H_∞ control of discrete-time uncertain periodic systems

N. Bougatef, M. Chaabane, O. Bachelier and D. Mehdi*

Abstract

This paper addresses the problem of H_∞ control for a class of linear discrete-time periodic system. The obtained results are then extended for the periodic system with Linear Fractional Representation (LFR) uncertainty. Furthermore, linear matrix inequality (LMI)-based sufficient conditions for H_∞ control are established. Numerical examples are introduced to illustrate the contribution of this paper.

Keyword *Discrete systems, periodic systems, state feedback stabilization, Linear Fractional Representation, H_∞ control, asymptotic stabilization, robustness.*

1. Introduction

The H_∞ norm as a measure of system performances has been thoroughly embedded in control theory during the last few years [1]. During the last decade, the H_∞ control theory has attracted a lot of attention and made significant progress [2]. Recently, a great number of research papers focused on the problem of robust H_∞ control for linear systems with parameter uncertainty. The objective is to design a control law to stabilize an uncertain system while satisfying in the same time a $H_\infty-$norm bound constraint with the aim to insure disturbance attenuation in the presence for all admissible uncertainties. The robust H_∞ control design for linear uncertain systems has been investigated in [3] with time-varying norm-bounded uncertainty using the notion of quadratic stabilization with an H_∞-norm bound. In terms of modified algebraic Riccati equations, necessary and sufficient conditions for H_∞ control of uncertain linear systems have been estabilshed, for instance, in [2] . In [4], it has been shown that the robust H_∞ analysis and synthesis problem of linear discrete-time system with norm-bounded and time-varying uncertainty is actually a scaled H_∞ analysis and synthesis problem of an auxiliary system, which is independent of uncertainties. In [5], the authors have extended the obtained results on the robust H_∞ control for delayed systems. Then, robust H_∞ state feedback control in which both robust stability and a prescribed H_∞ performance are required to be achieved irrespective of the uncertainty and unknown time-delay is investigated.

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The present paper investigates the robust H_∞ control problem for uncertain discrete-time periodic system. The main attention has been focused on a class of linear periodic systems with Linear Fractional Representation (LFR) of the uncertainty in both the state and input matrices. The formalism of Linear Matrix Inequalities (LMI) approach is used throughout the paper. The problem of robust stabilization and robust H_∞ control, where the control law is required to guarantee both robust stability and a prescribed robust H_∞ peroformance. The approach adopted here lies on the notion of quadratic stabilization with a H_∞-norm bound which is introduced in [3] and [6].

Notation: We denote by $X^\top$ the transpose of matrix X, by the Hermitian expression $\mathsf{Sym}\{\cdot\}$: $\mathsf{Sym}\{X\} = X + X^\top$.

Matrix inequalities are considered in the sense of Löwner *i.e.* "< 0" ("≤ 0") means negative (semi–)definite and "> 0" ("≥ 0") positive (semi-)definite. I and 0 are, respectively, the identity the null matrix of suitable dimension. $\|x\|_2$ denotes norm 2 of the signal x which is defined by $\|x\|_2 = \sqrt{\sum_{k=0}^{\infty} x_k^\top x_k}$.

2. Preliminaries

Consider the following linear uncertain p-periodic system:

$$x_{k+1} = (A_k + \Delta A_k)\, x_k \tag{2}$$

with

$$\Delta A_k = E_k \Delta_k (I - M_k \Delta_k)^{-1} N_k \tag{3}$$

where A_k, E_k, M_k and N_k are real p-periodic matrices with appropriate dimensions and Δ_k is an unknown matrix that satisfies

$$\Delta_k^\top \Delta_k \leq I,\ k \in \mathbb{N} \tag{4}$$

Furthermore, we assume that

$$\mathcal{M}_k = \begin{bmatrix} I & -M_k \\ -M_k^\top & I \end{bmatrix} > 0 \tag{5}$$

In this part, we will study the problem of asymptotic stability of uncertain periodic system such that in equations (2)-(4). Then, can we consider that the uncertainty matrix Δ_k is periodic?, to answer this question, let us consider that each matrix A_k, for $k = 0, \ldots, p-1$ is affected by a simple additive uncertainty, denoted ΔA_k. It comes then that from a periodic point of view, these uncertainties can be considered as periodic as is the case for matrix $A_k = A_{k+p}$, $k = 0, \ldots, p-1$.

