

# Design of Discrete Time Repetitive Control For Inversely Unstable Systems

Hammoud SAARI and Mohamed DJEMAI

**Abstract—** This paper deals with a discrete time repetitive control synthesis for possible non-minimum phase plants. The main design features of the repetitive controllers are discussed. More precisely one shows that one can realize two objectives: perfect and non-perfect tracking. Some illustrative examples are given to highlight the main features of the proposed approaches.

Keywords-Repetitive control; non-minimum phase; optimization

## I. INTRODUCTION

INDUSTRIAL processes make often repetitive or periodic tasks. Typical examples are industrial robots which perform repetitive or periodic tasks such as assembly, pick and place, painting,... etc. Repetitive control is an iterative approach that generates the process input so that the process output will be as close as possible to the desired output.

Repetitive control systems have been mainly studied in the last few years [1, 10 and 16]. Arimoto *et al.* [2] have proposed a continuous control algorithm based on the use of the past errors as a modification agent of the reference signal. Caron [4] has proposed a discrete control law based on the use of the previous controller output to modify the present control signal. This algorithm involves a classical closed loop system. It was shown for both algorithms that the asymptotic repetitive control inverts the process and hence the tracking error goes to zero, which seems to be difficult when the process contains unstable zeros [5, 6].

Much work has been done on model inversion for non-minimum phase systems [3, 8]. Among them, one mentions the work of Tomizuka [7, 14]. The design of the controller is based on the zero phase compensator where the dynamics of unstable zeros is compensated. Another form of this controller is proposed by Landau [9], where the unstable zeros and the delay are compensated.

Based on these studies, Tomizuka *et al.* [15] and Saari *et al.* [12] have developed other approaches of repetitive controller for non-minimum phase process based on the internal model structure. In this case, the asymptotic control signal inverts the process and hence the asymptotic tracking error is equal to zero. To overcome the process inversion,

promising approaches have been developed. Indeed, Moore *et al.* [11] and Saari *et al.* [12] have formulated the repetitive control objective as an optimization problem. In this case, the control signal does not invert the process and hence, the errors will not tend to zero.

Otherwise, there has been recently increasing interest in the connection between the control and the identification steps involved in a control systems design. In [13], a repetitive controller design taking into account both the control objectives and the model identification is presented. An adequate input-output identification filter is designed to minimize the difference between the nominal and the actual repetitive convergence conditions.

In this paper, we emphasize the main design features concerning the discrete time repetitive control for inversely unstable systems. The desired trajectory is supposed to be known and repetitive. The algorithm involves a classical closed loop system already stabilized by a simple controller. Repetitive control is based on the use of the previous controller input and output signals in order to modify the present control signal. More specifically, it is shown that one can realize two objectives; perfect or non-perfect tracking. This is obtained by the cancellation or not of unstable zeros of the process.

The paper is organized as follows: In section 2, the problem that we address is formulated. The repetitive control algorithm for inversely unstable systems is derived in section 3. Simulation and experimental results are provided in section 4 and 5. Finally, the paper is concluded with some remarks in section 6.

## II. PROBLEM FORMULATION

Consider the linear discrete time SISO system, described by the following transfer function:

$$G(z^{-1}) = \frac{z^{-d} B(z^{-1})}{A(z^{-1})} \quad (1)$$

$$\text{with: } \begin{cases} B(z^{-1}) = b_0 + b_1 z^{-1} + \dots + b_m z^{-m} \\ A(z^{-1}) = 1 + a_1 z^{-1} + \dots + a_n z^{-n} \end{cases}$$

where  $z$  is the Z-transform complex variable and  $d$  is the number of delay steps. The numerator  $B(z^{-1})$  can be factorized as  $B(z^{-1}) = B^+(z^{-1})B^-(z^{-1})$ , where  $B^+(z^{-1})$  of order  $m^+$  and  $B^-(z^{-1})$  of order  $m^-$  are respectively the inversely stable and inversely unstable parts of  $B(z^{-1})$ . In

