(X \upharpoonright F_{a^+})). \tag{10}$$

Proof. Obviously,

$$\bigvee_{a \in L \setminus \{\top\}} (a \otimes \mu(X \upharpoonright F_{a^+})) \leq \bigvee_{a \in L} (a \otimes \mu(X \upharpoonright F_a)),$$

since $F_{a^+} \subseteq F_a$ for any $a \in L \setminus \{\top\}$. Put $c = \bigvee_{a \in L}(a \otimes \mu(X \upharpoonright F_a))$ and suppose that there exists $c' \in L$ such that

$$\bigvee_{a \in L \setminus \{\top\}} (a \otimes \mu(X \upharpoonright F_{a^+})) < c' < c. \tag{11}$$

Let $a \in L$ be an element. Obviously, it is sufficient to consider $a > \bot$. Put $a_1 < a_2 < \cdots$ with $\bigvee_{n=1}^{\infty} a_n = a$. According to the presumption (11), we have

$$a_n \otimes \mu(X \upharpoonright F_{a_n^+}) < c'$$

for any $n = 1, 2, \ldots$ Since $F_a \subseteq F_{a_n^+}$ for any $n = 1, 2, \ldots$, then

$$a_n \otimes \mu(X \upharpoonright F_a) < c'$$

for any $n = 1, 2, \ldots$ and hence

$$\bigvee_{n=1}^{\infty} (a_n \otimes \mu(X \upharpoonright F_a)) = (\bigvee_{n=1}^{\infty} a_n) \otimes \mu(X \upharpoonright F_a) = a \otimes \mu(X \upharpoonright F_a) \leq c'.$$

However, this implies $\bigvee_{a \in L}(a \otimes \mu(X \upharpoonright F_a)) \leq c'$, a contradiction. □

Let f be $\mathcal{F}$-measurable and $X \in \mathcal{F}$. Put

$$\mathcal{G}_{f,X} = \{\mathrm{Supp}(X \upharpoonright F_a) \mid a \in L\}$$

and define $\nu_{f,X} : \mathcal{G}_f \to L$ (where $\mathcal{G}_f = \{\mathcal{G}_{f,X} \mid X \in \mathcal{F}\}$) by

$$\nu_{f,X}(\mathrm{Supp}(X \upharpoonright F_a)) = \mu(X \upharpoonright F_a).$$

Then (9) can be simply rewritten as

$$\int_X^{\otimes} f \; d\mu = \bigvee_{a \in L} (a \otimes \nu_{f,X}(\mathrm{Supp}(X \upharpoonright F_a))). \tag{12}$$

One can see now that the fuzzy integral can be expressed using sets (i.e. $\mathrm{Supp}(X \upharpoonright F_a)$). Unfortunately, the construction is based on the mapping $\nu_{f,X}$ which is not a fuzzy measure[7] and, moreover, it is dependent on f and X. A natural question arises, whether there exists an algebra of sets and a fuzzy measure defined on it which would ensure (12) in some generality. An answer is given in the rest of this section.

[7] In fact, $\nu_{f,X}$ may form only a part of a fuzzy measure defined on an algebra of sets as will be demonstrated in Lemma 3.5.

Let $(A, \mathcal{F}, \mu)$ be a fuzzy measurable space and $M = \mathrm{Supp}(A)$. Let $\mathcal{P}$ denote the σ-algebra of sets containing all supports of $\mathcal{F}$-measurable fuzzy sets and

$$\nu(Y) = \bigvee_{\substack{Z \in \mathcal{F} \\ \mathrm{Supp}(Z) \subseteq Y}} \mu(Z). \tag{13}$$

One can see that $(M, \mathcal{P}, \nu)$ is not, in general, a fuzzy measure space in our sense, because ν is not a continuous measure. We say that an $\mathcal{F}$-measurable fuzzy set X is *μ-dominating* in $\mathcal{F}$, if

$$\mu(Y) \leq \mu(X \upharpoonright T) \tag{14}$$

for arbitrary $T \subseteq M$ and $Y \in \mathcal{F}$ with $\mathrm{Supp}(Y) \subseteq \mathrm{Supp}(X \upharpoonright T)$, whenever $X \upharpoonright T$ is $\mathcal{F}$-measurable. It is easy to see that A and $1_{\emptyset}$ are the simplest μ-dominating fuzzy sets in $\mathcal{F}$. If $T \subseteq M$ and $A \upharpoonright T$ is $\mathcal{F}$-measurable, then $A \upharpoonright T$ is μ-dominating.

