Stability and Stabilization by output feedback control of positive Takagi-Sugeno fuzzy discrete-time systems with delay

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Abstract— **This paper deals with the problem of stabilization by output feedback control of Takagi-Sugeno (T-S) fuzzy discrete-time systems with a fixed delay by linear programming (LP) and cone complementarity while imposing positivity in closed-loop. The stabilization conditions are derived using the single Lyapunov-Krasovskii Functional (LKF). An example of a real plant is studied to show the advantages of the design procedure.**

Key-words: T-S fuzzy discrete-time systems, positive systems, Lyapunov-Krasovskii functional, stabilization, Linear programming, output feedback.

I. Introduction

The problem concerns a special class of nonlinear systems called Takagi-Sugeno models (T-S) [6]. From the historical of the approach, this class can be interpreted as a collection of linear models interconnected by nonlinear functions, called membership functions, which are dependent variables. The most delicate problem is the choice of premise variables that partition the space [5], [7].

Positive systems have been of great interest by researchers in recent years [8], [1], [3], [4] and [9]. The class of positive T-S fuzzy systems has been considered for the first time in [2]. The obtained results have been presented by using LMIs. In this paper, we study the conditions of stability and stabilization by output feedback control, while imposing positivity in closed-loop, of such discrete-time systems by using linear programming (LP). An application on the model of a real process is considered. A comparison between a cone complementarity approach and linear programming with imposing conditions of positivity in closed-loop, The rest of this paper is organized as follows: In section 2, we give the description of T-S fuzzy models with fixed state delay and the fuzzy control law based on the PDC structure. New delay independent stabilization conditions are established for positive systems in section 3. In section 4, an example of a real plant is given to show the need of such controllers. Some conclusions are given in section 5.

Notation:

- M^T denotes the transpose of a real matrix M.
- For a square matrix $Q \geq 0$, if $Q \in \mathbb{R}^{n*n}$ is positive definite.
- $A \succ 0$ stands for a positive matrix A, that is with nonegative elements: $a_{ij} \succ 0$.

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II. Problem formulation and preliminary results

Specifically, the Takagi-Sugeno fuzzy system is described by fuzzy IF-THEN rules, which locally represent linear input-output relations of a system. The fuzzy system is of the following form:

Rule i: IF $z_1(k)$ is F_i^1 and $\cdots$ and $z_p(k)$ is F_i^p Then:

$$x(k+1) = A_i\ x(k) + A_{i1}\ x(k-\tau) + B_i\ u(k) \quad (1)$$
$$x(k) = \Psi(k) \succ 0, k \in [-\tau, 0] \quad (2)$$
$$y(k) = C_i\ x(k) \quad (3)$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control input, $y(k) \in \mathbb{R}^l$ is the output, $A_i \in \mathbb{R}^{n*n}$, $B_i \in \mathbb{R}^{n*m}$, $C_i \in \mathbb{R}^{l*n}$, τ is a fixed delay, with $i = 1, 2, ..., r$, r is the number of IF-THEN rules, $z_1(k) \cdots z_p(k)$ and F_i^j are respectively the premise variable and the fuzzy sets.

The control law is chosen to be an output feedback one given by:

$$u(k) = K_i\ y(k), \quad (4)$$

The global system will be represented by T-S fuzzy models described by:

$$x(k+1) = \sum_{i=1}^{r} h_i(z(k))(A_i\ x(k) + A_{i1}\ x(k-\tau) + B_i\ u(k)) \quad (5)$$

The used control in this work is the so called PDC control:

$$u(k) = \sum_{i=1}^{r} h_i(z(k))K_i\ y(k), \quad (6)$$

where $h_i(z(k)) = \frac{w_i(z(k))}{\sum_{i=1}^{r} w_i(z(k))}$; $w_i(k) = \prod_{j=1}^{p} F_i^j(z(k))$,

with $h_i(z(k)) \succeq 0$; $\forall t \succ 0$; $\sum_{i=1}^{r} h_i(z(k)) = 1$,

$i = 1, 2, ..., r$ and $j = 1, 2, ..., p$.

By using (6), the closed-loop system (5) is then written as:

$$x(k+1) = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{s=1}^{r} h_i(z(k))h_j(z(k))h_s(z(k))\,[(A_i + B_iK_jC_s\ )\ x(k) + A_{i1}\ x(k-\tau)]$$

If the matrix C_s is common to all subsystems, the system described above will be:

$$x(k+1) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(k))h_j(z(k))\,[(A_i + B_iK_jC\ )\ x(k) + A_{i1}\ x(k-\tau)] \quad (7)$$
$$x(k) = \Psi(k) \succ 0, k \in [-\tau, 0]$$

The aim of this work is to present new sufficient conditions of stabilizability by output feedback control allowing the state to be always nonnegative for discrete-time fuzzy systems with simple fixed delays.

Assumptions:

H_1 : Matrices A_i, A_{i1} and B_i are positive.

