Regulator Problem for Linear Delayed Systems with Asymmetrical Constraints on the Control Vector and its Rate

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Abstract— **This paper deals with the problem of constraints on both control and its rate, for linear continuous-time delayed systems. For this, some known results dealing with delayed continuous-time systems with imposed constraints on their control vector and its rate are recalled. A pole assignment technique is then used to find a feedback gain matrix which guarantees the respect of the constraints without saturating neither the control vector, nor its rate. An illustrative example shows the application of the proposed method.**

keywords: Linear delay systems, asymmetric constrained control, asymmetric constrained rate control, invariance positive, pole assignment.

I. INTRODUCTION

Time delays are frequently encountered in many fields of science and engineering and they are often source of the degradation of performance. The instability of the time-delayed linear system is one of the most important issues in control theory, and there are significant contributions (for instance, [?], [?], [?], [?], [?] and references therein).

This paper deals with regulator problem for linear continuous-time delay systems with asymmetrical constrained control and its rate. The positive invariance approach is used in this paper, because it gives simple method to calculate constant state feedback controllers answering this objective. This approach is based on the constraint avoidance ([?], [?], [?] and [?]): preventing the saturation, the closed-loop system, therefore, stays in a region of linear behavior. Conditions guaranteeing the admissibility of the control vector and its rate are recalled for linear continuous-time systems. Furthermore, a link is made between a recent pole assignment procedure, developed by Baddou et al. [?], and these conditions to find state feedback stabilizing controllers, respecting control constraints and rate constraints also. This pole assignment technic uses a Sylvester equation solution to build a feedback gain matrix dealing with the *inverse procedure*. The proposed method provides answers in some cases where other methods fail.

The rest of the paper is organized as follows. In Section 2, the problem is formulated and related preliminaries are presented. In Section 3, the main results are given, then a numerical examples and simulation are presented to illustrate our proposed results are less conservative and more effective. Finally, the conclusion is given in Section 4.

A. Notations

- For two vectors x, y of $\mathbb{R}^n$, $x \leq y$ (respectively, $x < y$) if $x_i \leq y_i$ (respectively $x_i < y_i$), i = 1,...,n.
- $\mathbb{I}_n$ is the identity matrix of $\mathbb{R}^n$; A^T and $\sigma(A)$ denote respectively the transpose and the spectrum of matrix A; Re(λ) the real part of the eigenvalue λ and $\lambda_i(A)$ the ith eigenvalue of A.
- $x^+ = sup(x_i, 0), x^- = sup(-x_i, 0)$
- $int\mathbb{R}^m_+$ is the interior of $\mathbb{R}^m_+$
- If $H \in \mathbb{R}^{m\times m}$ and $P \in \mathbb{R}^{m\times n}$ then

$$\widetilde{H}^c = \begin{bmatrix} H_1 & H_2 \\ H_2 & H_1 \end{bmatrix}, \widetilde{P}^d = \begin{bmatrix} P^+ & P^- \\ P^- & P^+ \end{bmatrix}$$

$$H_1(i,j) = \begin{cases} h_{ij} \text{ if } i = j \\ h^+_{ij} \text{ if } i \neq j \end{cases} ;$$

$$H_2(i,j) = \begin{cases} 0 \text{ if } i = j \\ h^-_{ij} \text{ if } i \neq j \end{cases} , i,j = 1,...,m$$

$$P^+(i,j) = (P(i,j))^+ ;$$

$$P^-(i,j) = (P(i,j))^-, i = 1,...,m; j = 1,...,n$$

II. PROBLEM STATEMENT AND PRELIMINARY RESULTS

This paper is devoted to the study of linear continuous-time delay systems described by:

$$\begin{cases} \dot{x}(t) = A_o x(t) + A_1 x(t-h) + Bu(t), & t > t_o \\ x(\theta) = \psi(\theta), & \theta \in [t_o - h, t_o] \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ is the state and $h > 0$ is the time delay. Matrices A_o and B are constants, it is assumed that (A_o, B) is stabilizable. Consider the state feedback control law:

$$u(t) = Fx(t) \quad , F \in \mathbb{R}^{m\times n} \quad (2)$$

The control u is constrained to evolve in the set Ω defined by:

$$\Omega = \{u \in \mathbb{R}^m / -u_{min} \leq u \leq u_{max}\} \quad (3)$$

where u_{min}, $u_{max} \in int\mathbb{R}^m_+$ are given vectors. The control is admissible only if the state is constrained to evolve in a specified domain defined by:

$$\mathcal{D}(F, u_{min}, u_{max}) = \{x \in \mathbb{R}^n / -u_{min} \leq Fx \leq u_{max}\} \quad (4)$$

