Translation invariance and scale invariance of approximations of fuzzy numbers

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Abstract

We give some sufficient conditions such that a crisp, interval, triangular, trapezoidal, parametric approximation of a fuzzy number, with or without additional requirements, to have the properties of invariance to translations or scale invariance.

Keywords: Fuzzy number, Approximation, Scale invariance, Translation invariance.

1. Introduction

Lists of criteria which a crisp approximation (or defuzzification) and a trapezoidal approximation operator should or just possess were proposed in [13] and [10]. Scale invariance and translation invariance were included in these lists and then studied for recent introduced approximations of fuzzy numbers (e.g. [1] - [4], [18], [19]). The proofs are rather sophisticated when the approximation operators are given on cases. Our aim is to give some sufficient conditions, valid in a general framework, which ensure the scale and translation invariance.

2. Preliminaries

A fuzzy number A is a fuzzy subset of the real line $\mathbb{R}$ with the membership function μ_A which is normal, fuzzy convex, upper semicontinuous and supp A is bounded, where

$$\text{supp}A = cl\left\{x \in \mathbb{R} : \mu_A(x) > 0\right\}$$

and cl is the closure operator. A space of all fuzzy numbers will be denoted by $F(\mathbb{R})$.

Every α-cut, $\alpha \in]0,1]$, of $A \in F(\mathbb{R})$ is a closed interval $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, where

$$A_L(\alpha) = \inf\left\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\right\},$$
$$A_U(\alpha) = \sup\left\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\right\}.$$

We denote

$$A_0 = [A_L(0), A_U(0)] = \text{supp } A.$$

For $A \in F(\mathbb{R})$,

$$A_\alpha = [A_L(\alpha), A_U(\alpha)], \alpha \in [0,1]$$

and $z, \lambda \in \mathbb{R}$ we consider the translation $A + z$ and the scalar multiplication $\lambda \cdot A$ by

$$(A+z)_\alpha = A_\alpha + z = [A_L(\alpha) + z, A_U(\alpha) + z]$$

and

$$(\lambda \cdot A)_\alpha = \lambda A_\alpha = \begin{cases} [\lambda A_L(\alpha), \lambda A_U(\alpha)], \text{ if } \lambda \geq 0 \\ [\lambda A_U(\alpha), \lambda A_L(\alpha)], \text{ if } \lambda < 0. \end{cases}$$

Various characteristics of fuzzy numbers are useful in applications. Some of the most important are the expected interval $EI(A)$, expected value $EV(A)$, value $Val(A)$, ambiguity $Amb(A)$ and width $w(A)$ of a fuzzy number A. They are given by (see [6], [8], [12])

$$EI(A) = \left[\int_0^1 A_L(\alpha)\,d\alpha, \int_0^1 A_U(\alpha)\,d\alpha\right],$$
$$EV(A) = \frac{1}{2}\left(\int_0^1 A_L(\alpha)\,d\alpha + \int_0^1 A_U(\alpha)\,d\alpha\right),$$
$$Val(A) = \int_0^1 \alpha A_U(\alpha)\,d\alpha + \int_0^1 \alpha A_L(\alpha)\,d\alpha,$$
$$Amb(A) = \int_0^1 \alpha A_U(\alpha)\,d\alpha - \int_0^1 \alpha A_L(\alpha)\,d\alpha$$

and

$$w(A) = \int_0^1 A_U(\alpha)\,d\alpha - \int_0^1 A_L(\alpha)\,d\alpha.$$

The weighted L_2-distance on $F(\mathbb{R})$ is defined as (see e.g. [18])

$$d_\lambda^2(A,B) = \int_0^1 (A_L(\alpha) - B_L(\alpha))^2 \lambda_L(\alpha)\,d\alpha + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2 \lambda_U(\alpha)\,d\alpha, \quad (1)$$

where λ_L, λ_U are non-negative functions on $[0,1]$, called weighted functions, such that

$$\int_0^1 \lambda_L(\alpha)\,d\alpha > 0$$

and

$$\int_0^1 \lambda_U(\alpha)\,d\alpha > 0.$$

If $\lambda_L(\alpha) = \lambda_U(\alpha) = 1$, for every $\alpha \in [0,1]$, we obtain the L_2-distance on $F(\mathbb{R})$,

$$d^2(A,B) = \int_0^1 (A_L(\alpha) - B_L(\alpha))^2 \, d\alpha + \int_0^1 (A_U(\alpha) - B_U(\alpha))^2 \, d\alpha. \quad (2)$$