Definition 2.1 *[7] The linear p-periodic system* (2) *is robustly stable* `if and only if`*,*

$$\forall \Delta_k \in \nabla_k,\ \exists\, X_k(\Delta_k) = X_k^\top(\Delta_k) > 0,$$
$$\begin{bmatrix} I \\ A_k^\top(\Delta_k) \end{bmatrix}^\top \begin{bmatrix} -X_{k+1}(\Delta_{k+1}) & 0 \\ 0 & X_k(\Delta_k) \end{bmatrix} \begin{bmatrix} I \\ A_k^\top(\Delta_k) \end{bmatrix} < 0 \quad (6)$$

for all $k = 0, \ldots, p-1$.

Definition 2.2 *[8] The linear p-periodic system* (2) *is quadratically stable* `if and only if`*,*

$$\exists\ X_k = X_k^\top > 0, \quad \forall \Delta_k \in \nabla_k$$
$$\begin{bmatrix} I \\ A_k^\top(\Delta_k) \end{bmatrix}^\top \begin{bmatrix} -X_{k+1} & 0 \\ 0 & X_k \end{bmatrix} \begin{bmatrix} I \\ A_k^\top(\Delta_k) \end{bmatrix} < 0 \quad (7)$$

for all $k = 0, \ldots, p-1$.

Definition 2.3 *The linear p-periodic system* (2) *is asymptotically stable* `if`*,*

$$\exists\ X_k = X_k^\top > 0, \quad \forall \Delta_k \in \nabla_k$$
$$\begin{bmatrix} I \\ A_k^\top(\Delta_k) \end{bmatrix}^\top \begin{bmatrix} -X_{k+1} & 0 \\ 0 & X_k \end{bmatrix} \begin{bmatrix} I \\ A_k^\top(\Delta_k) \end{bmatrix} < 0 \quad (8)$$

for all $k = 0, \ldots, p-1$.

In the three definitions $A_k(\Delta_k)$ is defined by:

$$A_k(\Delta_k) = A_k + \Delta A_k$$
$$\Delta A_k \text{ is described by } (3) \quad (9)$$

Lemma 2.1 *Suppose that there exits a symmetric positive definite p-periodic matrix* P_k *such that the following matrix inequality holds, for all* $k = 0, 1, \ldots, p-1$*:*

$$A_k^\top P_{k+1} A_k - P_k < 0 \quad (10)$$

Then, the p-periodic system (2) *is asymptotically stable when* $\Delta A_k = 0$, $k \in \mathbb{N}$.

3. H_∞ control of periodic system

Consider the discrete-time p-periodic system

$$x_{k+1} = (A_k + \Delta A_k)\, x_k + (B_k + \Delta B_k)\, u_k + B_{wk} w_k$$
$$z_k = C_k x_k + D_k u_k \quad (11)$$

with ΔA_k is descibed by (3) and

$$\Delta B_k = E_k \Delta_k (I - M_k \Delta_k)^{-1} L_k \quad (12)$$

where $x_k \in \mathbb{R}^n$ is the state, $w_k \in \mathbb{R}^q$ is the disturbance input, $z_k \in \mathbb{R}^p$ is the controlled output and A_k, E_k, D_k, C_k, B_k , M_k, N_k and L_k are real p-periodic matrices with appropriate dimensions and Δ_k is an unknown matrix that satisfies (4).

In our paper, we are concerned with the problem of robust p-periodic state feedback control for the uncertain p-periodic system (11)-(4) for all admissible uncertainties. Our aim is to design a p-periodic state feedback controller

$$u_k = K_k x_k;\ K_{k+p} = K_k \ \textit{for all}\ p,\ k \in \mathbb{N} \quad (13)$$

such that for a given scalar $\gamma > 0$, for all nonzero $w_k \in l_2[0, +\infty)$ and for all parameter uncertainties satisfying (3), (12) and (4),

$$\|z\|_2 \le \gamma \|w\|_2 \quad (14)$$

In this situation, the p-periodic system (11) with the controller (13) is said to achieve a robust H_∞ performance (14).

