Nilpotent Minimum Fuzzy Description Logics

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Abstract

We define the fuzzy description logic $\mathcal{ALCH}_{\mathcal{NM}}$ based on the t-norm of Nilpotent Minimum and we prove its reduction to the crisp description logic $\mathcal{ALCH}$. Hence we prove decidability results for $\mathcal{ALCH}_{\mathcal{NM}}$.

Keywords: Fuzzy description logics, t-norm based fuzzy logics, Nilpotent Minimum.

1. Introduction

Description logics are studied in order to provide a logical formalism for semantic web and ontologies, in such a way to give a theoretical framework for dealing with structured representation of knowledge. For a comprehensive look on the subject we refer the reader to [4].

As a natural generalization of description logic, in order to deal with uncertainty, recently fuzzy description logics have been introduced. In particular, in the works by Straccia et al. (see [13], [12], [14] among others), a systematic generalization of description logics to the fuzzy case has been given, by considering different fuzzy logics based on different sets of connectives. In many cases a reduction of satisfiability of fuzzy description logics to the crisp case can be found, so that decidability results of fuzzy description logics follows from the corresponding one of the classical cases.

On this topic it is worth to mention the work [11] by Hajek, where the author investigates fuzzy description logic in the framework of predicate Basic logic. Following [11], in [9] the authors define a family of t-norm based description logics with truth constants and gave some results on completeness and decidability.

In this paper we shall define a fuzzy description logic based on Nilpotent Minimum and we shall give a reduction to a crisp description logic, following the same approach used for example in [12]. In the final section of remarks, we compare our approach with existing works in the literature.

Our choice for Nilpotent Minimum logic depends on the following facts:

- Like Gödel logic, the conjunction is modeled by a minimum-like connective, with the big advantage of being closed with respect to finite sets (unlike product, for example) and being always definable on every subset of $[0,1]$ (unlike Łukasiewicz logic);
- Unlike Gödel logic (and like Łukasiewicz logic), the negation is involutive and a number of useful laws are valid, like De Morgan law and interdefinability of quantifier;
- Recently many results have been given for Nilpotent Minimum Logic: in [15], [2], [6] the free algebras in the corresponding varieties are characterized, in [1] the spectral duality of the finitely presented algebra is investigated, in [10] the characterization of subvarieties is given and in [3] even the finitely additive measures on NM-algebras are studied.

Also note that the nilpotent minimum logic expanded by the Δ operator ($\Delta x = 1$ if $x = 1$, otherwise $\Delta x = 0$) is termwise equivalent to Gödel logic with extra involutive negation, that in turn can be seen as the closest approximation of the Zadeh fuzzy logic.

Due to the laws holding for nilpotent minimum logic and to some consideration on the set of truth values, we shall simplify the reduction given in [12] or in [13]. Please note that the description logic considered here are just case studies and that results similar to what presented in this paper could be found for different description logics more oriented to some direction of applications.

The Nilpotent Minimum t-norm $\odot$ was introduced by Fodor [8] as the first example of a left-continuous but non-continuous t-norm, and it is defined, for every $x, y \in [0,1]$, by

$$x \odot y = \begin{cases} \min\{x,y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

The propositional logic of Nilpotent Minimum, NM for short, is obtained by extending the monoidal t-norm based logic MTL, introduced by Esteva and Godo in [7], by the following axiom schemes:

$$\begin{array}{ll} \text{(WNM)} & \neg(\varphi \odot \psi) \vee ((\varphi \wedge \psi) \to (\varphi \odot \psi)), \\ \text{(INV)} & \neg\neg\varphi \to \varphi. \end{array}$$

We recall the reader that MTL and its extensions are based on the language having as primitive connectives the monoidal conjunction $\odot$, its residuum $\to$, the lattice conjunction $\wedge$, and the constant $\bot$. In each fixed standard $[0,1]$-semantics, $\odot$ is the t-norm, $\wedge$ is the minimum, and $\bot$ is 0. Usually derived connectives are $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$, the negation $\neg x = x \to \bot$, and the constant $\top = \neg\bot$. In each $[0,1]$-semantics $\vee$ is maximum and $\top$ is the constant 1.