H_2 : The matrix C_i is common to all subsystems: $C_1 = C_2 = \ldots = C_r = C$.

Definition 1: The T-S fuzzy system (5) is said to be controlled positive if, given any nonnegative initial state and any input function $u(k) \succ 0$, the corresponding trajectory remains in the positive orthant for all k: $x(k) \in R^n_+$.

Lemma 1: [9] The autonomous delayed system (5) is positive if and only if, given any nonnegative initial state, A_i and A_{i1} are a nonnegative matrix for $i = 1, ..., r$.

Now, the conditions of stability and stabilization of T-S fuzzy system (5) by using cone complementarity method as presented in [11] are recalled while adding the positivity conditions in closed-loop.

Theorem 1: The autonomous system (5) is asymptotically stable, if there exist a matrices $P = P^T \geq 0$, $Q = Q^T \geq 0$ and $R = R^T \geq 0$ such that the optimum of the following optimization problem is achievable and is equal to $2n$ for $i = 1, 2, ..., r$:

$$: \begin{cases} minimize\ Tr(PQ) \\ subject\ to: \\ \begin{pmatrix} P-R & 0 & A_i^T \\ * & R & A_{i1}^T \\ * & * & Q \end{pmatrix} \succ 0; \\ \begin{pmatrix} P & I \\ I & Q \end{pmatrix} \succ 0; \end{cases}$$

Theorem 2: If there exist a matrices $P = P^T \geq 0$, $Q = Q^T \geq 0$ and $R = R^T \geq 0$ such that the optimum of the following optimization problem is achievable and is equal to $2n$ for $i, j = 1, 2, ..., r$:

$$\begin{cases} minimize\ Tr(PQ) \\ subject\ to: \\ \begin{pmatrix} P-R & 0 & A_i^T + C^TK_j^TB_i^T \\ * & R & A_{i1}^T \\ * & * & Q \end{pmatrix} \succ 0\ ; \\ \begin{pmatrix} P & I \\ I & Q \end{pmatrix} \succ 0; \\ K_{ef}^j \succ 0\ ;\ e = 1, ..., m\ ;\ f = 1, ..., l \end{cases}$$

Then system (7) is asymptotically stable and positive.

The algorithm to solve the optimization described above, was made in [10].

To establish these conditions, the following Lyapunov-Krasovskii functional was used:

$$V(x(k)) = x(k)^TPx(k) + \sum_{d=1}^{\tau} x(k-d)^TRx(k-d) \quad (8)$$

Note that these results present a special case of the ones given by [11], adding only the positivity conditions.

III. Main results

This section concerns the study of conditions of stability and stabilization of fuzzy system (5) by using a linear program (LP) method.

Knowing that the dual system (5) is asymptotically stable, if and only if the system (5) is asymptotically stable, then we simply use the stability of the dual system.

Theorem 3: The autonomous system (5) is asymptotically stable for all $\tau \succ 0$ if there exist vectors $\lambda \in R^n$, $\lambda_j \in R^n$ / $j = 1, ..., r$; satisfying the following LPs:

$$\begin{cases} \left[\sum_{i=1}^{r}\sum_{j=1}^{r}(A_i + A_{i1})\lambda_j\right] - \lambda \prec 0\ ; \\ 0 \prec \lambda \prec \lambda_j;\quad j = 1, ..., r \end{cases}$$

Proof 1: The choice of the Lyapunov-Krasovskii functional in this case will be:

$V(x(k)) = x^T(k)\lambda + \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{d=1}^{\tau} x^T(k-d)A_{i1}\lambda_j\ ;\ \lambda \succ 0\ ;\ \lambda_j \succ 0\ ;\ \lambda \prec \lambda_j$

As noted above, we can deal with the stability of the autonomous dual system of (5) given by:

$$x(k+1) = \sum_{i=1}^{r} h_i(z(k))(A_i^T\ x(k) + A_{i1}^T\ x(k-\tau)) \quad (9)$$

The rate of increase of the Lyapunov-Krasovskii functional is:

$$\begin{aligned}\Delta V(x(k)) &= x^T(k+1)\lambda - x^T(k)\lambda \\ &+ \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{d=1}^{\tau} x^T(k+1-d)A_{i1}\lambda_j \\ &- \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{d=1}^{\tau} x^T(k-d)A_{i1}\lambda_j \qquad (10)\end{aligned}$$

Replacing the $x^T(k+1)$ by its expression of the autonomous dual system (9), the rate of increase of the functional will be of the form:

$$\Delta V(x(k)) = \sum_{i=1}^{r} h_i(z(k)) \left[x^T(k)A_i + x^T(k-\tau)A_{i1}\right]\lambda$$
$$+\sum_{i=1}^{r}\sum_{j=1}^{r}\left[x^T(k)A_{i1} - x^T(k-\tau)A_{i1}\right]\lambda_j - x^T(k)\lambda$$