Definition 3.1 *Let the constant* γ *be given. The uncertain p-periodic system* (11)-(4) *is said to be stabilizable with an* H_∞ *norm bound* γ *if there exists a p-periodic state feedback law* (13)*, such that for any admissible parameter uncertainty* ΔA_k *and* ΔB_k *the following conditions are satisfied.*

1. *The closed-loop p-periodic system is asymptotically stable when* $w_k = 0$
2. *Subject to the assumption of the zero initial condition, the controlled output* z_k *satisfies* (14)*.*

In the remainder of this part, we will establish the stabilization results associated with the nominal system of (11).

Theorem 3.1 *consider the p-periodic system* (11) *with* $\Delta A_k = 0$ *and* $\Delta B_k = 0$*. If there exists a p-periodic symmetric matrix* $X_k > 0$ *and a p-periodic matrix* Y_k *such that for a given scalar* $\gamma > 0$ *the following matrix inequality holds*

$$\begin{bmatrix} -X_k & 0 & \Gamma_{13}^\top(k) & \Gamma_{14}^\top(k) \\ 0 & -\gamma I & 0 & B_{wk}^\top \\ \Gamma_{13}(k) & 0 & -\gamma I & 0 \\ \Gamma_{14}(k) & B_{wk} & 0 & -X_{k+1} \end{bmatrix} < 0 \quad (15)$$

$$\Gamma_{13}(k) = (C_k X_k + D_k Y_k)$$
$$\Gamma_{14}(k) = (A_k X_k + B_k Y_k)$$

then, the p-periodic system is asymptotically stabilizable with an H_∞ *norm bound* γ *via a static p-periodic state feedback controller* (13)*.*
Furthermore, if the matrix inequality (15) *has a feasible solution* X_k*, then the p-periodic state feedback control law is given by:*

$$u_k = Y_k X_k^{-1} x_k \quad (16)$$

Proof:

Based on definitions 2.3 and 3.1 with $\Delta A_k = 0$ and $\Delta B_k = 0$, the asymptotic stabilizability of the p-periodic nominal system (11)-(13) with an H_∞ norm bound γ follows by considering the Lyapunov fonction $V_k = x_k^\top P_k x_k$ and the performance criterion

$$J = \sum_{k=0}^{\infty} \left(z_k^\top z_k - \gamma^2 w_k^\top w_k \right) \tag{17}$$

Now to establish the upper bound $\gamma \|w_k\|_2$ for the $l_2[0,\infty)$ norm of z_k, assume that $x_0 = 0$.

Since $x_0 = 0$, we have that for any nonzero $w_k \in l_2[0,\infty)$, J satisfies

$$\begin{aligned} J &= -x_\infty^\top P_k x_\infty + \sum_{k=0}^{\infty} \left(z_k^\top z_k - \gamma^2 w_k^\top w_k + \Delta V_k \right) \\ &= -x_\infty^\top P_k x_\infty + \sum_{k=0}^{\infty} x_k^\top \Big[A^{c\top}_k P_{k+1} A^c_k - P_k + C^{c\top}_k C^c_k + \\ &\quad A^{c\top}_k P_{k+1} B_{wk} \left(\gamma^2 I - B_{wk}^\top P_{k+1} B_{wk} \right)^{-1} B_{wk}^\top P_{k+1} A^c_k \Big] x_k \\ &\quad - \sum_{k=0}^{\infty} \left[w_k - R^{-1} B_{wk}^\top P_{k+1} A^c_k x_k \right]^\top R \left[w_k - R^{-1} B_{wk}^\top P_{k+1} A^c_k x_k \right] \end{aligned} \tag{18}$$

with $R = \left(\gamma^2 I - B_{wk}^\top P_{k+1} B_{wk} \right)$.
Hence, if

$$\begin{cases} \left[A^{c\top}_k P_{k+1} A^c_k - P_k + C^{c\top}_k C^c_k + A^{c\top}_k P_{k+1} B_{wk} R^{-1} B_{wk}^\top P_{k+1} A^c_k \right] < 0 \\ \textit{and} \\ \left(\gamma^2 I - B_{wk}^\top P_{k+1} B_{wk} \right) > 0 \end{cases} \tag{19}$$

then, $J < 0$, i.e., $\|z\|_2 \le \gamma \|w\|_2$ for any nonzero $w_k \in l_2[0,\infty)$.