Lemma 3.5. *Let* $(A, \mathcal{F}, \mu)$ *be a fuzzy measure space with* $M = \mathrm{Supp}(A)$*,* $(M, \mathcal{P}, \nu)$ *be a fuzzy measure space defined above and* X *be* μ*-dominating in* $\mathcal{F}$*. Then*

$$\nu_{f,X}(\mathrm{Supp}(X \upharpoonright F_a)) = \nu(\mathrm{Supp}(X \upharpoonright F_a)) \tag{15}$$

for any $\mathcal{F}$*-measurable mapping* f *on* X *and* $a \in L$*.*

Supposing fuzzy measure spaces without the presumption of continuity of fuzzy measures, then a simple consequence of the previous lemma is that each fuzzy integral defined for fuzzy sets on μ-dominating $\mathcal{F}$-measurable fuzzy sets can be transformed to fuzzy integrals defined only on (crisp) sets as the following corollary shows.

Corollary 3.6. *Let* **L** *be a complete MV-algebra,* $(A, \mathcal{F}, \mu)$ *be a fuzzy measure space with* $M = \mathrm{Supp}(A)$*,* $(M, \mathcal{P}, \nu)$ *be a fuzzy measure space defined above and* X *be* μ*-dominating. Then*

$$\int_X^{\otimes} f \; d\mu = \int_{\mathrm{Supp}(X)}^{\otimes} f \; d\nu \tag{16}$$

for any $\mathcal{F}$*-measurable mapping* f *on* X*.*

4. Convergence theorems for sequences of fuzzy integrals

4.1. Convergence of sequences in complete residuated lattices

Let us start with the convergence of values of a complete residuated lattice. We replace the notion of the absolute difference between two values by their similarity, naturally represented by the biresiduum. Of course, the values close to zero used in the case of the absolute difference have to be interpreted by the values close to one and the sign of inequality has to be changed.

Definition 4.1. Let $\{a_n\} \subset L$ be a sequence of elements and $b \in L$. We say that $a_1, a_2, \dots$ *converges to* b, if for any $a \in L$, $a < \top$, there exists a natural number n_0 such that

$$a_n \leftrightarrow b > a \tag{17}$$

for any $n > n_0$.

We will write $a_n \to b$, if the sequence $a_1, a_2, \dots$ converges to b. If $a_1, a_2, \dots$ is a non-increasing (non-decreasing) sequence converging to b, then we will write $a_n \searrow b$ ($a_n \nearrow b$). Let $f_1, f_2, \dots$ be a sequence of mappings from M to L and X be a fuzzy set. We say that $f_1, f_2, \dots$ (pointwise) converges to f on X, if $f_n(m) \to f(m)$ for any $m \in \mathrm{Dom}(X)$. We will write $f_n \to f$, if the sequence $f_1, f_2, \dots$ converges to f, and also $f_n \searrow f$ ($f_n \nearrow f$), whenever $f_1, f_2, \dots$ is a non-increasing (non-decreasing) convergent sequence of mappings.

Lemma 4.1. *Let* $\mathbf{L}$ *be a complete MV-algebra and* $a_1, a_2, \dots$ *be a non-increasing (non-decreasing) sequence. Then* $a_n \searrow b$ ($a_n \nearrow b$) *if and only if* $\bigwedge_{n=1}^{\infty} a_n = b$ ($\bigvee_{n=1}^{\infty} a_n = b$).