The $\delta_{p,q}$ distance, indexed by parameters $1 \leq p \leq \infty, 0 \leq q \leq 1$, is given as follows (see [9])

$$(\delta_{p,q}(A,B))^p = (1-q) \int_0^1 |A_L(\alpha) - B_L(\alpha)|^p \, d\alpha + q \int_0^1 |A_U(\alpha) - B_U(\alpha)|^p \, d\alpha, \quad (3)$$

for $1 \leq p < \infty$, and

$$\delta_{\infty,q}(A,B) = (1-q) \sup_{0<\alpha\leq 1} |A_L(\alpha) - B_L(\alpha)| + q \sup_{0<\alpha\leq 1} |A_U(\alpha) - B_U(\alpha)|. \quad (4)$$

The supremum distance d_∞ and $L_{p,\infty}$-distances $d_p, 1 \leq p < \infty$, on $F(\mathbb{R})$, involve the Hausdorff metric between the α-cuts of fuzzy numbers. They are defined by (see [7], p.52)

$$d_\infty(A,B) = \sup_{0\leq\alpha\leq 1} d_H(A_\alpha, B_\alpha), \quad (5)$$

$$d_p(A,B) = \left(\int_0^1 (d_H(A_\alpha, B_\alpha))^p \, d\alpha \right)^{1/p} \quad (6)$$

where

$$d_H(A_\alpha, B_\alpha) = \max\{|A_L(\alpha) - B_L(\alpha)|, |A_U(\alpha) - B_U(\alpha)|\}.$$

A class of distances between fuzzy numbers was introduced in [5] by

$$\widetilde{D}_{f,\varphi}(A,B) = \left(\int_0^1 \widetilde{D}_f^2(A_\alpha, B_\alpha) \, d\varphi(\alpha) \right)^{1/2}, \quad (7)$$

where

$$\widetilde{D}_f^2([a,b],[c,d]) = \int_0^1 (t|a-c| + (1-t)|b-d|)^2 \, df(t),$$

f is a normalized weight measure on $[0,1]$ and usually the function φ satisfies the conditions

$$\varphi(\alpha) \geq 0, \forall \alpha \in [0,1],$$
$$\alpha_1 \leq \alpha_2 \Rightarrow \varphi(\alpha_1) \leq \varphi(\alpha_2),$$
$$\int_0^1 \varphi(\alpha) \, d\alpha = 1.$$

On the basis of expressing the above metric $\widetilde{D}_f$ in terms of the mid and spread of intervals, the distance $D^*_{\psi,\theta}$ between fuzzy numbers was introduced in [16] as follows:

$$D^*_{\psi,\theta}(A,B) = \left(\int_0^1 (D^*_\theta(A_\alpha, B_\alpha))^2 \, d\psi(\alpha) \right)^{1/2}, \quad (8)$$

where $\theta \in (0,1], \psi$ is a weight probability measure on $[0,1]$,

$$(D^*_\theta([a,b],[c,d]))^2 = (mid[a,b] - mid[c,d])^2 + \theta(spr[a,b] - spr[c,d])^2,$$

$$mid[a_1,a_2] = \frac{a_1 + a_2}{2}$$

and

$$spr[a_1,a_2] = \frac{a_2 - a_1}{2}.$$

We recall, a distance D on $F(\mathbb{R})$ is translation invariant if

$$D(A+z, B+z) = D(A,B),$$

for every $A, B \in F(\mathbb{R}), z \in \mathbb{R}$ and scale invariant if

$$D(\lambda \cdot A, \lambda \cdot B) = |\lambda| D(A,B),$$

for every $A, B \in F(\mathbb{R}), \lambda \in \mathbb{R} \setminus \{0\}$.

3. Invariance to translations

The translation invariance of an approximation operator O on $F(\mathbb{R})$, that is

$$O(A+z) = O(A) + z,$$

for every $A \in F(\mathbb{R})$ and $z \in \mathbb{R}$, is considered a natural requirement. The following result gives sufficient conditions for the translation invariance of an operator which associates to a fuzzy number the nearest fuzzy number in a subset of fuzzy numbers, under additional conditions.

Theorem 1 *Let D be a translation invariant distance on $F(\mathbb{R})$ and $P_k, k \in \{1,...,n\}$ is a real parameter or interval associated with fuzzy numbers such that*

$$P_k(A+z) = P_k(A), \quad (9)$$

for every $A \in F(\mathbb{R})$ and $z \in \mathbb{R}$ or

$$P_k(A+z) = P_k(A) + z, \quad (10)$$

for every $A \in F(\mathbb{R})$ and $z \in \mathbb{R}$. If $\Omega \subset F(\mathbb{R})$ satisfies $z + \Omega = \Omega, \forall z \in \mathbb{R}$ and $\omega(A) \in \Omega$ is the nearest fuzzy number to a given $A \in F(\mathbb{R})$ (with respect to D) which preserves $P_k, k \in \{1,...,n\}$, that is

