Delay-Dependent Stability of Differential Linear Repetitive Processes with Delays Along Two Directions

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***Abstract*— Repetitive processes are a distinct class of 2D systems (i.e. information propagation in two independent directions) of both systems theoretic and applications interest. they cannot be controlled by direct extension of existing techniques from either standard (termed 1D here) or 2D systems theory. this paper deals with the stability of differential linear repetitive processes with delays in both directions of information propagation. Stability criteria are developed: is delay-dependent with tow approaches based on Lyapunov-Krasovskii functions and the resulting conditions are expressed in terms of linear matrix inequalities.**

Keywords*– *Differential repetitive processes, tow-dimensional systems, stability along the pass, delay-dependent stability.

I. Introduction

As one of important sources of instability and oscillation, time delay is unavoidable in technology and nature. It extensively exists in various engineering systems such as chemical processes, long transmission lines in pneumatic systems [1], and can make important effects on the properties of dynamic systems.

In this paper, the new results are one delay-dependent stability criteria for so-called differential linear repetitive processes which arise in the modeling of many physical examples(-see, for example [2]). The resulting conditions are formulated in terms of linear matrix inequalities (LMIs).

Notation : we use standard notation throughout this paper. The notation $P > 0 (< 0)$ is used for positive (negative) definite matrices. $*$ stands for the symmetric term of the diagonal elements of square symmetric matrix. I denotes the identity matrix with appropriate dimension. the superscript "T" represents the transpose.

II. BACKGROUND

In this paper we consider differential linear repetitive processes with delays in both directions of information propagation, i.e. along the pass and form pass to pass, delays whose dynamics are described by the following state space model over $0 \leq p \leq \alpha$, $k \geq 0$

$$\begin{array}{ll} \dot{x}_{k+1}(t) = & Ax_{k+1}(t) + A_{1d}x_{k+1}(t-d_1) + B_0 y_k(t) \\ & +B_{1d}y_{k-d2}(t) \\ y_{k+1}(t) = & Cx_{k+1}(t) + C_{1d}x_{k+1}(t-d_1) + D_0 y_k(t) \\ & +D_{1d}y_{k-d2}(t) \end{array} \tag{1}$$

Here on pass k, $x_k(t) \in \Re^n$ is the state vector and $y_k(t) \in \Re^m$ is the pass profile vector. The positive scalar d_1 and integer d_2 represent unknown but constant delays in the along the pass and pass to pass directions respectively and satisfy

$0 \leq d_1 \leq \overline{d_1} \leq \infty$; $0 \leq d_2 \leq \overline{d_2} \leq \infty$

Where $\overline{d_1}$ and $\overline{d_2}$ are delays upper bounds.

To complete the process description, it is necessary to specify the boundary conditions. i.e. the state initial vector on each pass and the initial pass profile. Here these are taken to be of the form

$$\begin{array}{l} x_{k+1}(\Theta) = d_{k+1}(\Theta), k \geq 0, \forall \Theta \in [-d_1, 0] \\ y_\vartheta(t) = f_\vartheta(t), \vartheta = -d_2, -d_2+1, ..., 0; \forall t : 0 \leq t \leq \alpha \end{array} \tag{2}$$

where $d_{k+1}(\Theta)$ are given continuously differentiable function on $[-d_1, 0]$ and $f_\vartheta(t)$ are known piece-wise continuous functions of t.

the goals of this paper are to develop solutions to the following questions:

✓When is a process with delays of the form considered here stable?

✓How large can the delays be before the stability property is lost?

The analysis in this paper will make extensive use of the following known results.

Lemma II.1 *[3] Assume that sequences $a(.) \in \Re^{n_a}$ and $b(.) \in \Re^{n_b}$ are defined on an interval Ω and $N \in \Re^{n_a \times n_b}$ is given. then for any matrices $X \in \Re^{n_a \times n_b}$, $Y \in \Re^{n_a \times n_b}$ and $Z \in \Re^{n_a \times n_b}$ such that*

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \geq 0$$

the following holds

$$\begin{array}{l} -2\int_\Omega a(\theta)^T N b(\theta) d\theta \\ \leq \int_\Omega \begin{bmatrix} a(\theta) \\ b(\theta) \end{bmatrix}^T \begin{bmatrix} X & Y-N \\ Y^T - N^T & Z \end{bmatrix} \begin{bmatrix} a(\theta) \\ b(\theta) \end{bmatrix} \end{array} \tag{3}$$

Lemma II.2 *[4] Assume that sequences $a(.) \in \Re^{n_a}$ and $b(.) \in \Re^{n_b}$ are defined on an interval Ω and $M \in \Re^{n_a \times n_b}$*