Proof. Let $a_n \searrow b$. Then $\bigvee_{n=1}^{\infty}(a_n \leftrightarrow b) = \top$. Since $a_n \geq b$ for any n, then also $\bigvee_{n=1}^{\infty}(a_n \to b) = (\bigwedge_{n=1}^{\infty} a_n) \to b = (\bigwedge_{n=1}^{\infty} a_n) \leftrightarrow b = \top$ which implies $\bigwedge_{n=1}^{\infty} a_n = b$. If $a_1, a_2, \dots$ is a non-increasing sequence, then $\bigwedge_{n=1}^{\infty} a_n = b$, then $(\bigwedge_{n=1}^{\infty} a_n) \to b = \bigvee_{n=1}^{\infty}(a_n \to b) = \bigvee_{n=1}^{\infty}(a_n \leftrightarrow b) = \top$. Then for any $a \in L$ with $a < \top$ there exists n_0 such that $a_n \to b = a_n \leftrightarrow b > a$ for any $n > n_0$ (note that $\to$ is a binary operation which is non-increasing in the first argument).

The proof for $a_n \nearrow b$ can be done by analogy, where the equality $b \to (\bigvee_{n=1}^{\infty} a_n) = \bigvee_{n=1}^{\infty}(b \to a_n)$ and the fact that $\to$ is a binary operation which is non-decreasing in the second argument are used. ☐

4.2. Globally convergent sequences of mappings

To investigate the convergence of the proposed fuzzy integral on general complete residuated lattices, we unfortunately cannot use the standard definition based on the pointwise convergence of mappings as the following example shows.

Example 4.1. Let $N = \{1, 2, \dots\}$ be the set of all natural numbers and $f_n : N \to [0, 1]$ be defined by

$$f_n(m) = \min\left(1, \frac{1}{2^{n-m}}\right) \tag{18}$$

for any $m \in N$. It is easy to see that $f_n(m) = 1$ for any $m \geq n$ and $f_{n+1} \subset f_n$ for any n, i.e., $f_1, f_2, \dots$ is a non-increasing sequence. Let $\mathbf{L}$ be the Łukasiewicz algebra (i.e., $b \leftrightarrow c = 1 - |b - c|$) and $a < 1$ be an arbitrary element of L. One can see that, for any $m \in N$, there exists n_0 such that $f_n(m) \leftrightarrow 0 = 1 - f_n(m) > a$ for any $n > n_0$ (this follows from the fact that $\lim_{n\to\infty} f_n(m) = 0$ for any m). Thus $f_1, f_2, \dots$ pointwise converges to f, where $f(m) = 0$ for any $m \in N$. Let $(N, \mathcal{P}(N), \mu)$ be a fuzzy measure space with a non-continuous measure, where

$$\mu(Y) = \begin{cases} 0, & \text{if } Y \text{ is finite,} \\ 0.5, & \text{if } Y \text{ is infinite and } Y \neq N, \\ 1, & \text{if } Y = N. \end{cases} \tag{19}$$

Since $f(m) = 0$, then one can simply verify that $\int^{\otimes} f \, d\mu = 0$. On the other hand, for any $n \in N$, we may construct the set $F^n = \{m \mid m \geq n\}$ of values of N for which $f_n(m) = 1$. Evidently, the set F^n is infinite for any n and $F^1 = N$. One can simply prove that $\int^{\otimes} f_1 \, d\mu = 1$ and $\int^{\otimes} f_n \, d\mu = \bigwedge_{m \in F^n} f_n(m) \otimes 0.5 = 1 \otimes 0.5 = 0.5$ for any $n > 1$. Hence, we obtain that the values of fuzzy integrals converge to 0.5 and thus $\int^{\otimes} f_n \, d\mu \nrightarrow \int^{\otimes} f \, d\mu$.

