L-fuzzy valued measure and integral

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Abstract

We continue to develop a construction of an L-fuzzy valued measure by extending a measure defined on a σ-algebra of crisp sets to an L-fuzzy valued measure defined on a T_M-tribe in the case when operations with L-sets and L-fuzzy numbers are defined by using the minimum triangular norm T_M. We introduce an L-fuzzy valued integral over an L-set with respect to an L-fuzzy valued measure, consider its properties and describe a method of L-fuzzy valued integration.

Keywords: L-set, L-fuzzy real number, L-fuzzy valued measure, L-fuzzy valued integral.

1. Introduction

One can find a lot of works regarding a fuzzy approach to measure and integral. The most important concepts and results concerning this topic are considered in [1], [2], [3]. Our interest is in developing a theory where not only sets are fuzzy, but also measure and integral take fuzzy real values. In the previous papers [4], [5] we suggested the construction that allows us to obtain an L-fuzzy valued measure defined on a T_M-clan of L-sets by extension of a measure defined on a σ-algebra of crisp sets. We continue to develop the results obtained before and describe how an L-fuzzy valued measure defined on a T_M-tribe can be obtained for a given σ-algebra $\Phi \subset 2^X$ and a finite measure $\nu : \Phi \to \mathbb{R}_+$. On the next stage we introduce the concept of an L-fuzzy valued integral over a measurable L-set. Some properties of L-fuzzy valued integral are considered. We suppose that L is a complete, completely distributive lattice (see e.g. [6]) and operations with L-sets and L-fuzzy numbers are defined by using the minimum triangular norm T_M.

We give our preference to the fuzzy real numbers as they were first defined by B. Hutton [7] and then studied thoroughly in a series of papers (see e.g. [8], [9], [10]). The preference of using this approach for defining fuzzy real numbers is motivated by our intention to develop results on approximation from [11], [12]. For problems that can be solved only approximately the notion of the error of a method of approximation plays the fundamental role. In order to estimate the quality of approximation on an L-fuzzy set, we need an appropriate L-fuzzy analogue of a norm. Our intension is to use the L-fuzzy valued integral to define an L-fuzzy norm for investigation of the error of approximation on an L-set.

2. Preliminaries

2.1. L-sets

Given a (crisp) universe X and a complete, completely distributive lattice $L(\wedge, \vee, 0_L, 1_L)$, an *L-subset* A of X (or, briefly, an L-set A) is a function $A : X \longrightarrow L$. The class of all L-subsets of X is denoted L^X. The operations with L-sets A, B are defined by using the minimum triangular norm T_M, its corresponding conorm S_M and decreasing involution N:

$$(A \wedge B)(x) = T_M(A(x), B(x)),$$
$$(A \vee B)(x) = S_M(A(x), B(x)),$$
$$A^c(x) = N(A(x)).$$

A finite family of L-sets $A_1, A_2, \ldots, A_n$ is said to be T_M*-disjoint* (see e.g.[1]) iff for each $k \in \{1, \ldots, n\}$ we have $(\bigvee_{i=1, i\neq k}^{n} A_i) \wedge A_k = \emptyset$. A countable family of L-sets is said to be T_M-disjoint iff every finite subfamily of this family is T_M-disjoint.

In order to consider an L-fuzzy valued T_M-measure we consider classes of L-sets called T_M-clans and T_M-tribes (see e.g. [1]).

Definition 2.1. *A subclass $\mathcal{A} \subset L^X$ is called a T_M-clan on X if the following properties are satisfied:*

- $\emptyset \in \mathcal{A}$;
- *for all $A \in \mathcal{A}$ we have $A^c \in \mathcal{A}$;*
- *for all $A, B \in \mathcal{A}$ we have $A \wedge B \in \mathcal{A}$.*

Definition 2.2. *A subclass $\Sigma \subset L^X$ is called a T_M-tribe on X if the following properties are satisfied:*

- $\emptyset \in \Sigma$;
- *for all $A \in \Sigma$ we have $A^c \in \Sigma$;*
- *for all sequences $(A_n)_{n\in N} \subset \Sigma$ we have $\bigwedge_{n=1}^{\infty} A_n \in \Sigma$.*

2.2. L-fuzzy real numbers

For our purposes we use the L-fuzzy real numbers as they were first defined by B. Hutton [7].