The Lindenbaum algebras of the logic NM constitute the variety of NM-algebras, which in turn is generated by the standard NM-algebra $[0,1]_\odot = \langle [0,1], \odot, \rightarrow, \wedge, \perp \rangle$, where

$$x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ \max\{1-x, y\} & \text{otherwise.} \end{cases}$$

Note that the negation $\neg x = x \rightarrow 0$ in the standard NM-algebra is given by the standard involutive negation $\neg x = 1 - x$. It follows that all De Morgan laws involving $\wedge$ and $\vee$ hold.

This is one of the nice features of the nilpotent minimum. While the minimum connective gives rise to a crisp negation function (mapping every value greater than 0 to 1 and mapping 0 to 1), the nilpotent minimum is a good approximation of a minimum conjunction having $1-x$ as negation. Further, just as it happens for the minimum, the nilpotent minimum is closed with respect to finite sets and indeed every linearly ordered set can be equipped with a structure of NM-algebra (for more details see [1]).

We are going to construct a fuzzy description logic based on this connective, following the work done in [13], [12], [14] for other fuzzy logics. In particular, we shall exhibit a reduction of the nilpotent minimum description logic to a crisp description logic, in order to establish decidability results for the first one. In doing this, we describe how to reduce the finite-valued truth tables of nilpotent minimum connectives to truth tables of classical connectives.

2. Description logic $\mathcal{ALCH}$

The description logic we consider in this paper is $\mathcal{ALC}$ plus role hierarchies, i.e. $\mathcal{ALCH}$. This logic is expressive enough for our aims to show how nilpotent minimum logic can be of interest for this field of applications. $\mathcal{ALCH}$ is a fragment of classical predicate logic, with no function symbols and where only unary and binary predicate are considered. However usually in the context of description logics a special nomenclature is considered, that we describe here.

Sintax: Let $\mathbf{A}$, $\mathbf{R}_A$ and $\mathbf{I}$ be non empty disjoint sets whose elements are called *atomic concepts* (cfr. unary predicates), *abstract roles* (cfr. binary predicates) and *abstract individuals* (cfr. constants), respectively.

Concepts are inductively built from atomic concepts, from top and bottom concepts $\top$ and $\perp$ and from roles in the following way:

- If A, B are concepts than $A \sqcup B$, $A \sqcap B$ and $\neg A$ are concepts.
- If A is a concept and R is an abstract role, then $\forall R.A$ and $\exists R.A$ are concepts.

An $\mathcal{ALCH}$ *knowledge base* (KB) is a triple $\mathcal{K} = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$ made of a *TBox*, an *RBox* and an *ABox.*

A *TBox* $\mathcal{T}$ is a finite set of *concept inclusion axioms* $\langle C \sqsubseteq D \rangle$, where $C, D \in \mathbf{A}$ are concepts.

A *RBox* $\mathcal{R}$ is a finite set of *role inclusion axioms* $\langle R \sqsubseteq R' \rangle$, where $R, R' \in \mathbf{R}$ are roles.

An *Abox* $\mathcal{A}$ is a finite set of *concepts assertions axioms* and *roles assertion axioms* (assertions, in brief) having the form $\langle a : C \rangle$ and $\langle (a,b) : R \rangle$, where $a, b \in \mathbf{I}$ are individuals, $C \in \mathbf{A}$ is a concept and $R \in \mathbf{R}_A$ is an abstract role.