Using Schur complement technique, condition (19) can be written as follow:

$$\begin{bmatrix} A^{c\top}_k P_{k+1} A^c_k - P_k + C^{c\top}_k C^c_k & A^{c\top}_k P_{k+1} B_{wk} \\ B_{wk}^\top P_{k+1} A^c_k & B_{wk}^\top P_{k+1} B_{wk} - \gamma^2 I \end{bmatrix} < 0 \tag{20}$$

with

$$A^c_k = A_k + B_k K_k \textit{ and } C^c_k = C_k + D_k K_k \tag{21}$$

Multiplying (20) on the left and on the right by

$$\begin{bmatrix} \gamma^{\frac{1}{2}} P_k^{-1} & 0 \\ 0 & \gamma^{-\frac{1}{2}} I \end{bmatrix}$$

and replacing $\gamma^{-1} P_k$ by P_k, we get:

$$\begin{bmatrix} -P_k^{-1} & 0 \\ 0 & -\gamma I \end{bmatrix} + \begin{bmatrix} P_k^{-1} C^{c\top}_k & P_k^{-1} A^{c\top}_k \\ 0 & B_{wk}^\top \end{bmatrix} \times \begin{bmatrix} \gamma^{-1} I & 0 \\ 0 & P_{k+1} \end{bmatrix} \begin{bmatrix} P_k^{-1} C^{c\top}_k & P_k^{-1} A^{c\top}_k \\ 0 & B_{wk}^\top \end{bmatrix}^\top < 0$$

Using the Schur complement, we get:

$$\begin{bmatrix} -P_k^{-1} & 0 & P_k^{-1} C^{c\top}_k & P_k^{-1} A^{c\top}_k \\ 0 & -\gamma I & 0 & B_{wk}^\top \\ C^c_k P_k^{-1} & 0 & -\gamma I & 0 \\ A^c_k P_k^{-1} & B_{wk} & 0 & -P_{k+1}^{-1} \end{bmatrix} < 0 \tag{22}$$

Let $X_k = P_k^{-1}$, for all $k = 0, 1, \ldots, p-1$ and using expressions of A^c_k and C^c_k given by (21), we get (15).

4. Robust controller design

In this part, we will introduce the notion of quadratic stabilization with an H_∞ norm bound which was introduced in [3] and [6].

Definition 4.1 *Let the constant γ be given. The uncertain p-periodic system* (11)-(4) *is said to be quadratically stabilizable with an H_∞ norm bound γ if there exists a linear p-periodic state feedback law* (13) *and a real symmetric positive definite p-periodic matrix P_k such that the inequality*

$$\begin{bmatrix} \Gamma_{11}(k) & \Gamma_{12}^\top(k) \\ \Gamma_{12}(k) & \Gamma_{22}(k) \end{bmatrix} < 0 \tag{23}$$

with

$$\begin{aligned} \Gamma_{11}(k) &= A^{c\top}_k(\Delta_k) P_{k+1} A^c_k(\Delta_k) - P_k + C^{c\top}_k C^c_k \\ \Gamma_{12}(k) &= B_{wk}^\top P_{k+1} A^c_k(\Delta_k) \\ \Gamma_{22}(k) &= B_{wk}^\top P_{k+1} B_{wk} - \gamma^2 I \end{aligned}$$

holds for any admissible uncertainty ΔA_k and ΔB_k.
where $A^c_k(\Delta_k) = A_k(\Delta_k) + B_k(\Delta_k) K_k$ and $C^c_k = C_k + D_k K_k$

Initially, we will show that quadratic stabilization with an H_∞ norm bound $\gamma > 0$ implies stabilization with the same H_∞ norm bound γ.

Lemma 4.1 *If the uncertain p-periodic system* (11)-(4) *is quadratically stabilizable with an H_∞ norm bound $\gamma > 0$, then it is also stabilizable with the same H_∞ norm bound γ.*

Proof:
By following similar arguments as in the proof of theorem 3.1, we obtain, for any admissible uncertainty ΔA_k and ΔB_k:

$$\begin{aligned} A^{c\top}_k(\Delta_k) P_{k+1} B_{wk} \left(\gamma^2 I - B_{wk}^\top P_{k+1} B_{wk} \right)^{-1} B_{wk}^\top P_{k+1} A^c_k(\Delta_k) + \\ A^{c\top}_k(\Delta_k) P_{k+1} A^c_k(\Delta_k) - P_k + C^{c\top}_k C^c_k < 0 \end{aligned} \tag{24}$$

which implies that $J < 0$, i.e., $\|z\|_2 \le \gamma \|w\|_2$ for any nonzero $w_k \in l_2[0,\infty)$.