As $0 \preceq h_i(z(k)) \preceq 1$, $A_{i1} \succ 0$, $x(k-\tau) \succeq 0$, $A_i \succ 0$, $x(k) \succeq 0$, $\lambda \prec \lambda_j \prec \sum_{j=1}^{r}\lambda_j$ it follows:

$$\sum_{i=1}^{r} h_i(z(k)) \left[x^T(k)A_i + x^T(k-\tau)A_{i1}\right]\lambda \prec$$
$$\sum_{i=1}^{r}\sum_{j=1}^{r}\left[x^T(k)A_i + x^T(k-\tau)A_{i1}\right]\lambda_j. \qquad (11)$$

Thus, $\Delta V(x(k)) \leq \sum_{i=1}^{r}\sum_{j=1}^{r}\left[x^T(k)A_i + x^T(k-\tau)A_{i1}\right]\lambda_j$

$$+\sum_{i=1}^{r}\sum_{j=1}^{r}\left[x^T(k)A_{i1} - x^T(k-\tau)A_{i1}\right]\lambda_j - x^T(k)\lambda$$
$$\leq x^T(k)\left[\sum_{i=1}^{r}\sum_{j=1}^{r}(A_i + A_{i1})\lambda_j - \lambda\right].$$

It is then obvious that $\sum_{i=1}^{r}\sum_{j=1}^{r}(A_i + A_{i1})\lambda_j - \lambda \prec 0$ implies $\Delta V(x(k)) \prec 0$. This result can be easily extended to design controllers ensuring asymptotic stability while imposing positivity in closed-loop. □

Theorem 4: The system (7) is asymptotically stable and positive if there exist a vector $\lambda = [\lambda_1 \ldots \lambda_n]^T \in R^n$,vectors $\lambda_j \in R^n$ and vectors $y_1^j, ..., y_l^j \in R^m \,/\, j = 1, ..., r$; satisfying the following LPs:

$$\begin{cases}\sum_{i=1}^{r}\sum_{j=1}^{r}(A_i + A_{i1})\lambda_j + B_i\sum_{s=1}^{l} y_s^j - \lambda \prec 0, i,j = 1,2,...,r, \\ y_s^j \succ 0,\ j \in \{1,2,...,r\}\ ;\ s \in \{1,2,...,l\} \\ 0 \prec \lambda \prec \lambda_j;\quad j \in \{1,2,...,r\}\end{cases}$$

with

$$K_j = \left[\frac{y_1^j}{\sum_{i=1}^{n} c_{1i}\lambda_i^j}, \frac{y_2^j}{\sum_{i=1}^{n} c_{2i}\lambda_i^j} \ldots, \frac{y_l^j}{\sum_{i=1}^{n} c_{li}\lambda_i^j}\right]; \qquad (12)$$
$$j = 1, ..., r;\ s = 1, \ldots, l;$$

with $C = \begin{bmatrix} c_{1i} \\ c_{2i} \\ \vdots \\ c_{li}\end{bmatrix}$; $i = 1, ..., n$

Proof 2: Following the same reasoning and replacing the $x^T(k+1)$ in equation (10) by its expression of the dual system of (7), it follows:

$$x(k+1) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(k))h_j(z(k))\left[(A_i \ + \right.$$
$$\left. B_iK_jC\ )^T\ x(k) + A_{i1}^T\ x(k-\tau)\right].$$

The expression of the rate of increase of the functional (10) becomes:

$$\Delta V(x(k)) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(k))h_j(z(k))\left[x^T(k)(A_i\right.$$

$+\mathrm{B}_iK_jC) + x^T(k-\tau)A_{i1}\big]\lambda - x^T(k)\lambda$
$+\sum_{i=1}^{r}\sum_{j=1}^{r}\left[x^T(k)A_{i1} - x^T(k-\tau)A_{i1}\right]\lambda_j$

That is $\Delta V(x(k)) \leq \sum_{i=1}^{r}\sum_{j=1}^{r}\left[x^T(k)(A_i + B_iK_jC)\right.$

$$\left. + x^T(k-\tau)A_{i1}\right]\lambda_j - x^T(k)\lambda$$
$$+\sum_{i=1}^{r}\sum_{j=1}^{r}\left[x^T(k)A_{i1} - x^T(k-\tau)A_{i1}\right]\lambda_j$$
$$\leq x^T(k)\left[\sum_{i=1}^{r}\sum_{j=1}^{r}(A_i + A_{i1} + B_iK_jC)\lambda_j - \lambda\right]$$