Definition 4.2. Let $\{f_n\} \subset \mathcal{F}(M)$ be a sequence of mappings and $f, X \in \mathcal{F}(M)$. We say that $f_1, f_2, \dots$ *globally converges to* f on X, if for any $a \in L$, $a < \top$, there exists a natural number n_0 such that

$$f_n(m) \leftrightarrow f(m) > a \tag{20}$$

for any $m \in \mathrm{Dom}(X)$ and $n > n_0$.

Analogously, we will write $f_n \xrightarrow{\mathrm{g}} f$, when the sequence $f_1, f_2, \dots$ globally converges to f on X. Now, let us define

$$f \leftrightarrow g = \bigwedge_{m \in M} (f(m) \leftrightarrow g(m)) \tag{21}$$

for any mappings $f, g \in \mathcal{F}(M)$. Then a relation between the closeness of functions f, g and the closeness of values of their integrals can be expressed as follows.

Lemma 4.2. *Let* $(A, \mathcal{F}, \mu)$ *be a fuzzy measure space and* $f, g : \mathrm{Dom}(A) \to L$. *Then*

$$\int_X^{\otimes} f \, d\mu \leftrightarrow \int_X^{\otimes} g \, d\mu \geq f \leftrightarrow g. \tag{22}$$

Proof. We may write

$$\int_X^{\odot} f \, d\mu \leftrightarrow \int_X^{\odot} g \, d\mu =$$
$$\bigvee_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \otimes \mu(Y)) \leftrightarrow$$
$$\bigvee_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (g(m) \otimes \mu(Y)) \geq$$
$$\bigwedge_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in M} (f(m) \otimes \mu(Y)) \leftrightarrow (g(m) \otimes \mu(Y)) \geq$$
$$\bigwedge_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in M} ((f(m) \leftrightarrow g(m)) \otimes (\mu(Y) \leftrightarrow \mu(Y)))$$
$$\bigwedge_{m \in M} ((f(m) \leftrightarrow g(m)) \otimes \top = f \leftrightarrow g,$$

where we use $\bigwedge_{i\in I}(a_i \leftrightarrow b_i) \leq (\bigvee_{i\in I} a_i) \leftrightarrow (\bigvee_{i\in I} b_i)$, $\bigwedge_{i\in I}(a_i \leftrightarrow b_i) \leq (\bigwedge_{i\in I} a_i) \leftrightarrow (\bigwedge_{i\in I} b_i)$ and $(a_1 \leftrightarrow b_1) \otimes (a_2 \leftrightarrow b_2) \leq (a_1 \otimes a_2) \leftrightarrow (b_1 \otimes b_2)$ holding in each complete residuated lattice. □

Assuming the global convergence of mappings, we may state a basic type of convergence of fuzzy integrals.

Theorem 4.3. *If* $f_n \xrightarrow{g} f$, *then* $\int_X^{\otimes} f_n \; d\mu \to \int_X^{\otimes} f \; d\mu$.

Proof. Let $f_1, f_2, \dots$ be a sequence with $f_n \to f$ and $a \in L$ with $a < \top$. We have to distinguish two cases. First, let us suppose that there is no element $a' \in L$ for which $a < a' < \top$. Since $f_n \to f$, then there exists n_0 such that $f_n(m) \leftrightarrow f(m) > a$ for any $m \in M$ and $n > n_0$. A simple consequence of the presumption is

$$f_n(m) \leftrightarrow f(m) = \top \tag{23}$$

for any $m \in M$ and $n > n_0$. Hence, $f_n \leftrightarrow f = \top$ for any $n > n_0$ (see the definition (21)). According to Lemma 4.2, we have

$$\int_X^{\otimes} f_n \; d\mu \leftrightarrow \int_X^{\otimes} f \; d\mu \geq f_n \leftrightarrow f = \top > a$$

for any $n > n_0$. Hence, $\int_X^{\otimes} f_n \; d\mu \to \int_X^{\otimes} f \; d\mu$. Further, let us suppose that there exists $a' \in L$ such that $a < a' < \top$. Then to the value a' there is n_0 such that $f_n(m) \leftrightarrow f(m) > a'$ for any $m \in M$ and $n > n_0$. We obtain $f_n \leftrightarrow f \geq a'$ for any $n > n_0$ and, according to Lemma 4.2, we may write

$$\int_X^{\odot} f_n \; d\mu \leftrightarrow \int_X^{\otimes} f \; d\mu \geq f_n \leftrightarrow f \geq a' > a$$

for any $n > n_0$. Hence, again $\int_X^{\otimes} f_n \; d\mu \to \int_X^{\otimes} f \; d\mu$ and the proof is finished. □

