On the variety of equality algebras*

Sándor Jenei[1] **László Kóródi**[2]

[1]Institute of Mathematics and Informatics, University of Pécs, Hungary
[1]Department of Knowledge-Based Mathematical Systems, Johannes Kepler University, Linz, Austria
[2]Institute of Mathematics and Informatics, University of Pécs, Hungary

Abstract

Equality algebras has recently been introduced. A subclass of equality algebras, called equivalential equality algebras is closely related to BCK-algebras with meet. We show that the variety of equality algebras has nice properties: We shall investigate their congruences and filters and prove that the variety of equality algebras is a **1**-regular, arithmetic variety.

Keywords: Equality algebra, BCK-algebra, **1**-regular, congruence permutable, congruence distributive variety

1. Introduction

The motivation for introducing equality algebras came from EQ-algebras [6]. In EQ-algebras, compared to equality algebras, there is an additional operation $\otimes$, called product, which is very loosely related to the other operations. Therefore, there might not exist deep algebraic characterizations of EQ-algebras, and our intention was to define a structure similar to EQ-algebras but without the product. That has lead to the following axioms:

Definition 1 An *equality algebra* [4] is an algebra $\mathcal{E} = \langle X, \sim, \wedge, \mathbf{1} \rangle$ of type (2, 2, 0) such that the following axioms are fulfilled for all $a, b, c \in X$:

(E1) $\langle X, \wedge, \mathbf{1} \rangle$ is a commutative idempotent integral monoid (i.e. $\wedge$-semilattice with top element $\mathbf{1}$),

(E2) $a \sim b = b \sim a$,

(E3) $a \sim a = \mathbf{1}$,

(E4) $a \sim \mathbf{1} = a$,

(E5) $a \leq b \leq c$ implies $a \sim c \leq b \sim c$ and $a \sim c \leq a \sim b$,

(E6) $a \sim b \leq (a \wedge c) \sim (b \wedge c)$,

(E7) $a \sim b \leq (a \sim c) \sim (b \sim c)$.

The operation $\wedge$ is called *meet (infimum)* and $\sim$ is an *equality* operation. We write $a \leq b$ iff $a \wedge b = a$, as usual and define the following two derived operations, the *implication* and the *equivalence* operation of the equality algebra $\mathcal{E}$ by

$$a \rightarrow b = a \sim (a \wedge b) \quad (1)$$

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a). \quad (2)$$

Call an equality algebra (and as well its equality operation $\sim$) *equivalential* if $\sim$ coincides with $\leftrightarrow$.

One can prove that equality algebras are exactly the $\otimes$-free subreducts of the so-called good EQ-algebras.

The general motivation for equality algebras from the side of logic was to define an algebraic structure which (with appropriate extensions) is suitable to axiomatize a large class of substructural logics based on an equivalence connective rather than implication. The very first step toward this aim has been done in [4] where it has been shown that

1. Equality algebras form a variety.

2. The class of equivalential equality algebras and the class of BCK-algebras with meet are term equivalent.

3. A generalization of a result of Kabziński and Wroński in [5] was obtained, namely, it holds true that $\sim$ can be represented as the equivalence operator of a BCK-algebra with meet (on a $\wedge$-semilattice $(X, \wedge)$ with top element $\mathbf{1}$) if and only if (E2)-(E4), (E6),

$$a \sim (a \wedge b \wedge c) \leq a \sim (a \wedge b),$$

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$$a \sim (a \wedge b) \leq (a \wedge c) \sim (a \wedge b \wedge c),$$
$$a \sim b = (a \sim (a \wedge b)) \wedge (b \sim (a \wedge b))$$

hold.

4. All totally ordered equality algebras are equivalential.

Point 3 above seems to be important since it tells us about an equational characterization of the equivalence operation of BCK semilattices, which may easily be arisen to an axiomatic description of the equivalential fragment of the related logic.

The term equivalence, which is mentioned above at point 2 is given by the following

Theorem 1 [4] The following two statements hold true:

i. For any equality algebra $\mathcal{E} = \langle X, \sim, \wedge, \mathbf{1} \rangle$, $\Psi(\mathcal{E}) = \langle X, \rightarrow, \wedge, \mathbf{1} \rangle$ is a BCK-algebra with meet.

ii. For any BCK-algebra with meet $\mathcal{B} = \langle X, \rightarrow, \wedge, \mathbf{1} \rangle$, $\Phi(\mathcal{B}) = \langle X, \leftrightarrow, \wedge, \mathbf{1} \rangle$ is an equality algebra, where $\leftrightarrow$ denotes the equivalence operation of $\mathcal{B}$. Moreover, the implication of $\Phi(\mathcal{B})$ coincides with $\rightarrow$, that is, we have

$$a \rightarrow b = a \leftrightarrow (a \wedge b). \tag{3}$$

Definition 2 Employing the notations of Theorem 1 call $\mathcal{B} = \Psi(\mathcal{E})$ the *underlying* BCK-algebra of $\mathcal{E}$ and call $\bar{\mathcal{E}} = \Phi(\mathcal{B})$ the *canonical* equality algebra of $\mathcal{B}$.