Semantics: An interpretation $\mathcal{I}$ is a pair $(\Delta^\mathcal{I}, \cdot^\mathcal{I})$ where $\Delta^\mathcal{I}$ (the domain of interpretation) is a nonempty set and $\cdot^\mathcal{I}$ is a function mapping:

- every abstract individual $a \in \mathbf{I}$ in an element $a^\mathcal{I} \in \Delta^\mathcal{I}$;
- every atomic concept $A \in \mathbf{A}$ in a subset of $\Delta^\mathcal{I}$, $\perp^\mathcal{I} = \emptyset$, $\top^\mathcal{I} = \Delta^\mathcal{I}$;
- every abstract role $R_A \in \mathbf{R}_A$ in a subset of $\Delta^\mathcal{I} \times \Delta^\mathcal{I}$.

The extension of interpretation to complex concepts is the usual one for conjunction, disjunction and negation. Quantifier are interpreted in the following way:

$$(\forall R.C)^\mathcal{I}(x) = \sup_{y \in \Delta^\mathcal{I}} (\neg R^\mathcal{I}(x,y) \vee C^\mathcal{I}(y))$$
$$(\exists R.C)^\mathcal{I}(x) = \inf_{y \in \Delta^\mathcal{I}} (R^\mathcal{I}(x,y) \wedge C^\mathcal{I}(y)).$$

An interpretation $\mathcal{I}$ *satisfies* $\langle a : C \rangle$ ($\langle (a,b) : R \rangle$) if and only if $a^\mathcal{I} \in C^\mathcal{I}$ ($(a^\mathcal{I}, b^\mathcal{I}) \in R^\mathcal{I}$).

The interpretation of the concept inclusion axioms is the following: $\mathcal{I}$ satisfies $C \sqsubseteq D$ if and only if $C^\mathcal{I} \subseteq D^\mathcal{I}$ and $\mathcal{I}$ satisfies a *concept definition* $C \equiv D$ iff $C^\mathcal{I} = D^\mathcal{I}$. Analogously for role axioms.

Example 2.1. Consider the concepts `rock`, `jazz`, `English`, `Japanese`, the role `play` and the individuals `Mirko`, `A`, `B` (we read `A` and `B` as names of music albums). Suppose we have the axioms $\langle$`A : rock`$\rangle$, $\langle$`B : jazz`$\rangle$, $\langle$`B : English`$\rangle$, $\langle$`(Mirko, A) : play`$\rangle$. Then every interpretation satisfying the above axioms also satisfies the assertion $\langle$`Mirko : play_EnglishRock`$\rangle$ where `play_EnglishRock` $\equiv \exists$`play.(English` $\sqcap$ `Rock)`.

3. Fuzzy description logic $\mathcal{ALCH}_{\mathcal{NM}}$

In our fuzzy extension of $\mathcal{ALCH}$, we fix a finite set of truth values

$$S_n = \left\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\right\}.$$

Further let $\bowtie \in \{\leq, \geq\}$. We are going to consider *signed* assertions, i.e. pairs of assertions and degree of truth.

Let $\mathbf{A}$, $\mathbf{R}_A$ and $\mathbf{I}$ be nonempty sets as before. Non-atomic concepts are defined as in the classical case.

The fuzzy $\mathcal{ALCH}$ *knowledge base* (fuzzy KB) $\mathcal{K} = \langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$ is made of a *fuzzy TBox* $\mathcal{T}$, *fuzzy RBox* $\mathcal{R}$ and a *fuzzy ABox* $\mathcal{A}$.

A *fuzzy TBox* $\mathcal{T}$ is a finite set of *signed fuzzy concept inclusion axioms* $\langle C \sqsubseteq D, \geq \alpha\rangle$ where C, D are concepts and $\alpha \in S_n$.

A *fuzzy RBox* $\mathcal{R}$ is a finite set of *signed fuzzy role inclusion axioms* $\langle R \sqsubseteq R', \geq \alpha\rangle$ where R, R' are roles and $\alpha \in S_n$.

A *fuzzy ABox* $\mathcal{A}$ is a finite set of *signed fuzzy concepts assertion axioms* and *signed fuzzy role assertion axioms* having the form $\langle a : C \bowtie, \alpha\rangle$ and $\langle (a, b) : R, \bowtie \alpha\rangle$, where $a, b \in \mathbf{I}$, C is a concept and R is a role.