4.3. Pointwise convergent sequences of mappings

To show convergence theorems for sequences of mappings that are not globally convergent, we will suppose that **L** is a complete linearly ordered MV-algebra which is, moreover, dense. Further, we will suppose that each fuzzy measure space has a continuous measure. Let us start our investigation with the following useful lemma (cf. [5]).

Lemma 4.4. *If* $f_n \searrow f$, *then* $\bigcap_{n=1}^{\infty} F_a^n = F_a$. *If* $f_n \nearrow f$, *then* $\bigcup_{n=1}^{\infty} F_{a^+}^n = F_{a^+}$.

Theorem 4.5. *Let* **L** *be a complete dense linearly ordered MV-algebra,* $X \in \mathcal{F}$ *and* $f_n \searrow f$ *on* X. *If there exists* n_0 *such that*

$$\mu(X \sqcap \{m \mid f_{n_0}(m) > \int_X^{\otimes} f \; d\mu\}) < \top, \tag{24}$$

then $\int_X^{\otimes} f_n \; d\mu \searrow \int_X^{\otimes} f \; d\mu$.

Proof. Without loss of generality, let us suppose that (24) is satisfied for $n_0 = 1$ (otherwise, put $g_1 = f_{n_0}, g_2 = f_{n_0+1}, \dots$). According to Lemma 4.1, we need to prove that $\bigwedge_{n\to\infty} \int_X^{\otimes} f_n \; d\mu = \int_X^{\otimes} f \; d\mu$. From the monotonicity of fuzzy integrals, we obtain

$$\bigwedge_{n\to\infty} \int_X^{\otimes} f_n \; d\mu \geq \int_X^{\otimes} f \; d\mu.$$

Suppose that $\bigwedge_{n\to\infty} \int_X^{\otimes} f_n \; d\mu > \int_X^{\otimes} f \; d\mu = c$. Let $c' \in L$ be such that $\bigwedge_{n=1}^{\infty} \int_X^{\otimes} f_n \; d\mu > c' > c$ (the existence of c' follows from the density of the MV-algebra). Hence, we obtain $\int_X^{\otimes} f_n \; d\mu > c'$ for any n. From the definition of fuzzy integral, for any n, there exists $a \in L$ such that $a \otimes \mu(X \sqcap F_a^n) \geq c'$. Denote

$$A^n = \{a \mid a \in L \text{ and } a \otimes \mu(X \sqcap F_a^n) \geq c'\}$$

and put $a^n = \bigwedge A^n$. From the monotonicity of the fuzzy measure μ, we have

$$\mu(X \sqcap F_{a^n}^n) \geq \mu(X \sqcap F_a^n)$$

(clearly $F_a^n \subseteq F_{a^n}^n$, then $X \sqcap F_a^n \subseteq X \sqcap F_{a^n}^n$) for any $a \in A^n$ and thus

$$a \otimes \mu(X \sqcap F_{a^n}^n) \geq a \otimes \mu(X \sqcap F_a^n) \geq c'$$

for any $a \in A^n$. Hence, we obtain

$$\bigwedge_{a\in A^n} (a \otimes \mu(X \sqcap F_{a^n}^n) = (\bigwedge_{a\in A^n} a) \otimes \mu(X \sqcap F_{a^n}^n) = a^n \otimes \mu(X \sqcap F_{a^n}^n) \geq c'$$