Definition 2.3. *An L-fuzzy real number is a function $z : \mathbb{R} \to L$ such that*

- *z is non-increasing;*
- $\bigwedge_t z(t) = 0_L$, $\bigvee_t z(t) = 1_L$;
- *z is left semi-continuous, i.e. for all $t_0 \in \mathbb{R}$ we have $\bigwedge_{t<t_0} z(t) = z(t_0)$.*

In the original papers of this subject (see [7], [8], [9]) L-fuzzy real numbers were defined not as order reversing functions, but as equivalence classes of such functions. However each class of equivalence has a unique left semi-continuous representative and therefore an L-fuzzy real number can be identified with this representative. A deep theoretical justification of viewing fuzzy numbers as distribution function was given by U. Höhle [10], who showed that such fuzzy real numbers can be obtained from the set of rational numbers $\mathbb{Q}$ by means of Dedekind completion in the same way as real numbers R are obtained from $\mathbb{Q}$ if one applies the multiple-valued logic, instead of the binary logic which stands behind the Dedekind completion in the classic case.

The set of all L-fuzzy real numbers is called *the L-fuzzy real line* and it is denoted by $\mathbb{R}(L)$. An L-fuzzy number z is called *non-negative* if $z(0) = 1_L$. We denote by $\mathbb{R}_+(L)$ the set of all non-negative L-fuzzy real numbers.

Operations with L-fuzzy real numbers such as addition $\oplus$ and multiplication by a real positive number r are defined as following:

$$(z_1 \oplus z_2)(t) = \bigvee_\tau \{z_1(\tau) \wedge z_2(t-\tau)\},\ (zr)(t) = z(\frac{t}{r}).$$

The supremum and the infimum of a set of non-negative L-fuzzy numbers $F \subset \mathbb{R}_+(L)$ are defined by the formulas (see e.g. [11], [12]):

$$(Inf\ F)(t) = \bigwedge\{z(t) \mid z \in F\},\ t \in \mathbb{R},$$

$$Sup\ F = Inf\{z \mid z \in \mathbb{R}(L),\ z \geq z' \text{ for all } z' \in F\}.$$

Taking into account that F is bounded from below it is easy to see that $Inf\ F$ is an L-fuzzy real number. In case F is bounded from above (i.e. there exists $z_0 \in \mathbb{R}(L)$ such that $z \leq z_0$ for all $z \in F$), $Sup\ F$ is an L-fuzzy real number, otherwise the condition

$$\bigwedge_t SupF(t) = 0_L$$

does not necessarily hold.

Going forward we will need also the countable addition of non-negative fuzzy real numbers. Given a sequence of non-negative fuzzy real numbers $(z_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+(L)$ we consider the countable sum

$$\bigoplus_{n=1}^{\infty} z_n = Sup\{z_1 \oplus z_2 \oplus ... \oplus z_n \mid n \in \mathbb{N}\}.$$

For $a \in \mathbb{R}_+$ and $\alpha \in L$ by $z(a,\alpha)$ we denote a special type of non-negative L-fuzzy real numbers

$$(z(a,\alpha))(t) = \begin{cases} 1,\ t \leq 0, \\ \alpha,\ 0 < t \leq a, \\ 0,\ t > a, \end{cases}$$

that will play an important role in our work.

2.3. *L*-fuzzy valued measure

We consider a measure that is defined on a T_M-tribe and takes values in $\mathbb{R}_+(L)$.

Definition 2.4. *Let Σ be a T_M-tribe. A function*

$$\mu : \Sigma \to \mathbb{R}_+(L)$$

is called an L-fuzzy valued measure if it satisfies the following conditions:

- $\mu(\emptyset) = z(0, 1_L)$;
- *μ is T_M-valuation, i.e. for all $A, B \in \Sigma$ it holds*

$$\mu(A) \oplus \mu(B) = \mu(A \wedge B) \oplus \mu(A \vee B);$$

- *μ is left T_M-continuous, i.e.*

$$Sup\{\mu(A_n) \mid n \in \mathbb{N}\} = \mu(A),$$

for all $(A_n)_{n\in N} \subset \Sigma$ and $A = \bigvee_{n\in\mathbb{N}} A_n$.