A fuzzy interpretation $\mathcal{I}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ (the domain of interpretation) is a nonempty set and $\cdot^{\mathcal{I}}$ is a function mapping:

- every abstract individual $a \in \mathbf{I}$ in an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$;
- every atomic concept $A \in \mathbf{A}$ in a function from $\Delta^{\mathcal{I}}$ to S_n ($\bot^{\mathcal{I}}$ is the function identically equal to 0 and $\top^{\mathcal{I}}$ is the function identically equal to 1);
- every abstract role $R_A \in \mathbf{R}_A$ in a function from $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ to S_n.

The extension of interpretation to complex concepts is the following:

$$
\begin{aligned}
(C \sqcap D)^{\mathcal{I}}(x) &= C^{\mathcal{I}}(x) \odot D^{\mathcal{I}}(x)\\
(\neg C)^{\mathcal{I}}(x) &= \neg(C^{\mathcal{I}}(x))\\
(\forall R.C)^{\mathcal{I}}(x) &= \sup_{y\in\Delta^{\mathcal{I}}} (R^{\mathcal{I}}(x, y) \to C^{\mathcal{I}}(y))\\
(\exists R.C)^{\mathcal{I}}(x) &= \inf_{y\in\Delta^{\mathcal{I}}} (R^{\mathcal{I}}(x, y) \wedge C^{\mathcal{I}}(y)).
\end{aligned}
$$

Interpretation of axioms is the following:

- $\mathcal{I}$ satisfies $\langle (a : C), \bowtie \alpha\rangle$ if and only if $C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie \alpha$.
- $\mathcal{I}$ satisfies $\langle ((a, b) : R), \bowtie \alpha\rangle$ if and only if $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \bowtie \alpha$.
- $\mathcal{I}$ satisfies $C \sqsubseteq D \geq \alpha$ if and only if $\inf_{x\in\Delta^{\mathcal{I}}} C^{\mathcal{I}}(x) \to D^{\mathcal{I}}(x) \geq \alpha$.
- $\mathcal{I}$ satisfies $R \sqsubseteq R' \geq \alpha$ if and only if $\inf_{(x,y)\in(\Delta^{\mathcal{I}})^2} R^{\mathcal{I}}(x, y) \to R'^{\mathcal{I}}(x, y) \geq \alpha$;

where $\odot$ and $\to$ are the nilpotent minimum and the associated residuum. Note that the interpretation of connectives of Nilpotent Minimum logic is stable with respect to the finite set of values S_n.

Example 3.1. Suppose to have the following fuzzy interpretation of the axiom concepts in Example 2.1. The individuals are `A,B,C,D, Mirko` (we are using here the same symbols for the sintax and for its interpretation). The concepts `rock, jazz, English, Japanese` are interpreted in accordance with the following table:

	A	B	C	D
`rock`	0.8	0.5	0.2	0.8
`jazz`	0.2	0.5	0.8	0.5
`English`	0.8	0.2	0.5	0.2
`Japanese`	0.3	0.8	0.5	0.2

We can say that A is an album of Japanese rock with degree of 0.3, while B is an album of English rock with degree 0. The role `play` is interpreted by the following table:

`play`	A	B	C	D
`Mirko`	0.8	0.3	0.2	0.8

Then in this interpretation we have that Mirko makes rock music with degree 0.7, i.e. the degree of assertion $\langle \texttt{Mirko} : \forall \texttt{play.rock}\rangle$ is 0.7, while the degree of assertion $\langle \texttt{Mirko} : \forall \texttt{play.}(\texttt{rock} \wedge \texttt{English})$ is 0.2.

Note also that this interpretation satisfies the set $\{\langle (a, b) : \texttt{play} \geq 0.8\rangle, \langle a : \texttt{rock} \geq 0.8\rangle, \langle b : \texttt{English} \geq 0.8\rangle, \forall\texttt{play.rock} \geq 0.7\}$.