and $a^n \in A^n$ for any n. Let us show that $a^1, a^2, \dots$ is a non-decreasing sequence. Suppose that $a^n > a^{n+1}$ for some n. Then $F_{a^{n+1}}^n \supseteq F_{a^{n+1}}^{n+1}$ (which follows from $f_n \supseteq f_{n+1}$). From the monotonicity of μ, we have

$$\mu(X \sqcap F_{a^{n+1}}^n) \geq \mu(X \sqcap F_{a^{n+1}}^{n+1}).$$

Since $a^{n+1} \in A^{n+1}$, then

$$a^{n+1} \otimes \mu(X \sqcap F_{a^{n+1}}^n) \geq a^{n+1} \otimes \mu(X \sqcap F_{a^{n+1}}^{n+1}) \geq c'$$

and thus $a^{n+1} \in A^n$ (i.e. $a^{n+1} \geq a^n$), but this is a contradiction with the presumption $a^n > a^{n+1}$. Put

$$Y^n = \bigcap_{k=1}^{\infty} F_{a^k}^n \tag{25}$$

for each n. Let us prove that $Y^n = F_a^n$, where $a = \bigvee_{k=1}^{\infty} a^k$. Since $F_{a^k}^n \supseteq F_a^n$ for any k, then $Y^n = \bigcap_{k=1}^{\infty} F_{a^k}^n \supseteq F_a^n$. Suppose that there exists $m \in Y^n$ and $m \notin F_a^n$, i.e. $f_n(m) < a$. Since $a = \bigvee_{k=1}^{\infty} a^k$, then there exists l and $f_n(m) < a^l$ which gives $m \notin F_{a^l}^n$ and, hence, $m \notin Y^n$, a contradiction. Thus we proved that

$$F_a^n = \bigcap_{k=1}^{\infty} F_{a^k}^n, \tag{26}$$

where $a = \bigvee_{k=1}^{\infty} a^k$. According to the presumption of the theorem and our convention on $n_0 = 1$, we have

$$\mu(X \upharpoonright F^n_{a^k}) \leq \mu(X \upharpoonright \{m \mid f_{n_0}(m) > c\}) < \top \quad (27)$$

for any n and k. In fact, since

$$a^1 \otimes \mu(X \upharpoonright F^1_{a^1}) \geq c' > c,$$

then $a_1 > c$ and hence $F^1_{a^1} \subseteq \{m \mid f_{n_0}(m) > c\}$. Thus $X \upharpoonright F^n_{a^k} \subseteq X \upharpoonright \{m \mid f_{n_0}(m) > c\}$ which implies (27). Since $a^1 \leq a^2 \leq \cdots$, then

$$X \upharpoonright F^1_{a^n} \subseteq X \upharpoonright F^1_{a^1} \subseteq X \upharpoonright \{m \mid f_{n_0}(m) > c\}$$

and, analogously, one can prove $X \upharpoonright F^n_{a^k} \subseteq X \upharpoonright \{m \mid f_{n_0}(m) > c\}$ using from the previous relation and $f_n \searrow f$. Hence, (27) is satisfied for any n and k. Thus, from the continuity of μ and the fact that $F^n_{a^1} \supseteq F^n_{a^2} \supseteq \cdots$, we may write

$$\bigwedge_{k=1} \mu(X \upharpoonright F^n_{a^k}) = \mu(\bigcap_{k=1}^{\infty} X \upharpoonright F^n_{a^k}) =$$
$$\mu(X \upharpoonright \bigcap_{k=1}^{\infty} F^n_{a^k}) = \mu(X \upharpoonright F^n_a).$$

Since $a > a^k$, then

$$a \otimes \mu(X \upharpoonright F^n_{a^k}) \geq a^k \otimes \mu(X \upharpoonright F^n_{a^k}) \geq c'$$

and, hence,

$$\bigwedge_{k=1}^{\infty} (a \otimes \mu(X \upharpoonright F^n_{a^k})) = a \otimes \bigwedge_{k=1}^{\infty} \mu(X \upharpoonright F^n_{a^k}) =$$
$$a \otimes \mu(X \upharpoonright F^n_a) > c'.$$