$$P_k(\omega(A)) = P_k(A), \forall k \in \{1,...,n\}$$

then $\omega(A) + z \in \Omega$ is the nearest fuzzy number to $A + z$ (with respect to D) which preserves $P_k, k \in \{1,...,n\}$, that is

$$P_k(\omega(A+z)) = P_k(A+z), \forall k \in \{1,...,n\}.$$

Proof. We have

$$D(A,\omega(A)) \le D(A,B),$$

for every $B \in \Omega$ such that $P_k(A) = P_k(B), \forall k \in \{1,...,n\}$ and

$$P_k(A) = P_k(\omega(A)), \forall k \in \{1,...,n\}.$$

Let $B' \in \Omega$ and let $z \in \mathbb{R}$ such that $P_k(A+z) = P_k(B'), \forall k \in \{1,...,n\}$. Let $B \in \Omega$ such that $B' = B+z$. We get $P_k(A+z) = P_k(B+z)$ and taking into account (9), (10) and the translation invariance of D we get

$$\begin{aligned} D(A+z,\omega(A)+z) &= D(A,\omega(A)) \le D(A,B) \\ &= D(A+z,B+z) = D(A+z,B'). \end{aligned}$$

In addition (taking into account (9) and (10) too),

$$P_k(A+z) = P_k(\omega(A)+z),$$

therefore $\omega(A)+z \in \Omega$ is the nearest element to $A+z$ (with respect to d) which preserves the parameters $P_k, k \in \{1,...,n\}$. In other words,

$$\omega(A+z) = \omega(A)+z$$

that is the operator ω is translation invariant. ■

Corollary 2 *Let $P_k : F(\mathbb{R}) \to \mathbb{R}, k \in \{1,...,n\}$ defined by*

$$\begin{aligned} P_k(A) &= a_k \int_0^1 A_L(\alpha)\,d\alpha + b_k \int_0^1 A_U(\alpha)\,d\alpha \\ &+ g_k \int_0^1 \alpha A_L(\alpha)\,d\alpha + h_k \int_0^1 \alpha A_U(\alpha)\,d\alpha, \end{aligned}$$

where $a_k, b_k, g_k, h_k \in \mathbb{R}$, $\Omega \subset F(\mathbb{R})$ with $z+\Omega = \Omega, \forall z \in \mathbb{R}$ and $\omega : F(\mathbb{R}) \to \Omega$ such that $\omega(A)$ is the nearest fuzzy number to a given A with respect to a translation invariant distance on $F(\mathbb{R})$, which, in addition, preserves the parameters $P_k, k \in \{1,...,n\}$. If

$$2(a_k+b_k)+g_k+h_k = 0 \tag{11}$$

or

$$2(a_k+b_k)+g_k+h_k = 2, \tag{12}$$

for every $k \in \{1,...,n\}$, then

$$\omega(A+z) = \omega(A)+z,$$

for every $z \in \mathbb{R}$.

Proof. In this particular case (9) is equivalent with (11) and (10) is equivalent with (12). ■

We denote

$$Int(\mathbb{R}) = \{[a,b] : a,b \in \mathbb{R}, a < b\}$$

Corollary 3 *Let $P_k : F(\mathbb{R}) \to Int(\mathbb{R}), k \in \{1,...,n\}$ defined by*

$$P_k(A) = [P_k'(A), P_k''(A)],$$

where

$$\begin{aligned} P_k'(A) &= a_k' \int_0^1 A_L(\alpha)\,d\alpha + b_k' \int_0^1 A_U(\alpha)\,d\alpha \\ &+ g_k' \int_0^1 \alpha A_L(\alpha)\,d\alpha + h_k' \int_0^1 \alpha A_U(\alpha)\,d\alpha, \end{aligned}$$

$$\begin{aligned} P_k''(A) &= a_k'' \int_0^1 A_L(\alpha)\,d\alpha + b_k'' \int_0^1 A_U(\alpha)\,d\alpha \\ &+ g_k'' \int_0^1 \alpha A_L(\alpha)\,d\alpha + h_k'' \int_0^1 \alpha A_U(\alpha)\,d\alpha, \end{aligned}$$

$a_k', b_k', g_k', h_k', a_k'', b_k'', g_k'', h_k'' \in \mathbb{R}$, $\Omega \subset F(\mathbb{R})$ such that $z+\Omega = \Omega$ and $\omega : F(\mathbb{R}) \to \Omega$ such that $\omega(A)$ is the nearest fuzzy number to A with respect to a translation invariant distance on $F(\mathbb{R})$ which, in addition, preserves $P_k, k \in \{1,...,n\}$. If

$$2(a_k'+b_k')+g_k'+h_k' = 0, \tag{13}$$
$$2(a_k''+b_k'')+g_k''+h_k'' = 0, \tag{14}$$

or

$$2(a_k'+b_k')+g_k'+h_k' = 2, \tag{15}$$
$$2(a_k''+b_k'')+g_k''+h_k'' = 2, \tag{16}$$

for every $k \in \{1,...,n\}$, then

$$\omega(A+z) = \omega(A)+z,$$

for every $z \in \mathbb{R}$.