The control rate is constrained as follows:

$$-\Delta_{min} \leq \dot{u}(t) \leq \Delta_{max} \quad (5)$$

We denote $U = \begin{bmatrix} u^T_{max} & u^T_{min} \end{bmatrix}^T$, $\Delta = \begin{bmatrix} \Delta^T_{max} & \Delta^T_{min} \end{bmatrix}^T$.

Tacking into account (**??**), system (**??**) becomes:

$$\dot{x}(t) = (A_o + BF)x(t) + A_1 x(t-h) \quad (6)$$

Let us make the change of variables,

$$z(t) = Fx(t), \quad F \in \mathbb{R}^{m\times n} \quad (7)$$

hence,

$$\dot{z}(t) = F(A_o + BF)x(t) + FA_1x(t-h) \quad (8)$$

If there exist matrices H and $G \in \mathbb{R}^{m\times m}$ such that

$$F(A_o + BF) = HF \quad (9)$$

and

$$FA_1 = GF \quad (10)$$

then,

$$\begin{cases} \dot{z}(t) &= Hz(t) + Gz(t-h), \quad t > t_o \\ z(s) &= \phi(s), \quad s \in [t_o - h, t_o] \end{cases} \quad (11)$$

Definition II.1. *A subset $\mathcal{D}$ of $\mathbb{R}^m$ is said to be positively invariant with respect to a given system (**??**), if for every $\phi(s) \in \mathcal{D}, (s \in [t_o - h, t_o])$, the motion $z(t, t_o, \phi) \in \mathcal{D}$, for every $t > t_o$.*

Generally, matrix F is chosen such that system (**??**), which is asymptotically stable when $A_1 = 0$ by virtue of (A_o+BF) is stable, i.e,

$$Re[\lambda_i(A_o + BF)] < 0 \ , \ i = 1, \ldots, n$$

is asymptotically stable independent of delay:

$$det(s\mathbb{I}_n - A_o - A_1e^{-sh} - BF) \neq 0 \text{ ,for } Re(s) \geq 0 \text{ ,} \forall h \geq 0$$

Let us recall the pole assignment procedure used in the so-called *inverse procedure* for constrained control linear systems (see [**?**]): Consider the system (**??**) with the following assumptions:

1) Matrix A_o has at least $(n-m)$ stable and desirable eigenvalues $\lambda_{m+1}, \ldots, \lambda_n$ that are associated with $(n-m)$ linearly independent eigenvectors $\xi_{m+1}, \ldots, \xi_n$. $\{\lambda_{m+1}, \ldots, \lambda_n\}$ is invariant under complex conjugation.
2) The subspace $Im(B)$ and the stable open-loop eigenspace are complementary subspaces of $\mathbb{R}^n$. In other words, matrix

$$[B, \xi] = [b_1, \ldots, b_m, \xi_{m+1}, \ldots, \xi_n]$$

is invertible, where b_i denote the column vectors of matrix B, $i = 1, ..., m$.

The subspace $\nu = vect(\xi_{m+1}, \ldots, \xi_n)^{\perp}$, the orthogonal complement of the open-loop stable eigenspace, is spanned by $V_1^T, \ldots, V_m^T$. (V_i is a row vector) and from now on, V is the $m \times n$ matrix whose rows are $V_1, \ldots, V_m$.