Theorem 4.1 *The uncertain p-periodic system* (11)-(4) *is said to be asymptotically stabilizable by the p-periodic state feedback controller* (13) *with an H_∞ norm bound γ if there*

exist a p-periodic symmetric matrix $X_k > 0$, a p-periodic matrix Y_k and a p-periodic scalar $\epsilon_k > 0$ such that the following matrix inequality:

$$\Psi(k) = \begin{bmatrix} \Psi_{11}(k) & \Psi_{12}(k) \\ \Psi_{12}^\top(k) & \Psi_{22}(k) \end{bmatrix} < 0 \tag{25}$$

holds with

$$\Psi_{11}(k) = \begin{bmatrix} -X_k & 0 & (C_k X_k + D_k Y_k)^\top \\ 0 & -\gamma I & 0 \\ C_k X_k + D_k Y_k & 0 & -\gamma I \end{bmatrix},$$

$$\Psi_{12}(k) = \begin{bmatrix} (A_k X_k + B_k Y_k)^\top & (N_k X_k + L_k Y_k)^\top & 0 \\ B_{wk}^\top & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Psi_{22}(k) = \begin{bmatrix} -X_{k+1} & 0 & \epsilon_k E_k \\ 0 & -\epsilon_k I & \epsilon_k M_k \\ \epsilon_k E_k^\top & \epsilon_k M_k^\top & -\epsilon_k I \end{bmatrix}.$$

Furthermore, if the matrix inequality (25) *is feasible for some ϵ_k and X_k, then the p-periodic state feedback control law is given by:*

$$u_k = Y_k X_k^{-1} x_k \tag{26}$$

Proof:
Using the same approach as in the proof of theorem 3.1, the uncertain p-periodic system is asymptotically stabilizable with an H_∞ norm bound γ if there exist a symmetric p-periodic matrix $P_k > 0$ such that

$$\begin{bmatrix} -P_k^{-1} & 0 & P_k^{-1} C_k^{c\top} & P_k^{-1} A_k^{c\top}(\Delta_k) \\ 0 & -\gamma I & 0 & B_{wk}^\top \\ C_k^c P_k^{-1} & 0 & -\gamma I & 0 \\ A_k^c(\Delta_k) P_k^{-1} & B_{wk} & 0 & -P_{k+1}^{-1} \end{bmatrix} < 0 \tag{27}$$

which is in fact condition (22) rewritten for the uncertain system, *i.e.*, A_k^c is replaced by

$$\begin{aligned} A^c{}_k(\Delta_k) &= A_k + E_k \Delta_k (I - M_k \Delta_k)^{-1} N_k + \\ &\quad \left(B_k + E_k \Delta_k (I - M_k \Delta_k)^{-1} L_k\right) K_k \\ &= \underbrace{(A_k + B_k K_k)}_{A_k^c} + E_k \Delta_k (I - M_k \Delta_k)^{-1} \underbrace{(N_k + L_k K_k)}_{N_k^c} \\ &= A_k^c + E_k \Delta_k (I - M_k \Delta_k)^{-1} N_k^c. \end{aligned} \tag{28}$$

Therefore, condition (27) can be displayed as follows

$$\begin{bmatrix} -P_k^{-1} & 0 & P_k^{-1} C_k^{c\top} & P_k^{-1} A_k^{c\top} \\ 0 & -\gamma I & 0 & B_{wk}^\top \\ C_k^c P_k^{-1} & 0 & -\gamma I & 0 \\ A_k^c P_k^{-1} & B_{wk} & 0 & -P_{k+1}^{-1} \end{bmatrix} + \mathrm{Sym}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ E_k \end{bmatrix} \Delta_k (I - M_k \Delta_k)^{-1} \begin{bmatrix} P_k^{-1} N_k^{c\top} \\ 0 \\ 0 \\ 0 \end{bmatrix}^\top \right\} < 0. \tag{29}$$