Proof. In this particular case (9) is equivalent with (13)-(14) and (10) is equivalent with (15)-(16). ■

4. Scale invariance

It is well-known that an operator O on $F(\mathbb{R})$ is scale invariant if

$$O(\lambda \cdot A) = \lambda \cdot O(A),$$

for every $A \in F(\mathbb{R})$ and $\lambda \in \mathbb{R}$. The scale invariance is a basic requirement for any approximation operator on $F(\mathbb{R})$ (see [10], [13]). The result corresponding to Theorem 1 is the following.

Theorem 4 *Let D be a scale invariant distance on $F(\mathbb{R})$ and $P_k, k \in \{1,...,n\}$ real parameters or intervals associated to fuzzy numbers such that*

$$P_k(\lambda \cdot A) = \lambda P_k(A), \tag{17}$$

for every $A \in F(\mathbb{R})$ and $\lambda \in \mathbb{R}$ or

$$P_k(\lambda \cdot A) = |\lambda| P_k(A), \tag{18}$$

for every $A \in F(\mathbb{R})$ and $\lambda \in \mathbb{R}$. If $\Omega \subset F(\mathbb{R}), \lambda \cdot \Omega \subset \Omega, \forall \lambda \in \mathbb{R}$ and $\omega(A) \in \Omega$ is the nearest fuzzy number to a given $A \in F(\mathbb{R})$ (with respect to D) which preserves $P_k, k \in \{1,...,n\}$, that is

$$P_k(\omega(A)) = P_k(A), \forall k \in \{1,...,n\}$$

then $\lambda \cdot \omega(A) \in \Omega$ is the nearest fuzzy number to $\lambda \cdot A$ (with respect to D) which preserves $P_k, k \in \{1,...,n\}$, that is

$$P_k(\omega(\lambda \cdot A)) = P_k(\lambda \cdot A), \forall k \in \{1,...,n\}$$

Proof. First, we consider the case $\lambda = 0$. Since $0 \in \Omega$, it follows that $\omega(0) = 0$. This implies $\omega(\lambda \cdot A) = \omega(0) = 0 = \lambda \cdot \omega(A)$. Therefore, in what follows we may suppose that $\lambda \neq 0$.
We have

$$D(A, \omega(A)) \leq D(A, M),$$

for every $M \in \Omega$ such that $P_k(A) = P_k(M), \forall k \in \{1,...,n\}$ and

$$P_k(A) = P_k(\omega(A)), \forall k \in \{1,...,n\}.$$

Let $A \in F(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Let $M' \in \Omega$ such that $P_k(\lambda \cdot A) = P_k(M'), \forall k \in \{1,...,n\}$. According to (17) and (18) we have $P_k(A) = P_k\left(\frac{1}{\lambda} \cdot M'\right), \forall k \in \{1,...,n\}$. We get

$$D(\lambda \cdot A, \lambda \cdot \omega(A)) = |\lambda| D(A, \omega(A))$$
$$\leq |\lambda| D\left(A, \frac{1}{\lambda} \cdot M'\right) = D(\lambda \cdot A, M')$$

and

$$P_k(\lambda \cdot A) = \lambda P_k(A)$$
$$= \lambda P_k(\omega(A)) = P_k(\lambda \cdot \omega(A))$$

or

$$P_k(\lambda \cdot A) = |\lambda| P_k(A)$$
$$= |\lambda| P_k(\omega(A)) = P_k(\lambda \cdot \omega(A)),$$

$\forall k \in \{1,...,n\}$, therefore $\lambda \cdot \omega(A) \in \Omega$ is the nearest fuzzy number to $\lambda \cdot A$ (with respect to D) which preserves the parameters $P_k, k \in \{1,...,n\}$. In other words,

$$\omega(\lambda \cdot A) = \lambda \cdot \omega(A).$$

■

Remark 5 *Note that the presumption $\lambda \cdot \Omega \subset \Omega, \forall \lambda \in \mathbb{R}$ in Theorem 4 (as the presumption $z + \Omega = \Omega, \forall z \in \mathbb{R}$ in Theorem 1) is important. Indeed, if $r, s > 0, r \neq s$ and*