3. Construction of *L*-fuzzy valued measure

3.1. Measurable *L*-sets

For a given σ-algebra $\Phi \subset 2^X$ and a finite measure $\nu : \Phi \to \mathbb{R}_+$ an L-fuzzy valued measure can be obtained by the following schema (see [4], [5]):

- For $M \in \Phi$, $\alpha \in L$ we define an L-fuzzy set

$$(A(M,\alpha))(x) = \begin{cases} \alpha, x \in M, \\ 0, x \notin M. \end{cases}$$

  All these L-sets form a class of L-sets that we denote by $\wp$:

$$\wp = \{A(M,\alpha) | M \in \Phi, \alpha \in L\}.$$

  Note that the following properties hold for all L-sets $A_1, A_2, ..., A_n \in \wp$:

  (i) $\bigwedge_{i=1}^{n} A_i \in \wp$;

  (ii) there exist such T_M-disjoint L-fuzzy sets $B_1, B_2, ..., B_k \in \wp$ that

$$\bigvee_{i=1}^{n} A_i = \bigvee_{i=1}^{k} B_i.$$

- Next we define an L-fuzzy valued function

$$m : \wp \to \mathbb{R}_+(L)$$

  by the formula

$$m(A(M,\alpha)) = z(\nu(M), \alpha).$$

  Obviously,

  (i) for all sets $A_i = A(M_i, \alpha) \in \wp,\ i = 1, 2:$

$$m(A_1) \oplus m(A_2) = m(A_1 \wedge A_2) \oplus m(A_1 \vee A_2);$$

  (ii) for all $(M_n)_{n\in N} \subset \Phi$ and $M = \bigcup_{n\in\mathbb{N}} M_n$ we have

$$Sup\{m(A(M_n,\alpha)) \mid n \in \mathbb{N}\} = m(A(M,\alpha));$$

(iii) for all pairwise disjoint sets $(M_n)_{n\in N} \subset \Phi$:

$$\bigoplus_{n=1}^{\infty} m(A(M_n,\alpha)) = m(A(\bigcup_{n\in\mathbb{N}} M_n,\alpha)).$$

- Now we extend m to the L-fuzzy valued function $m^* : L^X \to \mathbb{R}_+(L)$ as following:

$$m^*(E) = Inf\{\bigoplus_{n=1}^{\infty} m(E_n) \mid (E_n)_{n\in\mathbb{N}} \subset \wp : E \leq \bigvee_{n=1}^{\infty} E_n\}$$

(m^* is an L-fuzzy valued analogue of an outer measure).
Let us note that

(i) for all $E \in L^X$ there always exists such a sequence $(E_n)_{n\in\mathbb{N}} \subset \wp$ that $E \leq \bigvee_{n=1}^{\infty} E_n$;
(ii) m^* is bounded from above in the following sense:

$$m^*(E) \leq z(\nu(X), 1_L) \text{ for all } E \in L^X;$$

(iii) for all $E \in \wp$ we obtain $m^*(E) = m(E)$;
(iv) for L-sets $A, B \in L^X$ we have

$$m^*(A) \oplus m^*(B) \geq m^*(A \wedge B) \oplus m^*(A \vee B).$$

- Finally, we generalize to the fuzzy case the classical concept of m^*-measurabilty (in the sense of Caratheodory) and consider Σ - the class of all so called m^*-measurable L-sets .

Definition 3.1. *A set $E \in L^X$ is called a m^*-measurable if it satisfies the following conditions for all L-sets $B \in L^X$:*

$$m^*(E) \oplus m^*(B) = m^*(E \wedge B) \oplus m^*(E \vee B),$$

$$m^*(E^c) \oplus m^*(B) = m^*(E^c \wedge B) \oplus m^*(E^c \vee B).$$

Note that

(i) E^c is m^*-measurable for all m^*-measurable L-sets E;
(ii) all L-sets $E \in \wp$ are m^*-measurable.