Lemma II.1. *[**?**]: There is a matrix $\Lambda \in \mathbb{R}^{m\times m}$ such that $VA_o = \Lambda V$, and $\sigma(\Lambda) = \{\lambda_1, \ldots, \lambda_m\}$.*

Theorem II.1. *[**?**]: For any matrix F that satisfies (**??**), there exists an invertible matrix $K \in \mathbb{R}^{m\times m}$ such that:*

$$F = KV \quad (12)$$

Remark II.1.
Matrices Λ and V can be obtained using two methods:

1) *If A_o is diagonalizable then the rows of V are the left eigenvectors of matrix A_o and matrix $\Lambda = \Lambda_d$ is the diagonal matrix whose diagonal entries are the undesirable eigenvalues of A_o.*
2) *In the general case, V is formed by the first lines of the matrix $[B \quad \xi]^{-1}$. Indeed, $[B \quad \xi]^{-1}[B \quad \xi] = \begin{bmatrix} V \\ * \end{bmatrix} [B \quad \xi] = \mathbb{I}_n$ implies that $VB = \mathbb{I}_m$ and $V_i\xi_j = 0$, $i = 1, \ldots, (n-m); j = m+1, \ldots, n$. In this case $VA_o = \Lambda V$ leads to $\Lambda = VA_oB$.*
3) *The following considerations (used in [**?**]) will be also used here : $\Lambda = P\Lambda_dP^{-1}$, $L = \Lambda + VBK = PL_dP^{-1}$ and $H = KLK^{-1}$. Λ_d is defined in the first item of Remark (**??**) while L_d is the diagonal matrix whose diagonal entries are the chosen eigenvalues to be assigned.*

In order to compute V and Λ, the second item of Remark (**??**) will be used in this paper. The gain matrix F is given by $F = X^{-1}V = KV$, where X is the solution of the following equation, named Sylvester equation (See [**?**]):

$$\Lambda X - XH = -\mathbb{I}_m \quad (13)$$

First, recall the result given by Hmamed et al. [**?**] for delay system without constraints on its control rate.

Theorem II.2. *[**?**]: The polyhedral set $\mathcal{D}(F, u_{min}, u_{max})$ defined in (**??**) is positively invariant with respect to (**??**) if and only if there exist matrices H and $G \in \mathbb{R}^{m\times m}$ such that,*

1) *$FA + FBF = HF$ and $FA_1 = GF$*
2) *$\widetilde{H}^cU + \widetilde{G}^dU \leq 0$*

Second, we give in the following theorem the result given by Mesquine et al. [**?**], which deals with the positive invariance of linear system with constraints on control and its rate, but without delay, i.e., $A_1 = 0$.

Theorem II.3. *[**?**]: Domain $\mathcal{D}(F, u_{min}, u_{max})$ is positively invariant with respect to system $\dot{x}(t) = (A+BF)x(t)$ if and only if there exists $H \in \mathbb{R}^{m\times m}$ such that,*

1) *$FA + FBF = HF$*
2) *$\widetilde{H}^dU \leq \Delta$ and $\widetilde{H}^cU \leq 0$*

III. Main results

With this background, we are now able to solve the problem stated in the introduction section: under which conditions the asymmetrical domain $\mathcal{D}(F, u_{min}, u_{max})$ is positively invariant with respect to motions (**??**). Consider a stabilizable linear time delay system (**??**) with constraints on both control magnitude (**??**) and control rate (**??**). Using (**??**) as the feedback control law, closed-loop system is described by,

$$\dot{x}(t) = (A_o + BF)x(t) + A_1x(t-h)$$

The proposed method begins by the choice of H such that,

$$\widetilde{H}^cU < 0, \quad (14)$$

then compute matrices V and Λ as mentioned in the second item of Remark (**??**). Equation (**??**) is solved to obtain the feedback gain matrix F then compute the matrix G.

Combining conditions of Theorem (**??**) to those proposed in Theorem (**??**) enables us to claim the following result:

Proposition III.1. *: Let H chosen such that (**??**) is satisfied and $F = X^{-1}V$, X being the solution of (**??**), then, domain $\mathcal{D}(F, u_{min}, u_{max})$ is positively invariant with respect to (**??**) if and only if,*

1) $\widetilde{H}^c U + \widetilde{G}^d U < 0$

2) $\widetilde{H}^d U + \widetilde{G}^d U \leq \Delta$

where $G = KVA_1BK^{-1}$.