4. From fuzzy to crisp

In order to represent the information of a fuzzy KB of $\mathcal{ALCH}_{\mathcal{NM}}$ into a crisp KB of $\mathcal{ALCH}$ we have to create new concepts and roles. Let $A^{\mathcal{K}}$ e $R^{\mathcal{K}}$ be the set of atomic concepts and atomic roles of the fuzzy knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{R}, \mathcal{A}\rangle$.

For every $\alpha, \beta \in S_n$, and for every $A \in A^{\mathcal{K}}$ and $R \in R^{\mathcal{K}}$ we introduce two new atomic concepts $A_{\leq\alpha}$ and $A_{\geq\alpha}$ and two new atomic roles $R_{\leq\alpha}$ and $R_{\geq\alpha}$. Intuitively, $A_{\leq\alpha}$ represents the crisp concept satisfied by all those individuals who satisfy A with degree smaller or equal to α.

The semantics of these new concepts and roles is preserved by the following axioms: for every $\alpha \leq \beta$

$$
\begin{aligned}
A_{\geq\beta} \sqsubseteq A_{\geq\alpha} \quad & A_{\leq\alpha} \sqsubseteq A_{\leq\beta}\\
R_{\geq\beta} \sqsubseteq R_{\geq\alpha} \quad & R_{\leq\alpha} \sqsubseteq R_{\leq\beta}\,.
\end{aligned}
$$

We denote by $\mathcal{T}(n)$ and $\mathcal{R}(n)$ the corresponding set of axioms.

Example 4.1. Let $n = 4$ and consider the fuzzy assertion

$$\tau = \langle a : A \sqcap B \geqslant \frac{1}{2}\rangle.$$

Every model $\mathcal{I}$ of τ must satisfy $A^{\mathcal{I}}(a) \odot B^{\mathcal{I}}(a) \geqslant \frac{1}{2}$. By the properties of the t-norm, we have $A^{\mathcal{I}}(a) + B^{\mathcal{I}}(a) > 1$ and $\min\{A^{\mathcal{I}}(a), B^{\mathcal{I}}(a)\} \geqslant 1/2$. We do not know the degrees of $A^{\mathcal{I}}(a)$ and $B^{\mathcal{I}}(a)$ but since they belong to S_4 the only possibilities are those in table 1.

The crisp assertion $\langle a : (A_{\geqslant\frac{1}{2}} \wedge B_{\geqslant\frac{3}{4}}) \vee (A_{\geqslant\frac{1}{2}} \wedge B_{\geqslant 1}) \vee (A_{\geqslant\frac{3}{4}} \wedge B_{\geqslant\frac{1}{2}}) \vee (A_{\geqslant\frac{3}{4}} \wedge B_{\geqslant\frac{3}{4}}) \vee (A_{\geqslant\frac{3}{4}} \wedge B_{\geqslant 1}) \vee (A_{\geqslant 1} \wedge B_{\geqslant\frac{1}{2}}) \vee (A_{\geqslant 1} \wedge B_{\geqslant\frac{3}{4}}) \vee (A_{\geqslant 1} \wedge B_{\geqslant 1})\rangle$ is equivalent to the fuzzy one.

Note that such assertion can be simplified to $\langle a : (A_{\geqslant\frac{1}{2}} \wedge B_{\geqslant\frac{3}{4}}) \vee (A_{\geqslant\frac{3}{4}} \wedge B_{\geqslant\frac{1}{2}})\rangle$.