Finally,

$$\sum_{i=1}^{r}\sum_{j=1}^{r}(A_i + A_{i1} + B_iK_jC)\lambda_j - \lambda \prec 0\ ; \qquad (11)$$

implies $\Delta V(x(k)) \prec 0$. To ensure that the trajectory remains in the positive orthant, matrices $A_i + B_iK_jC$ must be positive. Now, by letting $K_j = [K_1^j\ K_2^j \ldots\ K_l^j]$ where K_s^j are vectors in R^m, one has $K_jC\lambda_j = \sum_{s=1}^{l} K_s^j[\sum_{i=1}^{n} c_{si}\lambda_i^j] = \sum_{s=1}^{l} y_s^j$, with $K_s^j[\sum_{i=1}^{n} c_{si}\lambda_i^j] = y_s^j$. Consequently, inequality (11) can be written as

$$\sum_{i=1}^{r}\sum_{j=1}^{r}(A_i + A_{i1})\lambda_j + B_i\sum_{s=1}^{l} y_s^j - \lambda \prec 0,$$

and

$$K_j = [\frac{y_1^j}{\sum_{i=1}^{n} c_{1i}\lambda_i^j}, \frac{y_2^j}{\sum_{i=1}^{n} c_{2i}\lambda_i^j} \ldots, \frac{y_l^j}{\sum_{i=1}^{n} c_{li}\lambda_i^j}]; j = 1, ..., r.$$

□

It is worth noting that conditions of stability and stabilization of the T-S fuzzy system without delay can be obtained as a particular case of the studied system with delay (7).

IV. APPLICATION TO A REAL PLANT MODEL

Consider the process composed of two linked tanks of capacity 22 liters each. This system can be described by:

$$\dot{x}_1(t) = u_1(t) - q_{12}(t) - q_1(t)$$
$$\dot{x}_2(t) = u_2(t) + q_{12}(t) - q_2(t)$$

where x_i holds for the level in liters of tank i, u_j represents the flow in liter/mn of pump j, q_{12} is the variation of the flow between the two tanks and q_i the loss flow of each tank. Applying the Torricelli law, one obtains:
$q_1 = \gamma_1\sigma_1\sqrt{2gx_1} = R_1\sqrt{x_1}$
$q_2 = \gamma_1\sigma_2\sqrt{2gx_2} = R_2\sqrt{x_2}$
$q_{12} = \gamma_{12}\sigma_1\sqrt{2g|x_1 - x_2|}sign(x_1 - x_2) = R_{12}\sqrt{|x_1 - x_2|}sign(x_1 - x_2)$

where γ_i and γ_{ij} are physical constants, σ_i is the tank section and g the gravity acceleration. The process model is then as follows:

$$\dot{x}_1(t) = u_1 - R_1\sqrt{x_1} - R_{12}\sqrt{|x_1 - x_2|}sign(x_1 - x_2)$$
$$\dot{x}_2(t) = u_2 - R_2\sqrt{x_2} + R_{12}\sqrt{|x_1 - x_2|}sign(x_1 - x_2)$$

The obtained model is then nonlinear. To obtain a T-S fuzzy representation for this nonlinear system, the classical transformation: $\sqrt{x_i} = \frac{x_i}{\sqrt{x_i}} = x_i z_i$ with $z_i = \frac{1}{\sqrt{x_i}}; \frac{1}{\sqrt{|x_1 - x_2}} = \frac{z_1 z_2}{\sqrt{|z_2^2 - z_1^2|}}$

The corresponding model is then given by: $\begin{cases}\dot{x}(t) = A(z_1, z_2)x(t) + Bu(t)\\ y(t) = Cx(t)\end{cases}$ where matrix $A(z_1, z_2)$ has the general following form:

$$A(z_1, z_2) = \begin{pmatrix} -R_1 z_1 - \frac{R_{12}}{\sqrt{|z_1^2 - z_2^2}} & \frac{R_{12} z_1 z_2}{\sqrt{|z_1^2 - z_2^2}} \\ \frac{R_{12} z_1 z_2}{\sqrt{|z_1^2 - z_2^2}} & -R_2 z_2 - \frac{R_{12}}{\sqrt{|z_1^2 - z_2^2}} \end{pmatrix}$$

$B = I_2$; $C = I_2$

The delayed model can be written as:

$$\begin{cases}\dot{x}(t) = (1 - \epsilon)A(z_1, z_2)x(t) + \epsilon|A(z_1, z_2)|x(t - \tau)\\ +Bu(t)\\ y(t) = Cx(t)\end{cases}$$

with $\epsilon \in [0, 1]$ and τ: fixed delay.