$$\Omega = \{A \in F(\mathbb{R}) : \exists a, b, \sigma, \beta \in \mathbb{R}, a \leq b, \sigma \geq 0, \beta \geq 0$$
$$\text{such that } A_L(\alpha) = a - \sigma(1-\alpha)^{1/r},$$
$$A_U(\alpha) = b + \beta(1-\alpha)^{1/s}, \forall \alpha \in [0,1]\Big\}$$

then $\lambda \cdot \Omega \subsetneq \Omega$ for $\lambda < 0$. The operator $\omega : F(\mathbb{R}) \to \Omega$ such that $\omega(A) \in \Omega$ is the nearest fuzzy number to given A with respect to distance d (see (2)) is not scale invariant (see [3], Theorem 12, (iii)) even if every other hypotheses in Theorem 4 are satisfied.

Corollary 6 *Let $P_k : F(\mathbb{R}) \to \mathbb{R}, k \in \{1,...,n\}$ defined by*

$$P_k(A) = a_k \int_0^1 A_L(\alpha) d\alpha + b_k \int_0^1 A_U(\alpha) d\alpha$$
$$+ g_k \int_0^1 \alpha A_L(\alpha) d\alpha + h_k \int_0^1 \alpha A_U(\alpha) d\alpha,$$

where $a_k, b_k, g_k, h_k \in \mathbb{R}$, $\Omega \subset F(\mathbb{R})$ with the property $\lambda \cdot \Omega \subset \Omega, \forall \lambda \in \mathbb{R}$ and $\omega : F(\mathbb{R}) \to \Omega$ such that $\omega(A)$ is the nearest fuzzy number to A with respect to a scale invariant distance D, which, in addition, preserves the parameters $P_k, k \in \{1,...,n\}$. If

$$a_k = b_k, \tag{19}$$
$$g_k = h_k \tag{20}$$

or

$$b_k = -a_k, \tag{21}$$
$$h_k = -g_k, \tag{22}$$

for every $k \in \{1,...,n\}$, then

$$\omega(\lambda \cdot A) = \lambda \cdot \omega(A),$$

for every $\lambda \in \mathbb{R}$.

Proof. In this particular case (17) is equivalent with (19)-(20) and (18) is equivalent with (21)-(22). ■

Corollary 7 *Let $P_k : F(\mathbb{R}) \to Int(\mathbb{R}), k \in \{1,...,n\}$ defined by*

$$P_k(A) = [P_k'(A), P_k''(A)],$$

where

$$P_k'(A) = a_k' \int_0^1 A_L(\alpha) d\alpha + b_k' \int_0^1 A_U(\alpha) d\alpha$$
$$+ g_k' \int_0^1 \alpha A_L(\alpha) d\alpha + h_k' \int_0^1 \alpha A_U(\alpha) d\alpha,$$

$$P_k''(A) = a_k'' \int_0^1 A_L(\alpha) d\alpha + b_k'' \int_0^1 A_U(\alpha) d\alpha$$
$$+ g_k'' \int_0^1 \alpha A_L(\alpha) d\alpha + h_k'' \int_0^1 \alpha A_U(\alpha) d\alpha,$$

$a_k', b_k', g_k', h_k', a_k'', b_k'', g_k'', h_k'' \in \mathbb{R}$, $\Omega \subset F(\mathbb{R})$ with the property $\lambda \cdot \Omega \subset \Omega, \forall \lambda \in \mathbb{R}$ and $\omega : F(\mathbb{R}) \to \Omega$ such that $\omega(A)$ is the nearest fuzzy number to A with respect to a scale invariant distance D which, in addition, preserves $P_k, k \in \{1,...,n\}$. If

$$a_k' = b_k'', \tag{23}$$
$$g_k' = h_k'', \tag{24}$$
$$a_k'' = b_k', \tag{25}$$
$$g_k'' = h_k' \tag{26}$$

or

$$a'_k = -b'_k, \quad (27)$$
$$g'_k = -h'_k, \quad (28)$$
$$a''_k = -b''_k, \quad (29)$$
$$g''_k = -h''_k \quad (30)$$

for every $k \in \{1, ..., n\}$, *then*

$$\omega(\lambda \cdot A) = \lambda \cdot \omega(A),$$

for every $\lambda \in \mathbb{R}$.

Proof. In this particular case (17) is equivalent with (23)-(26) and (18) is equivalent with (27)-(30). ■

5. Applications to crisp, interval, triangular, trapezoidal and parametric approximations of fuzzy numbers

With respect to the hypothesis in Theorems 1 and 4, let us remark that the distances $d_\lambda, d, \delta_{p,q}, \delta_{\infty,q}, d_\infty, d_p, \widetilde{D}_{f,\varphi}, D^*_{\psi,\theta}$ on $F(\mathbb{R})$, introduced by (1)-(8), are invariant to translations and $d, d_\infty, d_p, D^*_{\psi,\theta}$ are scale invariant. If $\lambda_L(\alpha) = \lambda_U(\alpha)$, for every $\alpha \in [0,1]$, then d_λ is scale invariant too.