Translation to crisp: For every fuzzy concept C, truth value $\alpha > 0$ and $\bowtie \in \{\leq, \geq\}$ we construct its crisp translation $\rho(C, \bowtie \alpha)$ (that is a concept in $\mathcal{ALCH}$) as following

- If C is an atomic concept then $\rho(C \bowtie \alpha) = C_{\bowtie\alpha}$;

$A^{\mathcal{I}}(a)$	$B^{\mathcal{I}}(a)$	$(A \sqcap B)^{\mathcal{I}}(a)$
1/2	3/4	1/2
1/2	1	1/2
3/4	1/2	1/2
3/4	3/4	3/4
3/4	1	3/4
1	1/2	3/4
1	3/4	3/4
1	1	1

Table 1: Example 4.1

- $\rho(A \sqcap B, \geq \alpha) = \bigsqcup_{\beta_1,\beta_2} \rho(A, \geq \beta_1) \sqcap \rho(B, \geq \beta_2)$ where $\beta_1, \beta_2 \geq \alpha$ and $\beta_1 + \beta_2 > 1$;
- $\rho(A \sqcap B, \leq \alpha) = \rho(A, \leq \alpha) \sqcap \rho(B, \leq \alpha)$;
- $\rho(\neg A, \geq \alpha) = \rho(A, \leq 1 - \alpha)$ and $\rho(\neg A, \leq \alpha) = \rho(A, \geq 1 - \alpha)$;
- $\rho(\forall R.C \geq \alpha) = \bigsqcup_\gamma \forall \rho(R \geq \gamma).\rho(C \geq \gamma) \sqcup \forall(\rho(R \geq (1-\alpha)_+).\rho(C \geq \alpha)$, where $(1-\alpha)_+$ is the successor of $1 - \alpha$.
- If $\alpha < 1/2$, $\rho(\forall R.C \leq \alpha) = \exists \rho(R \geq 1 - \alpha).\rho(C \leq \alpha)$, otherwise, for $\alpha \geq 1/2$, $\rho(\forall R.C \leq \alpha) = \bigwedge \exists \rho(R \geq 1 - \beta_2).\rho(C \leq \beta_1)$, where $\beta_1, \beta_2 \leq \alpha$ and $\beta_1 \leq 1 - \beta_2$.

Once we have translated the concepts, we can translate the axioms in the following way. For every concepts C, D and roles R, R' we have

- $\kappa(\langle a : C \bowtie \alpha \rangle) = \langle a : \rho(C \bowtie \alpha) \rangle$;
- $\kappa(\langle (a,b) : R \bowtie \alpha \rangle) = \langle (a,b) : \rho(R \bowtie \alpha) \rangle$;
- $\kappa(\langle C \sqsubseteq D \geq \alpha \rangle) = \bigcup_\gamma \{\rho(C \geq \gamma) \sqsubseteq \rho(D \geq \gamma)\} \cup \{\rho(C \geq (1-\alpha)_+) \sqsubseteq \rho(D \geq \alpha)\}$, where $(1-\alpha)_+$ is the successor of $1 - \alpha$;
- $\kappa(\langle R \sqsubseteq R' \geq \alpha \rangle) = \bigcup_\gamma \{\rho(R \geq \gamma) \sqsubseteq \rho(R' \geq \gamma)\} \cup \{\rho(R \geq (1-\alpha)_+) \sqsubseteq \rho(R' \geq \alpha)\}$, where $(1-\alpha)_+$ is the successor of $1 - \alpha$.

Definition 4.1. Given a fuzzy KB $\mathcal{K} = (\mathcal{T}, \mathcal{R}, \mathcal{A})$ we set $crisp(\mathcal{K}) = (\kappa(\mathcal{T}) \cup \mathcal{T}(n), \kappa(\mathcal{R}) \cup \mathcal{R}(n), \kappa(\mathcal{A}))$ the crisp knowledge base where $\kappa(\mathcal{S}) = \{\kappa(\tau) \mid \tau \in \mathcal{S}\}$ for any $\mathcal{S} \in \{\mathcal{A}, \mathcal{R}, \mathcal{T}\}$.

Theorem 4.1. The fuzzy knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{R}, \mathcal{A},)$ is satisfiable if and only if $crisp(\mathcal{K})$ is satisfiable.