Dr. Hammoud SAARI is with SETRAM, Ecole Nationale Supérieure Maritime, BP. 61, 42415 Bou-Ismaïl, ALGERIA, Email: hammoud.saari@yahoo.fr

Prof. M. DJEMAI is with Univ. Lille Nord de France, F-59000 Lille, France. UVHC, LAMIH, CNRS, UMR-8201, Campus du Mont Houy F-59313 Valenciennes, France. Email: Mohamed.Djemai@univ-valenciennes.fr

the sequel the operator  $z^{-1}$  will be omitted for the sake of simplification.

We will particularly consider the closed loop configuration of Fig. 1.

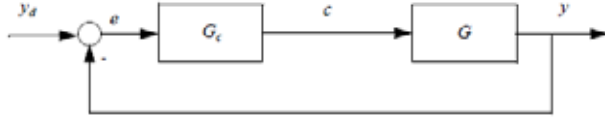


Figure 1. Closed loop control.

Here  $y_d$  is the reference signal and  $G_c$  is an a priori known controller that is designed to stabilize the considered system and to make the output  $y$  closer to the desired trajectory  $y_d$ .

It is clear, in the above control system that the output tracking will not be satisfactory due to two main reasons which are unavoidable in practice: the disturbances and the modelling errors. Furthermore, when the desired trajectory is repetitive or periodic, the control system will make the same errors. In fact, the control system does not take into account the errors made in the previous periods. Hence, it will be interesting to use all the information, obtained in the previous periods, in the actual control system to improve the output tracking.

Among that information, we will particularly use the previous output tracking errors and the control signal as shown in Fig. 2.

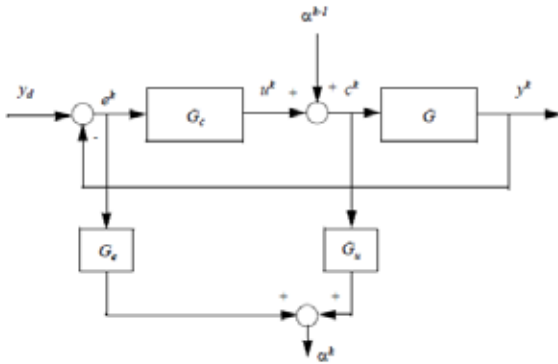


Figure 2. Repetitive control algorithm.

$k$  refers to the number of the period.  $y^k$ ,  $c^k$ , and  $e^k$  are respectively the output, the control and the tracking error signals at the  $k$ th period.  $\alpha^k$  is an anticipation signal obtained by filtering  $e^k$  and  $c^k$  by  $G_e$  and  $G_u$  respectively and will be applied at the next period ( $k+1$ ).

The design objective is to choose these two filters such that the asymptotic tracking error ( $k \rightarrow \infty$ ) goes to zero.

### III. REPETITIVE CONTROL

From Fig. 2, one can see that the repetitive control law is given by:

$$c^{k+1} = G_c(y_d - y^{k+1}) + G_u c^k + G_e(y_d - y^k) \quad (2)$$

$$\Rightarrow c^{k+1} = \frac{G_u - G_e G}{1 + G G_c} c^k + \frac{G_e + G_c}{1 + G G_c} y_d \quad (3)$$

$$\text{Let } D = \frac{G_u - G_e G}{1 + G G_c} \text{ and } F = \frac{G_e + G_c}{1 + G G_c}.$$

This implies

$$c^{k+1} = D c^k + F y_d \quad (4)$$

By developing the recurrence, one obtains

$$c^{k+1} = F(1 + D + \dots + D^k) y_d + D^{k+1} c^0 \quad (5)$$

Note that if  $\|D\|_\infty < 1$  i.e.:

$$\left\| \frac{G_u - G_e G}{1 + G G_c} \right\|_\infty < 1 \quad (6)$$

$$\text{then } c^\infty = \lim_{k \rightarrow \infty} c^k = \frac{F}{1 - D} y_d \quad (7)$$