Let us consider the following subsets of $F(\mathbb{R})$:

$$\mathbb{R}^c = \{A \in F(\mathbb{R}) : \exists c \in \mathbb{R} \text{ such that } A_L(\alpha) = A_U(\alpha) = c, \forall \alpha \in [0,1]\},$$

$$\mathbb{I} = \{A \in F(\mathbb{R}) : \exists c_L, c_U \in \mathbb{R} \text{ such that } A_L(\alpha) = c_L, A_U(\alpha) = c_U, \forall \alpha \in [0,1]\},$$

$$\triangle = \{A \in F(\mathbb{R}) : \exists t_1, t_2, t_3 \in \mathbb{R}, t_1 \le t_2 \le t_3 \text{ such that } A_L(\alpha) = t_1 + \alpha(t_2 - t_1), A_U(\alpha) = t_3 + \alpha(t_2 - t_3), \forall \alpha \in [0,1]\},$$

$$\triangle^s = \{A \in F(\mathbb{R}) : \exists t_1, t_2, t_3 \in \mathbb{R}, t_1 \le t_2 \le t_3, t_3 - t_2 = t_2 - t_1 \text{ such that } A_L(\alpha) = t_1 + \alpha(t_2 - t_1), A_U(\alpha) = t_3 + \alpha(t_2 - t_3), \forall \alpha \in [0,1]\},$$

$$\mathbb{T} = \{A \in F(\mathbb{R}) : \exists t_1, t_2, t_3, t_4 \in \mathbb{R}, t_1 \le t_2 \le t_3 \le t_4 \text{ such that } A_L(\alpha) = t_1 + \alpha(t_2 - t_1), A_U(\alpha) = t_4 + \alpha(t_3 - t_4), \forall \alpha \in [0,1]\}$$

and

$$\mathbb{P}_s = \Big\{A \in F(\mathbb{R}) : \exists a, b, \sigma, \beta \in \mathbb{R}, a \le b, \sigma \ge 0, \beta \ge 0 \text{ such that } A_L(\alpha) = a - \sigma(1-\alpha)^{1/s}, A_U(\alpha) = b + \beta(1-\alpha)^{1/s}, \forall \alpha \in [0,1]\Big\}$$

where $s > 0$ is fixed. In fact, $\mathbb{R}^c$ is the set of real numbers, $\mathbb{I}$ is the set of real intervals, $\triangle$ is the set of triangular fuzzy numbers, $\triangle^s$ is the set of symmetric triangular fuzzy numbers, $\mathbb{T}$ is the set of trapezoidal fuzzy numbers and $\mathbb{P}_s$ is the set of parametric (s,s)-fuzzy numbers (see [15]). If $\Omega \in \{\mathbb{R}^c, \mathbb{I}, \triangle, \triangle^s, \mathbb{T}, \mathbb{P}_s\}$ then $z + \Omega = \Omega$, for every $z \in \mathbb{R}$ and $\lambda \cdot \Omega \subset \Omega$, for every $\lambda \in \mathbb{R}$.

The following results are immediate consequences of Theorems 1 and 4.

Corollary 8 (*i*) *The operator* $O_{\mathbb{R}^c} : F(\mathbb{R}) \to \mathbb{R}^c$, *where* $O_{\mathbb{R}^c}(A)$ *is the nearest crisp value to* A *with respect to distance* d *in (2), is scale and translation invariant.*

(*ii*) *The operator* $O_{\mathbb{I}} : F(\mathbb{R}) \to \mathbb{I}$, *where* $O_{\mathbb{I}}(A)$ *is the nearest interval to* A *with respect to distance* d, *is scale and translation invariant.*

(*iii*) *The operator* $O_\triangle : F(\mathbb{R}) \to \triangle$, *where* $O_\triangle(A)$ *is the nearest triangular fuzzy number to* A *with respect to distance* d, *is scale and translation invariant.*

(*iv*) *The operator* $O_{\mathbb{T}} : F(\mathbb{R}) \to \mathbb{T}$, *where* $O_{\mathbb{T}}(A)$ *is the nearest trapezoidal fuzzy number to* A *with respect to distance* d, *is scale and translation invariant.*

(*v*) *The operator* $O_{\mathbb{P}_s} : F(\mathbb{R}) \to \mathbb{P}_s$, *where* $O_{\mathbb{P}_s}(A)$ *is the nearest* (s,s)*-fuzzy number to* A *with respect to distance* d, *is scale and translation invariant.*