Using Lemma 2.6 in [9], the condition (29) above is equivalent for all admissible uncertainties Δ_k to

$$\begin{bmatrix} -P_k^{-1} & 0 & P_k^{-1} C_k^{c\top} & P_k^{-1} A_k^{c\top} \\ 0 & -\gamma I & 0 & B_{wk}^\top \\ C_k^c P_k^{-1} & 0 & -\gamma I & 0 \\ A_k^c P_k^{-1} & B_{wk} & 0 & -P_{k+1}^{-1} \end{bmatrix} + \begin{bmatrix} \epsilon_k^{-\frac{1}{2}} P_k^{-1} N_k^{c\top} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \epsilon_k^{\frac{1}{2}} E_k \end{bmatrix} \mathcal{M}_k^{-1} \begin{bmatrix} \epsilon_k^{-\frac{1}{2}} P_k^{-1} N_k^{c\top} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \epsilon_k^{\frac{1}{2}} E_k \end{bmatrix}^\top < 0. \tag{30}$$

with $\mathcal{M}_k$ given by (5).
At this step, we notice that performing a Schur complement operation shows that condition (30) is in fact equivalent to

$$\begin{bmatrix} -P_k^{-1} & 0 & P_k^{-1} C_k^{c\top} & P_k^{-1} A_k^{c\top} & P_k^{-1} N_k^{c\top} & 0 \\ 0 & -\gamma I & 0 & B_{wk}^\top & 0 & 0 \\ C_k^c P_k^{-1} & 0 & -\gamma I & 0 & 0 & 0 \\ A_k^c P_k^{-1} & B_{wk} & 0 & -P_{k+1}^{-1} & 0 & \epsilon_k E_k \\ N_k^c P_k^{-1} & 0 & 0 & 0 & -\epsilon_k I & \epsilon_k M_k \\ 0 & 0 & 0 & \epsilon_k E_k^\top & \epsilon_k M_k^\top & -\epsilon_k I \end{bmatrix} < 0$$

where, in addition, we have multiplied both sides of the obtained condition by the diagonal matrix $\mathrm{diag}\{I, I, I, I, \epsilon_k^{\frac{1}{2}} I, \epsilon_k^{\frac{1}{2}} I\}$. Taking $X_k = P_k^{-1}$ and $Y_k = K_k P_k^{-1}$ for all $k = 0, 1, \ldots, p-1$, condition (25) is then recovered easily from above.

Example 1:
Consider the problem of H_∞ control of the discrete-time 2-periodic system (11) with $\Delta A_k = 0$ and $\Delta B_k = 0$ and

$$\left[\begin{array}{cc|cc|c} A_0 & C_0 & B_0 \\ A_1 & C_1 & B_1 \end{array}\right] = \left[\begin{array}{cc|cc|c} 0.1 & -0.2 & 0.2 & 0.1 & 0.5 \\ 0.41 & 0.2 & 0.12 & 0.2 & 0.2 \\ \hline 1.9 & 0.35 & 0.3 & 0.2 & 0.4 \\ 0.22 & 0.3 & 0.1 & 0.3 & 0.3 \end{array}\right];$$

$$\left[\begin{array}{cc|cc|cc|c} B_{w0} & B_{w1} & D_0 & D_1 \end{array}\right] = \left[\begin{array}{cc|cc|cc|cc} 0.2 & 0.3 & 0.3 & 0.35 & 0.15 & 0.12 \\ 0.15 & 0.1 & 0.22 & 0.3 & 0.3 & 0.4 \end{array}\right];$$

Let $\gamma = 0.4$.
we'll take the disturbance signal w_k as an exponential signal.
Using theorem 3.1, we obtain the following matrices:

$$\left[\begin{array}{cc|cc} X_0 & X_1 \\ Y_0^\top & Y_1^\top \end{array}\right] = \left[\begin{array}{cc|cc} 19.3349 & -5.8292 & 0.8816 & -2.6212 \\ -5.8292 & 2.9338 & -2.6212 & 22.6799 \\ & -7.4438 & 0.9923 & \\ & 2.2559 & -16.6127 & \end{array}\right]$$

Then, the state feedback gain matrices are given by:

$$\begin{bmatrix} K_0^\top & K_1^\top \end{bmatrix} = \left[\begin{array}{cc} -0.3307 & -1.6031 \\ 0.1799 & -0.9178 \end{array}\right];$$

Hence, we noticed that the statements of definition 3.1 are verified for $\Delta A_k = 0$ and $\Delta B_k = 0$.