Proof. We give here a sketch of the proof. What we want to prove is essentially a reduction rule for every connective of Nilpotent Minimum logic. Indeed, we can start by proving that $\langle a : C \bowtie \alpha \rangle$ is satisfiable in $\mathcal{K}$ if and only if its translation $\rho(\langle a : C \bowtie \alpha \rangle)$ is satisfiable in $crisp(\mathcal{K})$. Let us proceed by structural induction:

- If C is an atomic concept, the claim trivially follows.
- If $C = A \sqcap B$ then $\langle a : C \geq \alpha \rangle$ is satisfiable iff there exists an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \geq \alpha$, iff $A^{\mathcal{I}} + B^{\mathcal{I}} > 1$ and $max(A^{\mathcal{I}}, B^{\mathcal{I}}) \geq \alpha$, if and only if there are real numbers $\beta_1, \beta_2 \geq \alpha$ such that $A^{\mathcal{I}} \geq \beta_1$, $B^{\mathcal{I}} \geq \beta_2$ and $\beta_1 + \beta_2 > 1$, iff $\rho(A \sqcap B \geq \alpha)$ is satisfiable.
- If $C = \forall R.D$ first of all note the following behaviour of nilpotent minimum implication (see Figure 1).
  - $A \to B \geq \alpha$ if and only if either $A \leq B$ or $1 - A \geq \alpha$ or $B \geq \alpha$;
  - $A \to B \leq \alpha$ with $\alpha < 1/2$ iff $1 - A \leq \alpha$ and $B \leq \alpha$;
  - $A \to B \leq \alpha$ with $\alpha \geq 1/2$ iff $1 - A \leq \beta_1$ and $B \leq \beta_2$ where $\beta_1, \beta_2 \leq \alpha$ and $\beta_1 + \beta_2 \leq 1$.

  The case $C = \forall R.D$ hence follows from the conditions above, by noticing that for any formula A and nilpotent minimum interpretation $\mathcal{I}$,

$$(\forall x A(x))^{\mathcal{I}} \leq \alpha \quad \Leftrightarrow \quad \min_x A^{\mathcal{I}}(x) \leq \alpha$$
$$\Leftrightarrow \quad \exists x A_{\leq \alpha}(x) \text{ is satisfiable.}$$

The other cases can be handled analogously. In order to prove that also the translation of concept inclusion and role inclusion axioms is correct, we simply note that it is based on the translation of implication connective. □

Decidability results hence follows from corresponding results for $\mathcal{ALCH}$.

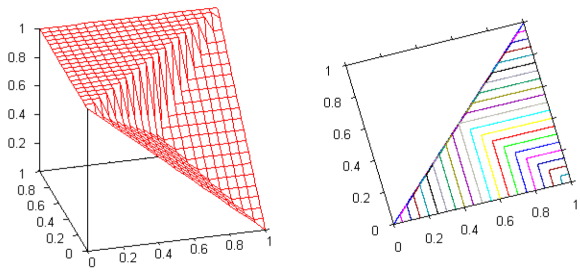

Figure 1: Nilpotent Minimum Implication.

Example 4.2. Let us consider the following fuzzy knowledge base:

$$\mathcal{K} = \{\langle a : \forall R.C \geq 0.75 \rangle, \langle (a,b) : R \geq 0.75 \rangle, \langle b : \neg C \geq 0.75 \rangle\}$$

with empty TBox and RBox, and suppose $n = 5$.

$\mathcal{K}$ is clearly unsatisfiable, since from the third assertion we have $C^{\mathcal{I}}(b^{\mathcal{I}}) \leq 0.25$ while from the first two we have that every fuzzy interpretation $\mathcal{I}$ must satisfy $C^{\mathcal{I}}(b^{\mathcal{I}}) \geq 0.75$, hence having a contradiction.