Thus

$$c^\infty = [(G_e + G_c)G + 1 - G_u]^{-1} (G_e + G_c) y_d \quad (8)$$

and the asymptotic output tracking error becomes:

$$e^\infty = \lim_{k \rightarrow \infty} (y_d - y^k) = [1 - G[(G_e + G_c)G + 1 - G_u]^{-1} (G_e + G_c)] y_d \quad (9)$$

Two cases can be distinguished depending on the filter  $G_u$ .

#### A. Perfect tracking

If the control filter  $G_u$  is unity ( $G_u = 1$ ), from (8), the control signal  $c^\infty$  becomes:

$$c^\infty = \frac{1}{G} y_d \quad (10)$$

It follows that

$$e^\infty = \lim_{k \rightarrow \infty} (y_d - y^k) = 0 \quad (11)$$

So the asymptotic tracking error is equal to zero. This convergence is ensured if the stability condition of the repetitive algorithm (6) is satisfied.

It follows from (10) that the control signal after an infinity number of period inverts the process, which seems to be impossible when the plant to be controlled exhibits unstable zeros. However, since  $y_d$  is a known signal, it will be possible to generate the control sequence such that the plant output follows  $y_d$ , even if the plant contains unstable zeros [9, 11].

In [14], it is shown that to satisfy the repetitive control convergence condition, the repetitive controller  $G_e(z^{-1})$  will contain the inverse of the process. The question is then, what can we do when the process contains unstable zeros? For this, one distinguishes three types of repetitive controllers [12]:

#### 1) Complete reverser controller

In this case, the repetitive controller  $G_e(z^{-1})$  is given by:

$$G_e(z^{-1}) = k_e \frac{z^d A(z^{-1}) B^-(z)}{b \cdot B^+(z^{-1})} \quad (12)$$

where  $b \geq \max_{\omega \in [0, \pi]} |B^-(e^{-j\omega})|^2$ ,  $k_e$  is the repetitive control gain and  $B^-(z)$  is obtained by replacing every  $z^{-l}$  in  $B^-(z^{-1})$  by  $z$ . The terms  $z^d$  and  $B^-(z)$  allow to realize a maximum advance equal to the number of unstable zeros plus the delay. The controller, in this case, uses the future input data to compute the output for the following period. This controller compensates also the poles and the cancellable zeros. The repetitive control condition is shown in [12].

#### 2) Partial reverser controller

In this case, the repetitive controller is derived from [9], and is given by:

$$G_e(z^{-1}) = k_e \frac{z^{d+m} A(z^{-1})}{b \cdot B^+(z^{-1})} \quad \text{where } b = B^-(1) \quad (13)$$

Note that, as in the previous case, one realizes an advance of  $d+m$  in order to compensate the delay and the unstable zeros. The repetitive control condition is shown in [12].

#### 3) Simple anticipative controller

In both previous cases, one introduces, in the repetitive controller  $G_e(z^{-1})$ , an advance equal to the number of the delays plus the number of unstable zeros. One introduces also the poles and the cancellable zeros to compensate them. The last cancellation can be avoided, because it is not necessary to incorporate complicated expressions in the repetitive controller when it is enough to compensate only the delay and the unstable zeros. The repetitive controller is then:

$$G_e(z^{-1}) = R(z^{-1}) \cdot z^{d+m} \quad (14)$$

where  $R(z^{-1})$  is a rational fraction, as simple as possible added in order to respect the repetitive control convergence condition [12].

#### 4) Comparison

Two remarks concerning these repetitive controllers can be made. First, they give quite the same asymptotic error. So, there is no difference between them from performance point of view. Second, the third controller is simpler than the others. In fact, it is not necessary to know exactly the process for designing the controller but it is sufficient to know the delay and the number of unstable zeros.