Proof: Using the feedback control law (**??**) one can write:

$$\begin{aligned} \dot{u}(t) &= F\dot{x}(t) \\ &= F(A_o x(t) + BFx(t) + A_1 x(t-h)) \\ &= F(A_o + BF)x(t) + FA_1 x(t-h) \end{aligned} \tag{15}$$

H is chosen such that (**??**) is satisfied. If $F = X^{-1}V = KV$, X being the solution of equation (**??**) then equation $FA_0 + FBF = HF$ is satisfied. Furthermore, matrix G is the solution of equation $FA_1 = GF$, which can be written as $KVA_1 = GKV$. Right multiplying this last equation by B yields $KVA_1B = GKVB = GK$. Finally $G = KVA_1BK^{-1}$. Then, equation (**??**) becomes:

$$\begin{aligned} \dot{u}(t) &= HFx(t) + GFx(t-h) \\ &= Hu(t) + Gu(t-h) \end{aligned} \tag{16}$$

One can easily obtain the result from Theorem **??** and Theorem **??**.

The following algorithm summaries the steps of the proposed method.

Algorithm III.1.

Step 1 *: Verify that A possesses at least $(n-m)$ stable eigenvalues.*

Step 2*: Compute matrices V and Λ using the second item of Remark (**??**).*

Step 3 *: Choose a matrix H for which $\sigma(H)$ is the chosen spectrum to be placed, such that,*

$$\widetilde{H}^c U < 0$$

Step 4*: Solve equation (**??**) to obtain X, compute $F = X^{-1}V = KV$ and compute $G = KVA_1BK^{-1}$.*

Step 5*: If conditions of Proposition (**??**) are satisfied, hence use F for the control, else return to step 3 and change H.*

Example III.1.

Consider the system described by the following;

$$A_o = \begin{bmatrix} -0.57 & 0.93 \\ -0.46 & 0.82 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.0803 & -0.0398 \\ 0 & -0.1 \end{bmatrix}$$

and $B = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}$

*Λ and V are computed using the second item of Remark **??**, that is,*

$$\Lambda = 0.36 \quad \textit{and} \quad V = \begin{bmatrix} 1.2534 & -2.5341 \end{bmatrix}$$

Assume that the control constraints and the control rate constrained are:

$$\begin{aligned} -5 \leq u(t) \leq 4.1 \\ -6 \leq \dot{u}(t) \leq 3 \end{aligned}$$

The eigenvalues of A_o are: $\sigma(A_o) = \{-0.11 \quad 0.36\}$, then we choose $H = -0.11$. Solving the Sylvester Equation *(**??**) leads to $X = -2.1277$, therefore $F = \begin{bmatrix} -0.5891 & 1.1910 \end{bmatrix}$ and $G = -0.0803$. Finally, the conditions $\widetilde{H}^c U + \widetilde{G}^d U < 0$ and $\widetilde{H}^d U + \widetilde{G}^d U \leq \Delta$ hold. The trajectories of the state and control components are presented in figure **??**.*

*Note that for the choice $H = -0.11$, the method presented in [**?**] can not be applied because H is an eigenvalue of A.*

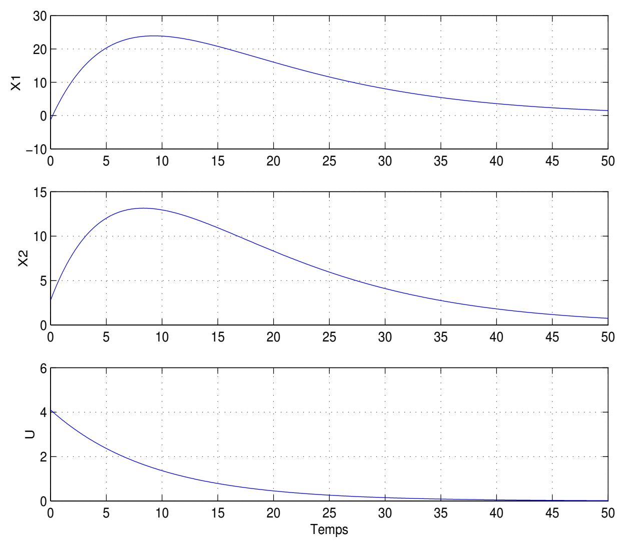

Fig. 1. The trajectories of the control and the state vectors.

IV. Conclusion

In this note, the regulator problem for delayed linear continuous-time systems with asymmetrical constrained control vector and its rate, is studied. The established result uses a Sylvester equation to find a feedback gain matrix for linear delayed continuous-time systems such that the obtained closed-loop system respects the imposed constraints. The illustrative example shows the efficiency of the presented technic.

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