In order to compute the crisp representation of $\mathcal{K}$ we firstly need the new atomic concepts $C_{\leq \alpha}$ and $C_{\geq \alpha}$ and the new roles $R_{\geq \alpha}$, $R_{\leq \alpha}$ for every $\alpha \in S_5$. The new axioms are

- $T(5) = \{C_{\geq 1} \sqsubseteq C_{\geq 0.75},\ C_{\geq 0.75} \sqsubseteq C_{\geq 0.5},\ C_{\geq 0.5} \sqsubseteq C_{\geq 0.25},\ C_{\geq 0.25} \sqsubseteq C_{\geq 0},\ C_{\leq 0} \sqsubseteq C_{\leq 0.25},\ C_{\leq 0.25} \sqsubseteq C_{\leq 0.5},\ C_{\leq 0.5} \sqsubseteq C_{\leq 0.75},\ C_{\leq 0.75} \sqsubseteq C_{\leq 1}\}$;
- $R(5) = \{R_{\geq 1} \sqsubseteq R_{\geq 0.75},\ R_{\geq 0.75} \sqsubseteq R_{\geq 0.5},\ R_{\geq 0.5} \sqsubseteq R_{\geq 0.25},\ R_{\geq 0.25} \sqsubseteq R_{\geq 0},\ R_{\leq 0} \sqsubseteq R_{\leq 0.25},\ R_{\leq 0.25} \sqsubseteq R_{\leq 0.5},\ R_{\leq 0.5} \sqsubseteq R_{\leq 0.75},\ R_{\leq 0.75} \sqsubseteq R_{\leq 1}\}$.

We can hence compute $\kappa(\mathcal{K}, \mathcal{A})$:

- $\kappa(\langle (a,b) : R \geq 0.75 \rangle) = \langle (a,b) : \rho(R, \geq 0.75) \rangle = \langle (a,b) : R_{\geq 0.75} \rangle$
- $\kappa(\langle b : \neg C \geq 0.75 \rangle) = \langle b : \rho(\neg C, \geq 0.75) \rangle = \langle b : \rho(C, \leq 0.25) = b : \neg C_{>0.25} \rangle$
- $\kappa(\langle a : \forall R.C \geq 0.75 \rangle) = \langle a : \rho(\forall R.C \geq 0.75) \rangle = \langle a : \bigvee_\gamma \forall R_{\geq \gamma}.C_{\geq \gamma} \vee \forall R_{\geq 0.5}.C_{\geq 0.75} \rangle$.

It is easy to check that $crisp(\mathcal{K}) = \langle T(5), R(5), \kappa(\mathcal{K}, \mathcal{A}) \rangle$is unsatisfiable.

5. Conclusions

We described a fuzzy description logic based on the logic of nilpotent minimum, that, according to us, has many interesting properties and combine the simplicity of the minimum connectives with the usefulness of De Morgan laws. In our approach we followed the works [13], [12], [14] but of course there are many connections with [11] and [9]. Indeed the language of our logic can be considered a fragment of the language considered in [9], since we allow constants only as signs. This is equivalent to say that we have constants only in expressions like $\varphi \rightarrow \alpha$ or $\alpha \rightarrow \varphi$ where φ do not contain constants. Further, since the nilpotent minimum is not a continuous t-norm or a divisible t-norm, results of decidability in [11] and [9] do not directly apply, even if their adaptation to our case seems not to be difficult.

This work is very preliminary and many other considerations can be made on the subject:

- The rules that we described for the reduction of the fuzzy knowledge base to the crisp one, can be used (if suitable arranged) also for describing a tableaux calculus for deciding the fuzzy description logic based on Nilpotent Minimum;
- we considered the description logic $\mathcal{ALCH}$ but it would be interesting to investigate more complex description logics;
- as also stressed by Hajek in [11], the decidability results of classical description logic relies on the fact that it can be reduced to the two variable fragment of classical predicate calculus that is decidable. What about the two variable fragment of fuzzy predicate calculi?
- we dealt with finite valued logic: even if from a theoretical point of view it could interesting to see what happens in the case of infinite-valued logic, from the point of view of applications it would not bring any new insight, since when dealing with a knowledge base having necessarily a finite number of formulas, only a finite number of truth values can appear.

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