#### B. Non perfect tracking

As it appears from (9) when ( $G_u \neq 1$ ), the asymptotic error does not have to be equal to zero. The task here is then to choose the filters  $G_e$  and  $G_u$  such that the norm of the final error is minimized. Note that the convergence condition, in

this case,  $\left\| \frac{G_u - G_e G}{1 + G G_c} \right\|_{\infty} < 1$  is much less restrictive than

$\left\| \frac{1 - G_e G}{1 + G G_c} \right\|_{\infty} < 1$  because we have the freedom to pick  $G_u$ .

Following the approach proposed in [11], the repetitive control algorithm can be cast as the following minimization problem [12]:

**Problem P1:** Given the desired trajectory  $y_d$ , the plant and the controller transfer functions  $G$  and  $G_c$ , we have to find the filters  $G_e^*$  and  $G_u^*$  to minimize the total energy of the error signal  $e^{\infty}(t)$ :

$$\min_{G_e, G_u} \left( \sum_t [e^{\infty}(t)]^2 \right)^{1/2} = \min_{G_e, G_u} \|e^{\infty}(t)\|_2 \quad (15)$$

which is equivalent to :

$$\min_{G_e, G_u} \left\| [1 - G[(G_e + G_c)G + 1 - G_u] \times (G_e + G_c)] y_d(t) \right\|_2 \quad (16)$$

subject to :  $\left\| \frac{G_u - G_e G}{1 + G G_c} \right\|_{\infty} < 1$

The solution of *P1* will give the repetitive control algorithm that produces the smallest final error energy. To state the solution of *P1*, let us introduce the following problem:

**Problem P2:** Given the desired trajectory  $y_d$  and the plant transfer function  $G$ , find the filter  $H^*$  to solve:

$$\min_H \|(1 - GH)y_d(t)\|_2 \quad (17)$$

The following theorem relates the solution of problems *P1* and *P2*.

**Theorem: (For the proof see [12]).**

Let  $H^*$  be the solution of P2 and  $G_c^*$  defined by the factorization:

$$H^* = T^* \cdot (G_c^* + G_c) \quad (18)$$

where  $T^*$  is an invertible filter to be determined which satisfy

$$\left\| 1 - \frac{(T^*)^{-1}}{1 + GG_c} \right\|_{\infty} < 1.$$

and let :

$$G_u^* = 1 - (T^*)^{-1} + G(G_c^* + G_c) \quad (19)$$

Then  $G_c^*$  and  $G_u^*$  are solutions of P1. ■

There are three remarks that can be made about this theorem. Firstly, to solve P1, we simply solve P2 which is equivalent to find an approximate inverse of the plant transfer function. Secondly, the factorization given in the theorem is not unique and hence several solutions of P1 may exist. Finally, we have formulated this problem for a fixed input  $y_d$ . If we want to solve the problem for any reference input signal, then we have to minimize the ratio of the error signal energy to the reference input energy. Then P1 becomes:

**Problem P1\*:** Given the plant and the controller transfer functions  $G$  and  $G_c$ , find the filters  $G_c^*$  and  $G_u^*$  to minimize the ratio of the final error signal energy to any non-zero reference signal energy [12]:

$$\min_{G_c, G_c^*} \left\{ \sup_{y_d(t) \neq 0} \left( \frac{\sum_t [e^{\infty}(t)]^2}{\sum_t [y_d(t)]^2} \right)^{1/2} \right\} \quad (20)$$

which is equivalent to:

$$\min_{G_c, G_c^*} \left\| 1 - G[(G_c + G_c^*)G + 1 - G_u^*]^{-1} (G_c + G_c^*) \right\|_{\infty} \quad (21)$$

$$\text{subject to : } \left\| \frac{G_u - G_c G}{1 + GG_c} \right\|_{\infty} < 1$$