3.2. *L*-fuzzy valued measure of measurable *L*-sets

We consider μ as the restriction of m^* to Σ:

$$\mu(E) = m^*(E) \text{ for all } E \in \Sigma.$$

Theorem 3.2. *μ is an L-fuzzy valued T_M-measure such that $\mu/\wp = m$.*

As it was shown in [5] all m^*-measurable L-sets form a T_M-clan. To obtain that the class Σ is a T_M-tribe, we consider a sequence $(E_n)_{n\in\mathbb{N}}$ of m^*-measurable L-sets. First we notice that

$$m^*(\bigvee_{n=1}^{\infty} E_n) = Sup\{m^*(\bigvee_{i=1}^{n} E_i) \mid n \in \mathbb{N}\}.$$

Now taking into account that $\bigvee_{i=1}^{n} E_i$ is m^*-measurable we obtain that for all L-sets $B \in L^X$ and for all $n \in \mathbb{N}$:

$$m^*(\bigvee_{i=1}^{n} E_i) \oplus m^*(B) =$$
$$m^*((\bigvee_{i=1}^{n} E_i) \wedge B) \oplus m^*((\bigvee_{i=1}^{n} E_i) \vee B).$$

This means that for all $n \in \mathbb{N}$

$$m^*(\bigvee_{i=1}^{n} E_i) \oplus m^*(B) \leq$$
$$m^*((\bigvee_{i=1}^{\infty} E_i) \wedge B) \oplus m^*((\bigvee_{i=1}^{\infty} E_i) \vee B),$$

and hence

$$Sup\{m^*(\bigvee_{i=1}^{n} E_i) \mid n \in \mathbb{N}\} \oplus m^*(B) \leq$$
$$m^*((\bigvee_{i=1}^{\infty} E_i) \wedge B) \oplus m^*((\bigvee_{i=1}^{\infty} E_i) \vee B).$$

Finally we obtain

$$m^*(\bigvee_{n=1}^{\infty} E_n) \oplus m^*(B) =$$
$$m^*((\bigvee_{n=1}^{\infty} E_n) \wedge B) \oplus m^*((\bigvee_{n=1}^{\infty} E_n) \vee B).$$

By analogy the result can be proved for $\bigwedge_{n=1}^{\infty} E_n$.

Thus by extension of a crisp measure ν we obtain L-fuzzy valued measure

$$\mu : \Sigma \to \mathbb{R}_+(L)$$

such that

(i) $\mu/\wp = m$;
(ii) $\mu/\Phi = \nu$.

The last equality means that for every $M \in \Phi$ it holds

$$\mu(A(M, 1_L)) = z(\nu(M), 1_L).$$

4. *L*-fuzzy valued integral

4.1. Definition of *L*-fuzzy valued integral

Our aim is to define an L-fuzzy valued integral

$$\int_E f \, d\mu,$$

where $E \in \Sigma$ and $f : X \to \mathbb{R}$ is a non-negative measurable function with respect to σ-algebra Φ.

By analogy with the classical case (see e.g. [13]) we define an L-fuzzy valued integral stepwise, first considering the case of simple non-negative measurable functions (for short SNMF):

$$\int_E (\sum_{i=1}^{n} c_i \chi_{C_i}) \, d\mu = \bigoplus_{i=1}^{n} (c_i \, \mu(C_i \wedge E)),$$

whenever

- $c_i \in \mathbb{R}_+, C_i \in \Phi$ for all $i = 1, ..., n$,
- χ_{C_i} is the characteristic function of C_i , $i = 1, ..., n$,

- $C_1, ..., C_n$ are pairwise disjoint sets.

Then considering the case of non-negative measurable functions f (for short NMF):

$$\int_E f\, d\mu = Sup\{\int_E g\, d\mu \mid g \leq f \text{ and } g \text{ is SNMF}\}.$$

For $\mathbb{I}_f = \int_E f\, d\mu$ due to properties of the supremum of a set of L-fuzzy numbers, we have

- $\mathbb{I}_f$ is non-increasing,
- $\bigvee_t \mathbb{I}_f(t) = 1_L$,
- $\mathbb{I}_f$ is left semi-continuous, i.e. $\bigwedge_{t<t_0} \mathbb{I}_f(t) = \mathbb{I}_f(t_0)$.

Definition 4.1. *We say that a non-negative measurable function f is L-fuzzy integrable iff*

$$\bigwedge_t \mathbb{I}_f(t) = 0_L.$$

4.2. Properties of L-fuzzy valued integral

For L-fuzzy integrable non-negative functions $f, f_1, f_2, ..., f_n, ...$ and measurable L-sets $E, E_1, E_2, ..., E_n, ... \in \Sigma$ the following properties of L-fuzzy valued integral are true.