To conclude this section, the basic properties of equality algebras are summarized:

Proposition 2 [4] Let $\mathcal{E} = \langle X, \sim, \wedge, \mathbf{1} \rangle$ be an equality algebra. Then the followings hold for all $a, b, c, d \in X$:

(a) $a \sim b \leq a \leftrightarrow b \leq a \rightarrow b$,

(b) $a \leq (a \sim b) \sim b$,

(c) $a \sim b = \mathbf{1}$ iff $a = b$,

(d) $a \rightarrow b = \mathbf{1}$ iff $a \leq b$,

(e) $a \rightarrow b = \mathbf{1}$ and $b \rightarrow a = \mathbf{1}$ imply $a = b$,

(f) $\mathbf{1} \rightarrow a = a$, $a \rightarrow \mathbf{1} = \mathbf{1}$, $a \rightarrow a = \mathbf{1}$,

(g) $a \leq b \rightarrow a$, $a \rightarrow (b \rightarrow a) = \mathbf{1}$,

(h) $a \leq (a \rightarrow b) \rightarrow b$, $a \rightarrow ((a \rightarrow b) \rightarrow b) = \mathbf{1}$,

(i) $a \rightarrow b \leq (b \rightarrow c) \rightarrow (a \rightarrow c)$, $(a \rightarrow b) \rightarrow ((b \rightarrow c) \rightarrow (a \rightarrow c)) = \mathbf{1}$

(j) $a \leq b \rightarrow c$ iff $b \leq a \rightarrow c$,

(k) $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$,

(l) $a \leftrightarrow a = \mathbf{1}$, $a \leftrightarrow \mathbf{1} = a$,

(m) $b \leq a$ implies $a \leftrightarrow b = a \rightarrow b = a \sim b$.

2. Filters and congruences in equality algebras

Next, we investigate the filter theory of equality algebras.

Definition 3 Let $\mathcal{E} = \langle E, \sim, \wedge, \mathbf{1} \rangle$ be an equality algebra and $F \subseteq E$.

1. F is called a *deductive system* or *filter* of $\mathcal{E}$ if for all $a, b \in E$ we have

   (i) $\mathbf{1} \in F$,

   (ii) $a \in F,\ a \leq b \Rightarrow b \in F$,

   (iii) $a,\ a \sim b \in F \Rightarrow b \in F$,

   Denote $Fil(\mathcal{E})$ the set of all filters of $\mathcal{E}$. Clearly, $Fil(\mathcal{E})$ closed under arbitrary intersections and $\{\mathbf{1}\} \in Fil(\mathcal{E})$, so $\langle Fil(\mathcal{E}), \subseteq \rangle$ is a complete lattice. A filter F of an equality algebra $\mathcal{E}$ is *proper* if $F \neq E$. A proper filter F is called *maximal* if $F \subseteq G \subseteq E$ implies $F = G$ for all G proper filter of $\mathcal{E}$.

2. A subset Θ of $E \times E$ is called *congruence* of $\mathcal{E}$, if it is an equivalence relation on E and for all $a, a', b, b' \in E$ such that $(a, b), (a', b') \in \Theta$, it holds that

   (i) $(a \wedge a', b \wedge b') \in \Theta$,

   (ii) $(a \sim a', b \sim b') \in \Theta$.

   Denote $Con(\mathcal{E})$ the set of all congruences of $\mathcal{E}$.

3. For $F \in Fil(\mathcal{E})$ define the following relations on E:

$$(x, y) \in \Theta_{\overrightarrow{F}} \iff \{x \rightarrow y, y \rightarrow x\} \subseteq F,$$
$$(x, y) \in \Theta_F \iff x \sim y \in F.$$

The following proposition states that the set of filters of an equality algebra coincide with the set of (BCK-algebra) filters of its underlying BCK-algebra.

Proposition 3 *Let $\mathcal{E} = \langle E, \sim, \wedge, \mathbf{1}\rangle$ be an equality algebra. $F \in Fil(\mathcal{E})$ iff for all $a, b \in E$,*

(i') $\mathbf{1} \in F$,

(ii') $a, a \to b \in F \Rightarrow b \in F$

holds.

Proposition 4 *If $\mathcal{E}$ is an equality algebra and $F \in Fil(\mathcal{E})$ then $\Theta_F \in Con(\mathcal{E})$ and $\Theta_F = \Theta_{\overrightarrow{F}}$.*

Lemma 5 *For $\Theta \in Con(\mathcal{E})$ we have $(a, b) \in \Theta$ iff $(a \sim b, \mathbf{1}) \in \Theta$*

The next theorem establishes a connection between $Fil(\mathcal{E})$ and $Con(\mathcal{E})$.

Theorem 6 *Let $\mathcal{E} = \langle E, \sim, \wedge, \mathbf{1}\rangle$ be an equality algebra, $\Theta, \Psi \in Con(\mathcal{E})$, $F \in Fil(\mathcal{E})$. Then*

(a) $[\mathbf{1}]_\Theta \in Fil(\mathcal{E})$, *where* $[\mathbf{1}]_\Theta = \{a \mid (a, \mathbf{1}) \in \Theta\}$,

(b) $\Theta_{[\mathbf{1}]_\Theta} = \Theta$,

(c) $[\mathbf{1}]_{\Theta_F} = F$,

(d) **(1-regularity)** *if* $[\mathbf{1}]_\Theta = [\mathbf{1}]_\Psi$, *then* $\Theta = \Psi$.

Lemma 7 *The variety of equality algebras is congruence permutable and congruence distributive.*

Remark 8 Every variety in which (E3), (E4), and (b) (or (E7) and (E2) instead of (b)) holds is congruence permutable. The term $m(x, y, z) = ((x \sim y) \sim z) \wedge ((y \sim z) \sim x)$. testifies it.

Summing up, we have obtained that

Theorem 9 *The variety of equality algebras is a* $\mathbf{1}$*-regular, arithmetical variety.*

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