This system can be described as T-S fuzzy model by considering that $z_i \in [a_i; b_i]; i = 1, 2$. The four following rules are taken into account:

$$\begin{cases} If\ z_1\ is\ a_1\ and\ z_2\ is\ a_2\ Then : \dot{x} = A_1 x\\ If\ z_1\ is\ a_1\ and\ z_2\ is\ b_2\ Then : \dot{x} = A_2 x\\ If\ z_1\ is\ b_1\ and\ z_2\ is\ a_2\ Then : \dot{x} = A_3 x\\ If\ z_1\ is\ b_1\ and\ z_2\ is\ b_2\ Then : \dot{x} = A_4 x\end{cases}$$

The membership functions are given by:
$h_1(k) = f_{11}(k)f_{21}(k);$
$h_2(k) = f_{11}(k)f_{22}(k);$
$h_3(k) = f_{12}(k)f_{21}(k);$
$h_4(k) = f_{12}(k)f_{22}(k);$
Where $f_{i1}(k) = \frac{z_i(k) - b_i}{a_i - b_i}$ and $f_{i2}(k) = 1 - f_{i1}(k) = \frac{a_i - z_i(k)}{a_i - b_i}$; $i = 1, 2$

The membership functions are finally as:
$h_1(k) = \frac{(z_1(k) - b_1)(z_2(k) - b_2)}{(a_1 - b_1)(a_2 - b_2)};$
$h_2(k) = \frac{(z_1(k) - b_1)(a_2(k) - z_2(k))}{(a_1 - b_1)(a_2 - b_2)};$
$h_3(k) = \frac{(a_1 - z_1(k))(z_2(k) - b_2)}{(a_1 - b_1)(a_2 - b_2)};$
$h_4(k) = \frac{(a_1 - z_1(k))(a_2 - z_2(k))}{(a_1 - b_1)(a_2 - b_2)};$

The obtained matrices A_i of the subsystems are:

$$A_1 = \begin{pmatrix} -R_1 a_1 - \frac{R_{12} a_1 a_2}{\sqrt{|a_1^2 - a_2^2|}} & \frac{R_{12} a_1 a_2}{\sqrt{|a_1^2 - a_2^2|}} \\ \frac{R_{12} a_1 a_2}{\sqrt{|a_1^2 - a_2^2|}} & -R_2 a_2 - \frac{R_{12} a_1 a_2}{\sqrt{|a_1^2 - a_2^2|}} \end{pmatrix};$$

$$A_2 = \begin{pmatrix} -R_1 a_1 - \frac{R_{12} a_1 b_2}{\sqrt{|a_1^2 - b_2^2|}} & \frac{R_{12} a_1 b_2}{\sqrt{|a_1^2 - b_2^2|}} \\ \frac{R_{12} a_1 b_2}{\sqrt{|a_1^2 - b_2^2|}} & -R_2 b_2 - \frac{R_{12} a_1 b_2}{\sqrt{|a_1^2 - b_2^2|}} \end{pmatrix};$$

$$A_3 = \begin{pmatrix} -R_1 b_1 - \frac{R_{12} b_1 a_2}{\sqrt{|b_1^2 - a_2^2|}} & \frac{R_{12} b_1 a_2}{\sqrt{|b_1^2 - a_2^2|}} \\ \frac{R_{12} b_1 a_2}{\sqrt{|b_1^2 - a_2^2|}} & -R_2 a_2 - \frac{R_{12} b_1 a_2}{\sqrt{|b_1^2 - a_2^2|}} \end{pmatrix};$$

$$A_4 = \begin{pmatrix} -R_1 b_1 - \frac{R_{12} b_1 b_2}{\sqrt{|b_1^2 - b_2^2|}} & \frac{R_{12} b_1 b_2}{\sqrt{|b_1^2 - b_2^2|}} \\ \frac{R_{12} b_1 b_2}{\sqrt{|b_1^2 - b_2^2|}} & -R_2 b_2 - \frac{R_{12} b_1 b_2}{\sqrt{|b_1^2 - b_2^2|}} \end{pmatrix};$$

For the discrete-time system, we apply the Euler discretization.

One can notice that matrices B and C in this example is common, which reduces considerably the number of the LMIs to be solved. The obtained T-S fuzzy model without delay is given by:

$$\begin{cases} x(k+1) = \sum_{i=1}^{4} h_i(z(k))(A_i x(k) + B_i u(k)) \\ y(k) = \sum_{i=1}^{4} h_i(z(k))C_i x(k) \end{cases} \quad (12)$$

The corresponding T-S model with fixed delay can be given as follows:

$$\begin{cases} x(k+1) = \sum_{i=1}^{4} h_i(z(k))((1 - \epsilon)A_i x(k) + \epsilon|A_i|x(k - \tau) \\ \qquad +B_i u(k)) \\ y(k) = \sum_{i=1}^{4} h_i(z(k))C_i x(k) \end{cases}$$

(13)

The objective is to design controllers ensuring stabilization of systems (13) associated to the real plant model for which matrices A_i, A_{i1} and B_i are positive, using the conditions of Theorem 2 and Theorem 4.