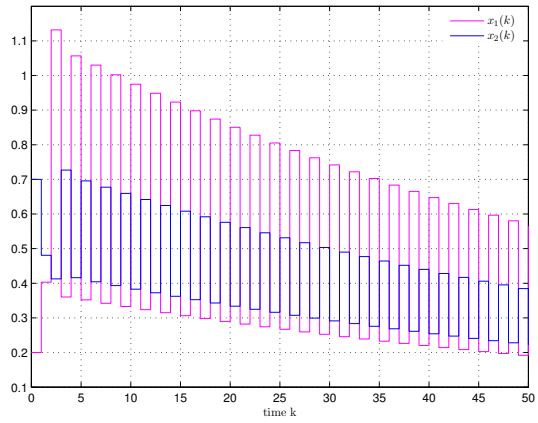

Figure 1. State trajectory x_k from random initial value

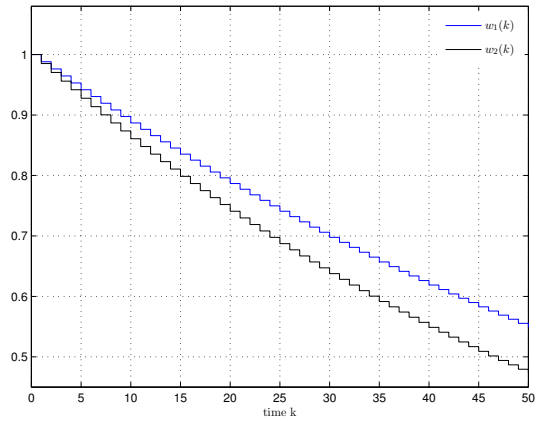

Figure 2. Disturbance input trajectory w_k

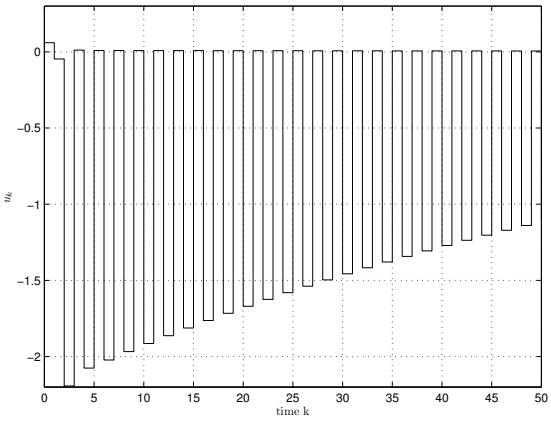

Figure 3. Control signal

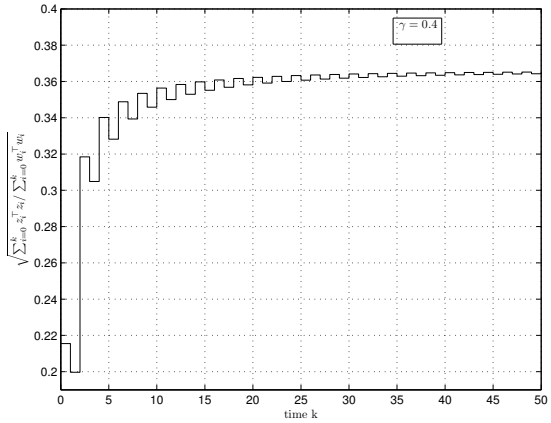

Figure 4. Variation of the ratio $\sqrt{\dfrac{\sum_{i=0}^{k} z_i^\top z_i}{\sum_{i=0}^{k} w_i^\top w_i}}$.

Figure 4 shows that the 2-periodic system (11) with the controller (13) achieves a H_∞ norm bound less than $\gamma = 0.4$ complying with the performance index (14).