As in the previous case, one can show that solving P1\* is equivalent to solve the following problem:

**Problem P2\*:** Given the plant transfer function  $G$ . Find the filter  $H^*$  to solve:

$$\min_H \left\| 1 - GH \right\|_{\infty} \quad (22)$$

### C. Proposed solution and convergence analysis

We have seen that to solve P1\*, it is sufficient to solve P2\* which is equivalent to find an approximate inverse of the plant transfer function. From (23) it is clear that the approximate inverse of this transfer function has the form [9]:

$$H^*(z^{-1}) = \frac{z^{d+m^-} A(z^{-1})}{B^-(1)B^+(z^{-1})} \quad (23)$$

Notice that  $H^*$  compensates the stable parts of  $G$  and keeps the unstable part of  $G$ . We have previously seen that any solution must satisfy the convergence condition:

$$\left\| \frac{G_u - G_c G}{1 + GG_c} \right\|_{\infty} < 1 \quad (24)$$

Taking into account (18) and (19), the convergence condition (24) becomes:

$$\left\| 1 - \frac{(T^*)^{-1}}{1 + GG_c} \right\|_{\infty} < 1 \quad (25)$$

Let  $(M, \varphi)$  and  $(\Gamma, \theta)$  be respectively the gain and the phase of  $GG_c(e^{-j\omega})$  and  $(T^*(e^{-j\omega}))^{-1}$ . The convergence condition (25) leads to:

$$[\Gamma - 2 \cos \theta - 2M \cos(\varphi - \theta)]_{\omega \in [0, \pi]} < 0 \quad (26)$$

If the filter  $T^*$  is chosen to be constant, then (26) becomes:

$$0 < \Gamma < \min_{\omega \in [0, \pi]} [2(1 + M \cos \varphi)] = f(\omega) \quad (27)$$

Hence, the repetitive control algorithm can be applied if the filter  $T^*$  satisfies (27).

## IV. SIMULATIONS RESULTS

To illustrate the features of the perfect tracking algorithm, let us take a simple example. The process to be controlled is given by:

$$G(z^{-1}) = \frac{z^{-1}(0.05 + 0.09z^{-1})}{1 - 0.3z^{-1}}$$

This is a first order transfer function with an unstable zero ( $z = -1.8$ ). The controller  $G_c(z^{-1})$  is set to 1.  $G_c(z^{-1})$  is chosen as a simple anticipative controller such that the convergence condition (6) is satisfied :

$$G_c(z^{-1}) = 5z^2$$

The reference input is shown in Fig. 3. Fig. 4 shows the error energy versus the number of period. It is clear that it tends to zero. Fig. 5 shows the control signal after 30 periods. Note that in spite that the signal is finite, there are large oscillations near the discontinuities that can damage the actuator in real applications. To avoid these problems, we propose to apply the non-perfect tracking algorithm.

To illustrate the features of the non-perfect tracking repetitive algorithm with regard to the previous one (perfect tracking algorithm), let us take the same example.

Fig. 6 shows the behaviour of  $f(\omega)$  (27). One can see that to ensure the convergence condition, we must take  $\Gamma < 1.8044$ . In our case we have chosen  $\Gamma = 1$  which

correspond to a filter  $T^* = 1$ .