($\mathbb{I}$1) $\int_E d\mu = \mu(E)$

($\mathbb{I}$2) $r \in \mathbb{R}_+ \Rightarrow \int_E rf d\mu = r\int_E f d\mu$

($\mathbb{I}$3) $f_1 \leq f_2 \Rightarrow \int_E f_1 d\mu \leq \int_E f_2 d\mu$

($\mathbb{I}$4) $E_1 \leq E_2 \Rightarrow \int_{E_1} f\, d\mu \leq \int_{E_2} f\, d\mu$

($\mathbb{I}$5) $\int_E (f_1 + f_2) d\mu = \int_E f_1 d\mu \oplus \int_E f_2 d\mu$

($\mathbb{I}$6) $E_1 \wedge E_2 = \emptyset \Rightarrow \int_{E_1 \vee E_2} f d\mu = \int_{E_1} f d\mu \oplus \int_{E_2} f d\mu$

($\mathbb{I}$7) $(E_n)_{n\in\mathbb{N}} : E_n \leq E_{n+1}$ and $\bigvee_{n\in\mathbb{N}} E_n = E \Rightarrow$
$\int_E f d\mu = Sup\{\int_{E_n} f d\mu \mid n \in \mathbb{N}\}$

($\mathbb{I}$8) $(f_n)_{n\in\mathbb{N}} : f_n \leq f_{n+1}$ and $\lim_{n\to\infty} f_n = f \Rightarrow$
$\int_E f\, d\mu = Sup\{\int_E f_n\, d\mu \mid n \in \mathbb{N}\}$

5. Integration over a measurable fuzzy set

In this section we suggest a method of calculation of the fuzzy valued integral over a measurable fuzzy set E in the case when $L = [0,1]$ and E is NMF (i.e. E is measurable with respect to σ-algebra Φ).

The main idea of the method is based on the following reasoning. The fuzzy set we want to integrate over can be viewed as a non-negative function. Let us assume that this function is measurable with respect to σ-algebra Φ. It is known that every non-negative measurable function can be presented as a limit of a non-decreasing sequence of SNMF. Obviously, every fuzzy set that is SNMF can be presented as the union of T_M-disjoint fuzzy sets from the class $\wp$. And the L-fuzzy valued integral over an element from the class $\wp$ can be easily calculated.

This observation gives a reason for the following theorem.

Theorem 5.1. *If $E : X \to [0,1]$ is a measurable function with respect to σ-algebra Φ, then fuzzy set E is measurable with respect to T_M-tribe Σ.*

We describe the method gradually depending on the type of a fuzzy set E: first considering the case when E is an element of the class $\wp$, then extend it to the case when E is SNMF or a finite union of elements from the class $\wp$ and, finally, the case when E is NMF.

5.1. Integration over $A(M,\alpha)$

To show that for all $A(M,\alpha) \in \wp$ it holds

$$\int_{A(M,\alpha)} f d\mu = z(\int_M f\, d\nu, \alpha),$$

we use some special properties of the addition of fuzzy numbers $z(a,\alpha)$ described in subsection 2.2.

- $a_1, a_2 \in \mathbb{R}_+ \Rightarrow z(a_1,\alpha) \oplus z(a_2,\alpha) = z(a_1 + a_2, \alpha)$;
- $c \in \mathbb{R}_+ \Rightarrow cz(a,\alpha) = z(ca,\alpha)$;
- $a_i \in \mathbb{R}_+, i \in J \Rightarrow Sup\{z(a_i,\alpha) \mid i \in J\} =$
$= z(sup\{a_i \mid i \in J\}, \alpha)$.