From the continuity of μ, Lemma 4.4 and the fact that $F^1_a \supseteq F^2_a \supseteq \cdots$, we obtain

$$\bigwedge_{n=1}^{\infty} (a \otimes \mu(X \upharpoonright F^n_a)) = a \otimes \bigwedge_{n=1}^{\infty} \mu(X \upharpoonright F^n_a) =$$
$$a \otimes \mu(\bigcap_{n=1}^{\infty} X \upharpoonright F^n_a) = a \otimes \mu(X \upharpoonright \bigcap_{n=1}^{\infty} F^n_a) =$$
$$a \otimes \mu(X \upharpoonright F_a) \geq c',$$

but this is a contradiction with $c' > \int_X^{\otimes} f \, d\mu$. □

The following example is a modification of Example 7.5 in [5] demonstrating the importance of the presumption (27) in the previous theorem. Without this presumption, the conclusion of that theorem does not hold in general.

Example 4.2. Let $t : [0, +\infty] \to [0, 1]$ be a transformation defined by

$$t(x) = \begin{cases} \frac{2}{\pi} \arctan x, & x < +\infty; \\ 1, & \text{otherwise.} \end{cases} \quad (28)$$

Let $\mathbf{L}_Ł$ be the Łukasiewicz algebra, $M = [0, +\infty]$, $\mathcal{F}$ be the class of all Borel sets that are in M (namely, $\mathcal{F} = \mathcal{B} \cap M$) and $\nu = t \circ \mu$, where μ is the Lebesque measure.[8] Take $f_n(m) = t(\frac{m}{n})$ for any $m \in M$ and $n = 1, 2, \ldots$, then $f_n \searrow f$, where $f(m) = 0$ for any $m \in M$. Obviously, $\int^{\otimes} f \, d\nu = 0$. The sequence of measurable mappings $f_1, f_2, \ldots$ does not satisfy (27). In fact, we have

$$\nu(\{m \mid f_n(m) > 0\}) = t \circ \mu(]0, +\infty]) = 1$$

for any $n = 1, 2, \ldots$ For arbitrary n and $a \in [0, 1]$, there exists m_0 such that $f(m) \geq \alpha$ holds for $m > m_0$ and hence $\nu(F_a) \geq \nu([m_0, +\infty]) = 1$. According to Theorem 3.3, we obtain

$$\int^{\otimes} f_n \, d\nu = \bigvee_{a \in [0,1]} a \otimes \nu(F_a) = \bigvee_{a \in [0,1]} a = 1.$$

Hence, $\bigwedge_{n=1}^{\infty} \int^{\otimes} f_n \, d\nu \neq \int^{\otimes} f \, d\nu$.

Theorem 4.6. *Let* $\mathbf{L}$ *be a complete dense linearly ordered MV-algebra,* $X \in \mathcal{F}$ *and* $f_n \nearrow f$ *on* X. *Then* $\int_X^{\otimes} f_n \, d\mu \nearrow \int_X^{\otimes} f \, d\mu$.

Proof. This proof is analogous to the proof of Theorem 4.5. □

Lemma 4.7. *Let* $\mathbf{L}$ *be a complete dense linearly ordered MV-algebra and* $X \in \mathcal{F}$. *Let* $f_1, f_2, \ldots$ *be a sequence of* $\mathcal{F}$*-measurable mappings on* X *and* $g(m) = \bigwedge_{n=1}^{\infty} f_n(m)$ *and* $h(m) = \bigvee_{n=1}^{\infty} f_n(m)$ *for any* $m \in M$. *Then* g *and* h *are* $\mathcal{F}$*-measurable on* X.