The objective is that the output y tracks a given reference y_r. the following control is used: $u(k) = K(\theta)Cx(k) + L(\theta)y_r$, where controller gain $K(\theta)$ ensures the asymptotic stability together with the positivity in closed-loop while controller gain $L(\theta)$ achieves the tracking objective, one obtains: $X(z) = (zI - \hat{A}(\theta) - A_\tau(\theta)z^{-\tau})^{-1}BL(\theta)\frac{z}{z-1}y_r$; so: $Y(z) = C(zI - \hat{A}(\theta) - A_\tau(\theta)z^{-\tau})^{-1}BL(\theta)\frac{z}{z-1}y_r$ Using the final value theorem, one can deduce:$y(\infty) = -C[I - \hat{A}(\theta) - A_\tau(\theta)]^{-1}BL(\theta)y_r$ with $\hat{A}(\theta) = (1-\epsilon)A + BK(\theta)C$;$A_\tau(\theta) = \epsilon|A|$. If one chooses $L_i(\theta) = (C(I - \hat{A}_i(\theta) - A_{i1}(\theta))^{-1}B)^{-1} = (C(I-(1-\epsilon)A_i - \epsilon|A_i|)^{-1}B)^{-1}$;$i = 1,..,4$, the tracking objective will be reached with $y(\infty) = y_r$.

A. Simulation results of the system without delay

The use of the Cone complementarity method without delay of Theorem 2 leads to the following results:

$$P=\begin{pmatrix} 1 & 2.477 * e^{-11} \\ 2.477 * e^{-11} & 1 \end{pmatrix};$$

$$Q=\begin{pmatrix} 1 & 2.475 * e^{-11} \\ 2.475 * e^{-11} & 1 \end{pmatrix}$$

$$K_1=\begin{pmatrix} 0.0858 & 0.0945 \\ 0.0945 & 0.0926 \end{pmatrix};K_2=\begin{pmatrix} 0.0892 & 0.1239 \\ 0.1239 & 0.1371 \end{pmatrix};$$

$$K_3=\begin{pmatrix} 0.1527 & 0.1378 \\ 0.1378 & 0.1033 \end{pmatrix};K_4=\begin{pmatrix} 0.1837 & 0.1490 \\ 0.1490 & 0.1746 \end{pmatrix}$$

Matrices in closed-loop are obtained as:

$$\hat{A}_1=\begin{pmatrix} 0.6337 & 0.3336 \\ 0.3336 & 0.6070 \end{pmatrix};\hat{A}_2=\begin{pmatrix} 0.7326 & 0.2681 \\ 0.2681 & 0.6026 \end{pmatrix};$$

$$\hat{A}_3=\begin{pmatrix} 0.5547 & 0.3083 \\ 0.3083 & 0.6875 \end{pmatrix};\hat{A}_4=\begin{pmatrix} 0.2239 & 0.6820 \\ 0.6820 & 0.2525 \end{pmatrix}$$

The obtained solutions of the LP method are as follows:

$\lambda=[0.0003, 0]^T$
$\lambda_1=[0.0302, 0.0299]^T$; $\lambda_2=[0.0734, 0.0730]^T$
$\lambda_3=[0.1139, 0.1133]^T$; $\lambda_4=[0.1840, 0.1833]^T$

$$K_1=\begin{pmatrix} 0.1103 & 0.1116 \\ 0.1088 & 0.1101 \end{pmatrix}; K_2=\begin{pmatrix} 0.0454 & 0.0457 \\ 0.0448 & 0.0450 \end{pmatrix};$$

$$K_3=\begin{pmatrix} 0.0293 & 0.0294 \\ 0.0289 & 0.0290 \end{pmatrix};K_4=\begin{pmatrix} 0.0181 & 0.0182 \\ 0.0179 & 0.0179 \end{pmatrix}$$

Matrices in closed-loop are obtained as:

$$\hat{A}_1=\begin{pmatrix} 0.6586 & 0.3510 \\ 0.3481 & 0.6249 \end{pmatrix};\hat{A}_2=\begin{pmatrix} 0.6879 & 0.1883 \\ 0.1874 & 0.5086 \end{pmatrix};$$

$$\hat{A}_3=\begin{pmatrix} 0.4288 & 0.1977 \\ 0.1972 & 0.6117 \end{pmatrix};\hat{A}_4=\begin{pmatrix} 0.0550 & 0.5486 \\ 0.5483 & 0.0926 \end{pmatrix}$$

The results of the simulation, with the following data: initial point $x_0 = [7, 8]^T$ and the trajectory reference $y_r = [14, 15]^T$, are obtained as follows:

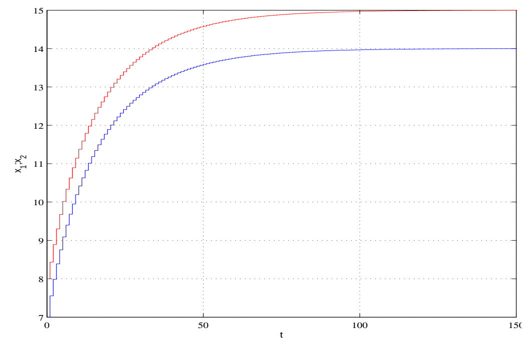

Fig. 1. This figure plots the evolution of the states x_1 and x_2(ConeC)