Example 2:

Consider the problem of robust H_∞ control of the discrete-time 2-periodic system (11)-(4) with:

$$\begin{bmatrix} A_0 & B0 & C0 \\ A_1 & B_1 & C_1 \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 1 & -0.4 & 0.1 & 0.2 & 0.2 & 0.3 \\ 0.3 & 0.1 & 0.4 & 0.12 & 0.1 & 0.2 \\ 1.2 & 0.5 & 0.2 & 0.21 & 0.3 & 0.1 \\ 0.2 & -0.3 & 0.2 & 0.2 & 0.2 & 0.3 \end{array}\right];$$

$$\begin{bmatrix} D_0 & E_0 & M_0 \\ D_1 & E_1 & M_1 \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 0.15 & 0.3 & 0.1 & 0.1 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.2 & 0.15 & 0.3 & 0.2 \\ 0.12 & 0.4 & 0.3 & 0.12 & 0.1 & 0.2 \\ 0.3 & 0.5 & 0.1 & 0.22 & 0.22 & 0.3 \end{array}\right];$$

$$\begin{bmatrix} N_0 & L_0 & B_{w0} \\ N_1 & L_1 & B_{w1} \end{bmatrix} = \left[\begin{array}{cc|cc|cc} 0.2 & 0.2 & 0.2 & 0.5 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0.12 & 0.2 & 0.15 & 0.1 \\ 0.3 & 0.2 & 0.3 & 0.2 & 0.2 & 0.3 \\ 0.22 & 0.3 & 0.1 & 0.3 & 0.22 & 0.3 \end{array}\right];$$

Let $\gamma = 0.8$.

we'll take the disturbance signal w_k as an exponential signal. Using theorem 4.1, we obtain the following matrices:

$$\begin{bmatrix} X_0 & Y_0 \\ X_1 & Y_1 \end{bmatrix} =$$

$$\left[\begin{array}{cc|cc} 2.1338 & 2.6545 & -4.0881 & -3.0149 \\ 2.6545 & 4.3619 & -0.7266 & -2.2446 \\ \hline 1.1535 & -1.5153 & -0.7814 & -0.2389 \\ -1.5153 & 2.7508 & 0.2821 & -0.3925 \end{array}\right]$$

and the scalars $\epsilon_0 = 0.4997$ and $\epsilon_1 = 1.2468$.
Then, the state feedback gain matrices are given by:

$$\begin{bmatrix} K_0 & K_1 \end{bmatrix} = \left[\begin{array}{cc|cc} -4.3469 & 1.9541 & -2.8645 & -1.6649 \\ 1.2334 & -1.2652 & 0.2068 & -0.0288 \end{array}\right];$$

Hence, we noticed that the statements of definition 3.1 are verified for any admissible parameter uncertainty ΔA_k and ΔB_k.

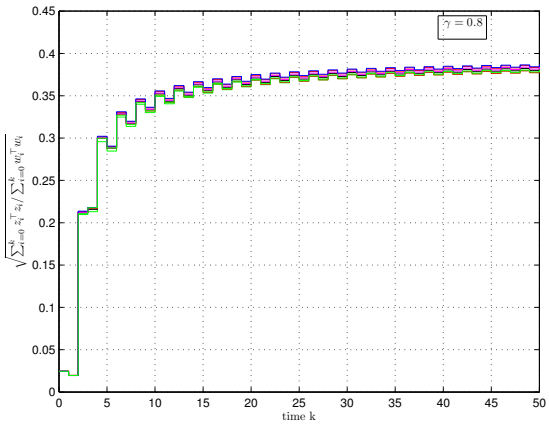

Figure 5. The variation of $\sqrt{\dfrac{\sum_{i=0}^{k} z_i^\top z_i}{\sum_{i=0}^{k} w_i^\top w_i}}$ for some values of Δ_k satisfying (4)

Figure 5 shows the evolution of $\sqrt{\dfrac{\sum_{i=0}^{k} z_i^\top z_i}{\sum_{i=0}^{k} w_i^\top w_i}}$ for some values of Δ_k satisfying (4). The figure shows explicitly that the closed-loop uncertain 2-periodic system exhibits a H_∞ norm bound less than $\gamma = 0.8$.

5. Conclusion

In this paper, a sufficient condition for H_∞ control problem of discrete-time periodic system has been presented in terms of Linear Matrix Inequality (LMI). The obtained results are then extended for the uncertain periodic system. The uncertainties are of the form of Linear Fractional Representation type which includes the case of norm bounded uncertainties as a special case. numerical examples are given to illustrate the proposed results.

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