From (23),  $H^*$  is given by: 
$$H^*(z^{-1}) = \frac{z^2(1-0.3z^{-1})}{0.14}$$

Hence: 
$$G_c^* = -1 - 2.14z - 7.14z^2$$
  

$$G_u^* = 0.64 + 0.36z$$

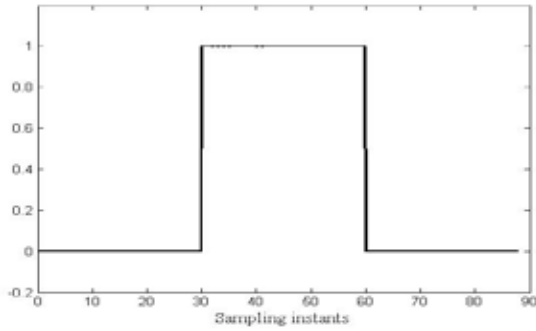


Figure 3. Reference input

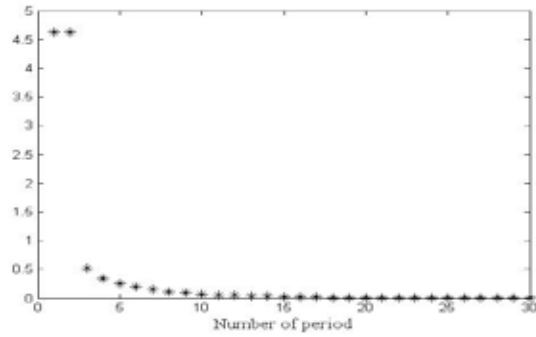


Figure 4.  $\|e\|_2$  behaviour versus the number of period (Perfect tracking)

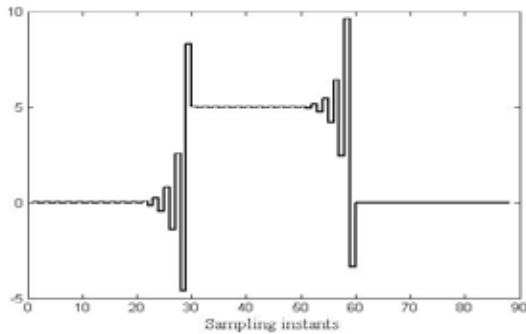


Figure 5. Control behaviour at the 30-th period (Perfect tracking)

Fig. 7 shows the behaviour of the error energy versus the number of periods. It does not tend to zero. Fig. 8 shows the control behaviour after 30 periods. One can see that the control signal does not present any oscillations near the discontinuities as in the previous case.

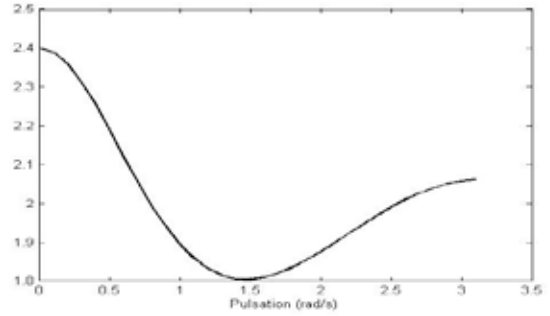


Figure 6. Behaviour of  $f(\omega)$  (Non perfect tracking)

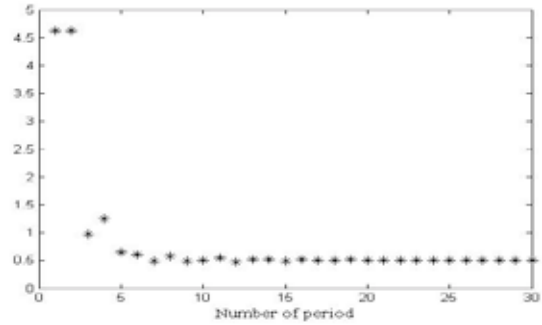


Figure 7.  $\|e\|_2$  behaviour versus the number of period (Non perfect tracking)

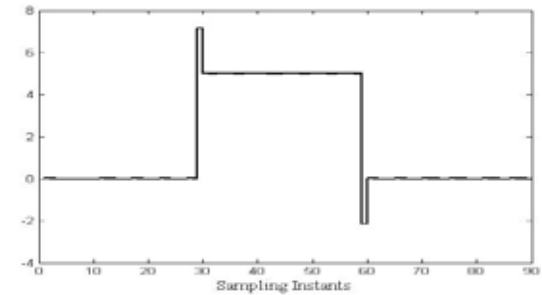


Figure 8. Control behaviour at the 30-th period (Non perfect tracking)