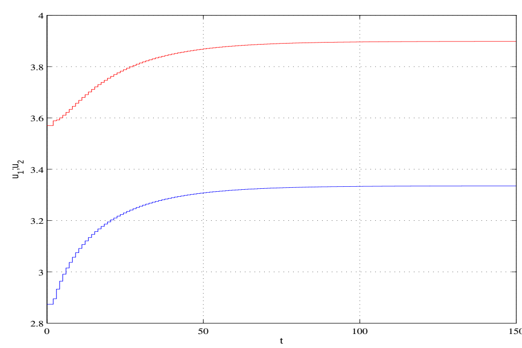

Fig. 2. This figure plots the evolution of the two pump flows(ConeC)

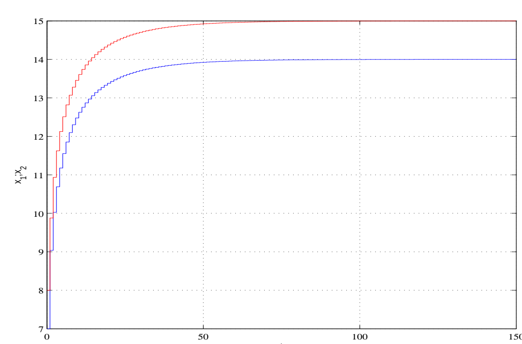

Fig. 3. This figure plots the evolution of the states x_1 and x_2(LP)

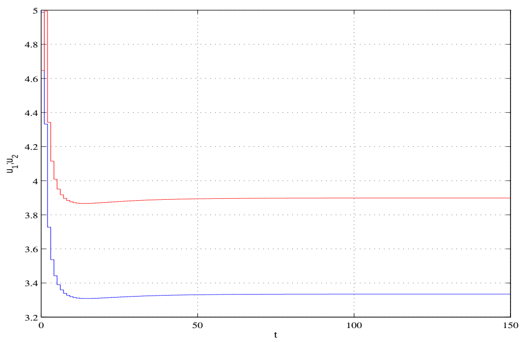

Fig. 4. This figure plots the evolution of the two pump flows(LP)

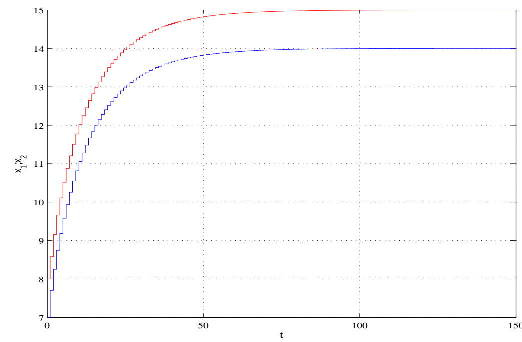

Fig. 5. This figure plots the evolution of the states x_1 and x_2(ConeC)

B. Simulation results of the system with fixed delay

The use of the Cone complementarity method with fixed delay of Theorem 2 leads to the following results:

$$P=\begin{pmatrix}1.0000 & 0\\ 0 & 1.0000\end{pmatrix} ;\ Q=\begin{pmatrix}0.9999 & 0\\ 0 & 0.9999\end{pmatrix} ;$$
$$R=\begin{pmatrix}0.3149 & -0.1684\\ -0.1684 & 0.3233\end{pmatrix}$$

$$K_1=\begin{pmatrix}0.0767 & 0.0884\\ 0.0879 & 0.0833\end{pmatrix};K_2=\begin{pmatrix}0.0733 & 0.1230\\ 0.1133 & 0.1275\end{pmatrix};$$
$$K_3=\begin{pmatrix}0.1439 & 0.1244\\ 0.1360 & 0.0861\end{pmatrix};K_4=\begin{pmatrix}0.1826 & 0.1268\\ 0.1303 & 0.1702\end{pmatrix}$$

Matrices in closed-loop are obtained as:

$$\hat{A}_1=\begin{pmatrix}0.5697 & 0.3036\\ 0.3031 & 0.5463\end{pmatrix};\hat{A}_2=\begin{pmatrix}0.6522 & 0.2531\\ 0.2431 & 0.5465\end{pmatrix};$$
$$\hat{A}_3=\begin{pmatrix}0.5059 & 0.2779\\ 0.2897 & 0.6117\end{pmatrix};\hat{A}_4=\begin{pmatrix}0.2191 & 0.6065\\ 0.6100 & 0.2406\end{pmatrix}$$

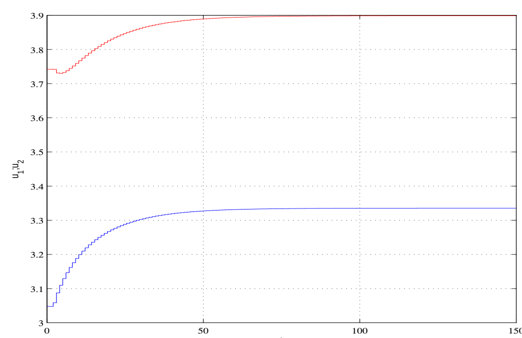

Fig. 6. This figure plots the evolution of the two pump flows(ConeC)