For $f = \sum_{i=1}^{n} c_i \chi_{C_i}$ we get

$$\int_{A(M,\alpha)} \sum_{i=1}^{n} c_i \chi_{C_i} d\mu = \bigoplus_{i=1}^{n} (c_i\, \tilde{\mu}(C_i \wedge A(M,\alpha))) =$$

$$= \bigoplus_{i=1}^{n} (c_i\, z(\nu(M \cap C_i), \alpha)) =$$

$$= z(\sum_{i=1}^{n} c_i\, \nu(M \cap C_i), \alpha) = z(\int_M f d\nu, \alpha).$$

In the case when f is NMF we have

$$\int_{A(M,\alpha)} f\, d\mu = Sup\{z(\int_M g\, d\nu, \alpha) \mid g \leq f \text{ and } g \text{ is SNMF}\} =$$

$$= z(sup\{\int_M g\, d\nu, \alpha \mid g \leq f \text{ and } g \text{ is SNMF}\}, \alpha) =$$

$$= z(\int_M f d\nu, \alpha).$$

5.2. Integration over SNMF E

If E is SNMF then $E(\mathbb{R}) = \{\alpha_1, ..., \alpha_n\}$. We assume that

$$\alpha_1 > \alpha_2 > ... > \alpha_n$$

and denote

$$M_i = E^{-1}(\alpha_i),\ i = 1, ..., n.$$

Then

- $i \neq j \Rightarrow M_i \cap M_j = \emptyset$;
- $\bigcup_{i=1}^{n} M_i = \mathbb{R}$;
- $E = \bigvee_{i=1}^{n} A(M_i, \alpha_i)$;
- $E^{\alpha_i} = \bigcup_{j=1}^{i} M_j$
  where E^{α_i} as the α_i-cut of fuzzy set E .

Taking into account the property of addition of fuzzy numbers:
$(\bigoplus_{i=1}^{n} z(a_i, \alpha_i))(t) =$

$$= \begin{cases} 1, t \leq 0, \\ \alpha_1,\ 0 < t \leq a_1, \\ \dots \\ \alpha_{i+1},\ a_1 + \dots + a_i < t \leq a_1 + \dots + a_{i+1}, \\ \dots \\ 0,\ t > a_1 + \dots + a_n, \end{cases}$$

we obtain

$$\int_E f\, d\mu = \bigoplus_{i=1}^{n} \int_{E(\alpha_i, M_i)} f\, d\mu = \bigoplus_{i=1}^{n} z(\int_{M_i} f d\nu, \alpha_i) =$$

$$= \begin{cases} 1,\ t \leq \int_{M_1} f d\nu, \\ \dots \\ \alpha_i,\ \sum_{j=1}^{i} \int_{M_j} f d\nu < t \leq \sum_{j=1}^{i+1} \int_{M_j} f d\nu, \\ \dots \\ 0,\ t > \sum_{j=1}^{n} \int_{M_j} f d\nu, \end{cases} =$$

$$= \begin{cases} 1,\ t \leq \int_{E^{\alpha_1}} f d\nu, \\ \dots \\ \alpha_i,\ \int_{E^{\alpha_i}} f d\nu < t \leq \int_{E^{\alpha_{i+1}}} f d\nu, \\ \dots \\ \alpha_n,\ \int_{E^{\alpha_{n-1}}} f d\nu < t \leq \int_{E^{\alpha_n}} f d\nu, \\ 0,\ otherwise. \end{cases}$$

5.3. Integration over NMF E

As was already mentioned every NMF E can be presented as the limit of a non-decreasing sequence of SNMF. To describe this sequence we use the same logic as in the previous subsection. Let us take a sequence $(E_n)_{n \in \mathbb{N}}$ such as:

- for all $n \in \mathbb{N} : E_n(\mathbb{R}) = \{\alpha_1^n, \dots, \alpha_{k_n}^n\}$;
- for all $n \in \mathbb{N}$ $\alpha_i^n > \alpha_{i+1}^n$, $i = 1, \dots, k_n - 1$;
- $M_1^n = \{x \mid E(x) = \alpha_1^n\}$,
  $M_i^n = \{x \mid \alpha_i^n \leq E(x) < \alpha_{i-1}^n\}$, $i = 2, \dots, k_n$;
- $E_n = \bigvee_{i=1}^{k_n} E(\alpha_i^n, M_i^n)$, $n \in \mathbb{N}$;
- $E = \bigvee_n E_n$.

Denoting $I = \int_E f\, d\mu$ and $I_n = \int_{E_n} f\, d\mu$ we get

$$I = Sup\{\int_{E_n} f\, d\mu \mid n \in \mathbb{N}\} = Sup\{I_n \mid n \in \mathbb{N}\}.$$

From the last equality we can get an approximate value of I by fixing n. Obviously, the integral accuracy in this case will be dependent on n .

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