(*vi*) *The operator* $O_{\triangle^s} : F(\mathbb{R}) \to \triangle^s$, *where* $O_{\triangle^s}(A)$ *is the nearest symmetric triangular fuzzy number to* A *with respect to distance* d, *is scale and translation invariant.*

In fact, $O_{\mathbb{R}^c}(A) = EV(A)$ and $O_{\mathbb{I}}(A) = EI(A)$, for every $A \in F(\mathbb{R})$, therefore the scale and translation invariance in (*i*) and (*ii*) are consequences of the properties of the expected value and expected interval, respectively. The scale and translation invariance of $O_\triangle, O_{\mathbb{T}}, O_{\mathbb{P}_s}$ are already proved in [17] and [3]. The operator $O_{\triangle^s}$ was determined in [14], but its properties have not been studied yet.

Corollary 9 (*i*) *The operator* $O^w_{\mathbb{R}^c} : F(\mathbb{R}) \to \mathbb{R}^c$, *where* $O^w_{\mathbb{R}^c}(A)$ *is the nearest crisp value to* A *with respect to distance* d_λ *in (1), is translation invariant. If* $\lambda_L(\alpha) = \lambda_U(\alpha)$, *for every* $\alpha \in [0,1]$, *then it is scale invariant too.*

(*ii*) *The operator* $O^w_{\mathbb{I}} : F(\mathbb{R}) \to \mathbb{I}$, *where* $O^w_{\mathbb{I}}(A)$ *is the nearest interval to* A *with respect to distance* d_λ, *is translation invariant. If* $\lambda_L(\alpha) = \lambda_U(\alpha)$, *for every* $\alpha \in [0,1]$, *then it is scale invariant too.*

(*iii*) *The operator* $O^w_\triangle : F(\mathbb{R}) \to \triangle$, *where* $O^w_\triangle(A)$ *is the nearest triangular fuzzy number to* A *with respect to distance* d_λ, *is translation invariant. If* $\lambda_L(\alpha) = \lambda_U(\alpha)$, *for every* $\alpha \in [0,1]$, *then it is scale invariant too.*

(*iv*) *The operator* $O^w_{\mathbb{T}} : F(\mathbb{R}) \to \mathbb{T}$, *where* $O^w_{\mathbb{T}}(A)$ *is the nearest trapezoidal fuzzy number to* A *with respect to distance* d_λ, *is translation invariant. If* $\lambda_L(\alpha) = \lambda_U(\alpha)$, *for every* $\alpha \in [0,1]$, *then it is scale invariant too.*

(*v*) *The operator* $O^w_{\mathbb{P}_s} : F(\mathbb{R}) \to \mathbb{P}_s$, *where* $O^w_{\mathbb{P}_s}(A)$ *is the nearest* (s,s)*-fuzzy number to* A *with respect to distance* d_λ, *is translation invariant. If* $\lambda_L(\alpha) = \lambda_U(\alpha)$, *for every* $\alpha \in [0,1]$, *then it is scale invariant too.*

The translation and scale invariance of $O^w_\triangle$ and $O^w_\mathbb{T}$ are already studied in [18], and the properties of $O^w_{\mathbb{P}_s}$ in [19]. Even if the nearest crisp value and the nearest interval to a given $A \in F(\mathbb{R})$, with respect to d_λ, are not determined yet, their existence and uniqueness are obvious and the results in (*i*) and (*ii*) are valid for these approximation operators too.

The existence and uniqueness of the nearest crisp value, interval, triangular fuzzy number, trapezoidal fuzzy numbers or parametric (s,s)-fuzzy number with respect to distances $\delta_{p,q}, \delta_{\infty,q}, d_\infty, d_p, \widetilde{D}_{f,\varphi}, D^*_{\psi,\theta}$ on $F(\mathbb{R})$, introduced by (3)-(8), could lead towards analogous results with Corollaries 8 and 9.

In the sequel we give some results related with trapezoidal approximations of fuzzy numbers.

With the notations in Theorems 1 and 4, Corollaries 3 and 7, we consider $n = 1$ and

$$P_1(A) = [P_1'(A), P_1''(A)]$$
$$= EI(A) = \left[\int_0^1 A_L(\alpha)\,d\alpha, \int_0^1 A_U(\alpha)\,d\alpha\right],$$

that is

$$a_1' = 1, b_1' = g_1' = h_1' = 0,$$
$$a_1'' = g_1'' = h_1'' = 0, b_1'' = 1.$$

The following result, already proved in [2] is immediate.