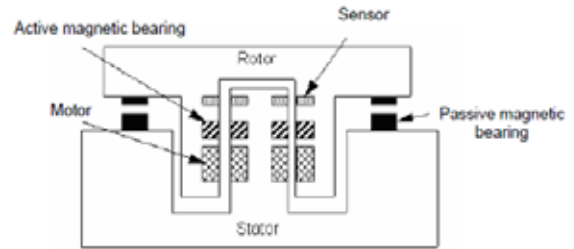


Figure. 9 Scheme of the inertial wheel.

## V. APPLICATION EXAMPLE

This section concludes with an application of the proposed control algorithm to magnetic bearings [6, 12], which is a device made of two main parts: an inertial wheel (rotor) and a stator (Fig. 9).

The guiding forces between the fixed part and the moving part are magnetic: the vertical sustentation is ensured by the passive magnetic bearing and the positioning in the horizontal plan is mainly due to two active magnetic bearings. When the rotor turns at high speed, there is an unbalanced movement of the inertial wheel induced by the non-concordance between the geometric and inertial centers. This negative effect produces a repetitive disturbance that has to be rejected.

The discrete time plant model is obtained for a particular polarization point around which the system is considered to be linear with a sampling period of  $T=0.8\text{ ms}$ . It is given by:

$$G(z^{-1}) = -9.29 \cdot 10^{-9} \frac{z^{-1}(1 + 4.22z^{-1})(1 + 0.25z^{-1})}{(1 - 1.03z^{-1})(1 - 0.97z^{-1})(1 - 0.89z^{-1})}$$

The stabilizing controller is then:

$$G_c(z^{-1}) = -10 \frac{(1 - 0.984z^{-1})}{(1 - 0.584z^{-1})}$$

The results obtained for this application are given on Fig. 10. They show that the repetitive algorithm is able to improve the centring of the inertial wheel. Fig. 11 gives the evolution of the peak-to-peak error during the ten periods of trial where the repetitive algorithm has been applied.

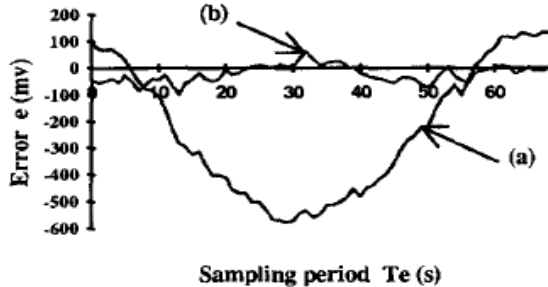


Figure 10. Behaviour of the error positioning (1 turn): (a) without repetitive algorithm; (b) with repetitive algorithm after the 10th turn.

## VI. CONCLUSION

In this paper a treatment of repetitive control in a closed loop configuration was studied. A rather complete study was made for the synthesis of the repetitive controllers. We were especially interested for systems with unstable zeros. Two objectives were realized: perfect and non perfect tracking. The obtained results from the application of this kind of control algorithm to magnetic bearings are promising.

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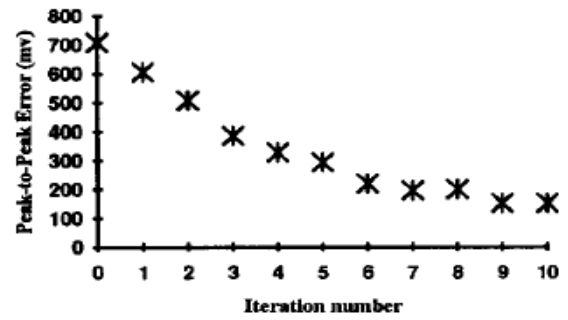


Figure 11. Evolution of the peak-to-peak error on ten periods.

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