Theorem 4.8. *Let* $\mathbf{L}$ *be a complete dense linearly ordered MV-algebra,* $X \in \mathcal{F}$ *and* $f_n \to f$ *on* X. *If there exists* n_0 *satisfying the condition (24), then* $\int_X^{\otimes} f_n \, d\mu \to \int_X^{\otimes} f \, d\mu$.

Proof. Put $g_n(m) = \bigwedge_{k \geq n} f_k(m)$ and $h_n(m) = \bigvee_{k \geq n} f_k(m)$ for any $m \in M$ and $n = 1, 2, \ldots$. Notice that $g_n \leq g_{n+1}$ and $h_{n+1} \leq h_n$. According to Lemma 4.7, g_n and h_n are $\mathcal{F}$-measurable on X for any $n = 1, 2, \ldots$ and, moreover, $g_n(m) \leq f_n(m) \leq h_n(m)$ for any $m \in M$ and $n = 1, 2, \ldots$ First, we will show that $g_n \nearrow f$. Let $c \in L$, $c < \top$, and $m \in M$. From the density of lattice, consider $c' \in L$ such that $c < c' < \top$. Since $f_n \to f$, then for c' there exists n_0 such that

$$f_n(m) \leftrightarrow f(m) > c' > c$$

for any $n > n_0$. Then we have

$$g_n(m) \leftrightarrow f(m) = (\bigwedge_{k \geq n} f_k(m)) \leftrightarrow (\bigwedge_{k \geq n} f(m)) \geq$$
$$\bigwedge_{k \geq n} (f_k(m) \leftrightarrow f(m)) \geq c' > c$$

[8]For simplicity, we will write $\nu(X)$ instead of $\nu(1_X)$ for $X \in \mathcal{F}$.

for any $n > n_0$ and $g_n \nearrow f$. Analogously, one can prove that $h_n \searrow f$. Now, from the monotonicity of the fuzzy integrals, we may write

$$\int_X^{\otimes} g_n\ d\mu \leq \int_X^{\otimes} f_n\ d\mu \leq \int_X^{\otimes} h_n\ d\mu$$

for any $n = 1, 2, \ldots$. Putting $c = \int_X^{\otimes} f\ d\mu$, then, according to the presumption of the theorem, there exists n_0 such that

$$\mu(X \upharpoonright \{m \mid f_{n_0} > c\}) < \top$$

and $g_{n_0}(m) = \bigwedge_{k \geq n_0} f_k(m) \leq f_{n_0}(m)$ for any $m \in M$. Then

$$\{m \mid g_{n_0} > c\} \subseteq \{m \mid f_{n_0} > c\}$$

and from the monotonicity of μ we obtain

$$\mu(X \upharpoonright \{m \mid g_{n_0} > c\}) < \top.$$

Let $a \in L$, $a < \top$. From Theorems 4.5 and 4.6, there exists n_0, m_0 such that for any $n \geq \max(n_0, m_0)$ we have

$$a < \int_X^{\otimes} f\ d\mu \to \int_X^{\otimes} g_n\ d\mu$$

and also

$$a < \int_X^{\otimes} h_n\ d\mu \to \int_X^{\otimes} f\ d\mu.$$

From the monotonicity of the residuum in its arguments and $g_n \leq f_n \leq h_n$, we obtain

$$a < \int_X^{\otimes} f\ d\mu \to \int_X^{\otimes} f_n\ d\mu$$

and

$$a < \int_X^{\otimes} f_n\ d\mu \to \int_X^{\otimes} f\ d\mu$$

which implies

$$a < (\int_X^{\otimes} f\ d\mu \to \int_X^{\otimes} f_n\ d\mu) \wedge (\int_X^{\otimes} f_n\ d\mu \to \int_X^{\otimes} f\ d\mu) = (\int_X^{\otimes} f_n\ d\mu \leftrightarrow \int_X^{\otimes} f\ d\mu)$$

for any $n > \max(n_0, m_0)$. Hence, we obtain $\int_X^{\otimes} f_n\ d\mu \to \int_X^{\otimes} f\ d\mu$. □

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