Corollary 10 *The operator* $O_{EI} : F(\mathbb{R}) \to \mathbb{T}$, *where* $O_{EI}(A)$ *is the nearest trapezoidal fuzzy number to* A *(with respect to distance* d *in (2)), which preserves the expected interval, is scale and translation invariant.*

With the notations in Theorems 1 and 4, Corollaries 2 and 6, we consider $n = 2$ and

$$P_1(A) = Amb(A)$$
$$= \int_0^1 \alpha A_U(\alpha)\,d\alpha - \int_0^1 \alpha A_L(\alpha)\,d\alpha,$$
$$P_2(A) = Val(A)$$
$$= \int_0^1 \alpha A_U(\alpha)\,d\alpha + \int_0^1 \alpha A_L(\alpha)\,d\alpha,$$

that is

$$a_1 = b_1 = 0, g_1 = -1, h_1 = 1,$$
$$a_2 = b_2 = 0, g_2 = h_2 = 1.$$

The following result, already proved in [4] is immediate.

Corollary 11 *The operator* $O_{Amb,Val} : F(\mathbb{R}) \to \mathbb{T}$, *where* $O_{Amb,Val}(A)$ *is the nearest trapezoidal fuzzy number to* A *(with respect to distance* d *in (2)), which preserves the ambiguity and value, is scale and translation invariant.*

We consider the operator $O_{core} : F(\mathbb{R}) \to \mathbb{T}$, where $O_{core}(A)$ is the nearest trapezoidal fuzzy number to A (with respect to distance d in (2)), which preserves the core, that is

$$core(O_{core}(A)) = core(A) = [A_L(1), A_U(1)].$$

If $n = 1$ and $P_1(A) = [A_L(1), A_U(1)]$ then

$$P_1(A + z) = [A_L(1) + z, A_U(1) + z]$$
$$= P_1(A) + z,$$

for every $z \in \mathbb{R}$,

$$P_1(\lambda \cdot A) = [\lambda A_L(1), \lambda A_U(1)] = \lambda P_1(A),$$

for every $\lambda \in \mathbb{R}, \lambda \geq 0$ and

$$P_1(\lambda \cdot A) = [\lambda A_U(1), \lambda A_L(1)] = \lambda P_1(A),$$

for every $\lambda \in \mathbb{R}, \lambda < 0$, therefore from Theorems 1 and 4 we obtain the following result, already proved in [1].

Corollary 12 *The operator* $O_{core} : F(\mathbb{R}) \to \mathbb{T}$ *is scale and translation invariant.*

An important benefit of the results in Theorems 1 and 4 is the possibility to deduce the properties of scale and translation invariance even if the approximations are actually unknown. For example, the calculus of the nearest trapezoidal fuzzy number to a fuzzy number, which preserves the ambiguity and the nearest trapezoidal fuzzy number to a fuzzy number, which preserves the value, both with respect to distance d in (2) were not performed yet. With the notations in Theorems 1 and 4, Corollaries 2 and 6, we consider $n = 1$ and

$$P_1(A) = Amb(A)$$
$$= \int_0^1 \alpha A_U(\alpha)\,d\alpha - \int_0^1 \alpha A_L(\alpha)\,d\alpha,$$

then

$$P_1(A) = Val(A)$$
$$= \int_0^1 \alpha A_U(\alpha)\,d\alpha + \int_0^1 \alpha A_L(\alpha)\,d\alpha,$$

that is

$$a_1 = b_1 = 0, g_1 = -1, h_1 = 1,$$

then

$$a_1 = b_1 = 0, g_1 = h_1 = 1,$$

respectively. We obtain (Corollaries 2 and 6 are used here)

Corollary 13 *The operators* $O_{Amb} : F(\mathbb{R}) \to \mathbb{T}$ *and* $O_{Val} : F(\mathbb{R}) \to \mathbb{T}$, *where* $O_{Amb}(A)$ *and* $O_{Val}(A)$ *are the nearest trapezoidal fuzzy numbers to* A *(with respect to distance* d *in (2)), which preserves the ambiguity and value, respectively, are scale and translation invariant.*

6. Conclusion

In the last few years many researchers focused on the approximation of fuzzy numbers. Between the most important requirements that an approximation operator should satisfy are: the invariance to translations, the scale invariance, additivity or continuity. One would expect that an approximation operator would posses as many of this properties as possible. In the present contribution we found sufficient conditions for approximation operators over the space of fuzzy numbers (with respect to well-known distances) to be invariant to translations and scale invariant, respectively. An immediate consequence of these results is that most of the approximation operator proposed so far in the literature satisfy these requirements. But, according to Theorems 1 and 4, the class of approximation operators which posses this two properties is even larger. This could be an useful tool in finding new approximation operators so that most of the basic requirements are fulfilled. The next goal is to characterize the general class of the additive approximations and/or continuous approximations of fuzzy numbers.

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