The use of the LP method with fixed delay of Theorem 4 leads to the following results:

$$\lambda=[0.0003, 0]^T$$
$$\lambda_1=[0.0302, 0.0299]^T\ ;\ \lambda_2=[0.0734, 0.0730]^T$$
$$\lambda_3=[0.1139, 0.1133]^T\ ;\ \lambda_4=[0.1840, 0.1833]^T$$
$$K_1=\begin{pmatrix}0.1103 & 0.1116\\ 0.1088 & 0.1101\end{pmatrix};K_2=\begin{pmatrix}0.0454 & 0.0457\\ 0.0448 & 0.0450\end{pmatrix};$$
$$K_3=\begin{pmatrix}0.0293 & 0.0294\\ 0.0289 & 0.0290\end{pmatrix};K_4=\begin{pmatrix}0.0181 & 0.0182\\ 0.0179 & 0.0179\end{pmatrix}$$

Matrices in closed-loop are obtained as:

$$\hat{A}_1=\begin{pmatrix}0.6040 & 0.3273\\ 0.3244 & 0.5737\end{pmatrix};\hat{A}_2=\begin{pmatrix}0.6237 & 0.1742\\ 0.1732 & 0.4624\end{pmatrix};$$
$$\hat{A}_3=\begin{pmatrix}0.3889 & 0.1809\\ 0.1804 & 0.5534\end{pmatrix};\hat{A}_4=\begin{pmatrix}0.0514 & 0.4956\\ 0.4953 & 0.0852\end{pmatrix}$$

The results of simulation with the following data: $\epsilon = 0.1$; initial points $\Psi(k) = [7, 8]^T, k \in [-\tau, 0]$ and the trajectory reference $y_r = [14, 15]^T$ are obtained as:

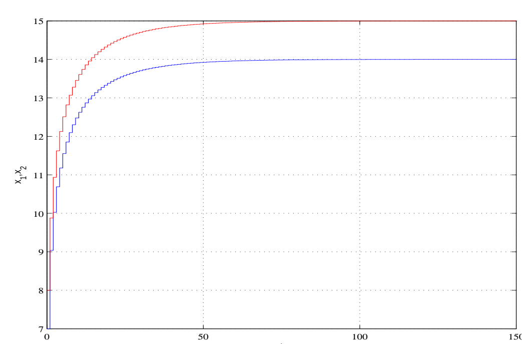

Fig. 7. This figure plots the evolution of the states x_1 and x_2(LP)

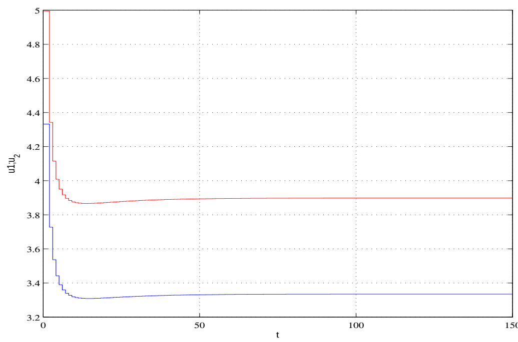

Fig. 8. This figure plots the evolution of the two pump flows(LP)

C. Comparison between the Cone complementarity and LP methods:

In this section, a comparison between the feasibility of the results of Theorem 2 and the ones of Theorem 4 is presented based on the real plant model.

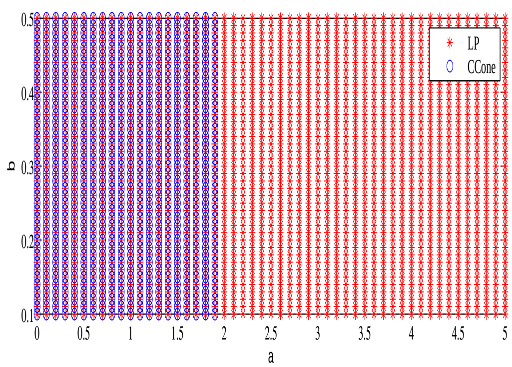

Fig. 9. Comparing the field feasibility of the ConeC and LP

Based on the comparison of the two presented methods, the Cone complementarity and linear programming, we note that the domain of feasibility of conditions based on linear programming is very large than the based Cone complementarity ones.

V. Conclusion

In this paper, we are concerned with the study of positive nonlinear systems. To obtain conditions of stability and stabilization by output feedback of nonlinear systems, while imposing positivity in closed-loop, the T-S fuzzy techniques are used. The study is performed by using a linear programming method. Finally, an application to a real model of a process with two tanks was presented together with a comparison between our results.

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