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is given. then for any matrices $R \in \Re^{n_a \times n_b}$, $S \in \Re^{n_a \times n_b}$ *and* $Q \in \Re^{n_a \times n_b}$ *such that*

$$\left[\begin{array}{cc} R & S \\ S^T & Q \end{array} \right] \geq 0$$

the following holds

$$\begin{array}{l} -2\sum_\Omega a(m)^T M b(m) \\ \leq \sum_\Omega \left[\begin{array}{c} a(m) \\ b(m) \end{array} \right]^T \left[\begin{array}{cc} R & S-M \\ S^T - M^T & Q \end{array} \right] \left[\begin{array}{c} a(m) \\ b(m) \end{array} \right] \end{array} \quad (4)$$

III. STABILITY THEORY

The stability theory for constant pass length linear repetitive processes consists of tow distinct concepts termed asymptotic stability and stability along the pass respectively. Noting again the unique control problem for these processes, this theory demands that bounded sequences of inputs produce bounded sequences f pass profiles where here bounded is defined in term of the norm on the underlying function space. The essential difference between them is that asymptotic stability demands this property over the finite pass length whereas stability along the pass is stronger in that it demands this property uniformly, i.e. independent of the pass length.

Consider now the case when no delays are present and hence the process dynamics are described by

$$\begin{array}{rcl} \dot{x}_{k+1}(t) = & Ax_{k+1}(t) + B_0 y_k(t) \\ y_{k+1}(t) = & Cx_{k+1}(t) + D_0 y_k(t) \end{array} \quad (5)$$

Necessary and sufficient conditions for stability along the pass of process described by this model are available in several forms but here it is the following which is relevant.

Lemma III.1 *[2] Differential linear repetitive processes described by (5) are stable along the pass if, and only if, the 2D characteristic polynomial*

$$C(s, z_2) := \det \left[\begin{array}{cc} sI - A & -B_0 \\ -z_2 C & I - z_2 D_0 \end{array} \right] \quad (6)$$

satisfies

$C(s, z_2) \neq 0$, $\forall (s, z_2)$: $Re(s_1) \geq 0$ *and* $\mid z_2 \mid \leq 1$

Several sets of tests for the necessary and sufficient condition of this last result are known. However, none of them have provided a basis on which to design a control law for stability and/or performance. This has led to the use of a sufficient based test which provides this essential property and here will be the route to developing answers to the questions posed in the previous section. The route is to use a Lyapunov function interpretation of stability along the pass which leads to LMI based conditions.

The Lyapunov function of interest here is given by

$$\begin{array}{rl} V_1(k,t) & = V_{11}(k,t) + V_{12}(k,t) \\ & = x_{k+1}^T(t) P_1 x_{k+1}(t) + y_k^T(t) P_2 y_k(t) \end{array} \quad (7)$$

where $P_1 > 0$ and $P_2 > 0$. (This function is a combination of tow independent function to take account of the tow directions of information propagation). Also

$$\begin{array}{rcl} \dot{V}_{11}(k,t) & = & \dot{x}_{k+1}^T(t) P_1 x_{k+1}(t) + x_{k+1}^T(t) P_1 \dot{x}_{k+1}(t) \\ \Delta V_{12}(k,t) & = & y_{k+1}^T(t) P_2 y_{k+1}(t) - y_k^T(t) P_2 y_k(t) \end{array}$$

and hence the associated increment for (7) is given by

$$\Delta V_1(k,t) = \dot{V}_{11}(k,t) + \Delta V_{12}(k,t) \quad (8)$$

Then we have the following result.

Lemma III.2 *[5] A differential linear repetitive processes described by (5) is stable along the pass if*

$$\Delta V_1(k,t) < 0 \quad (9)$$

It is possible to write the condition of this result in terms of an LMI see [5] for the details. Note also that if stability along the pass holds then $lim_{p \longrightarrow \infty} x_k(p) = 0$ *and* $lim_{k \longrightarrow \infty} y_k(p) = 0$.

Now we proceed to extend this last result to delay-dependent of the questions posed in the previous section using a Lyapunov-Krasovskii function approach which are then refined to LMI based conditions.

Consider the delay-dependent case. Then the candidate Lyapunov-Krasovskii function is

$$V_2(k,t) = V_{21}(k,t) + V_{22}(k,t) + V_{23}(k,t) + V_{24}(k,t) \quad (10)$$

where

$$\begin{array}{rcl} V_{21}(k,t) & = & x_{k+1}^T(t) P_1 x_{k+1}(t) \\ V_{22}(k,t) & = & \displaystyle\int_{-d_1}^{0} \int_{t+\vartheta}^{t} \dot{x}_{k+1}^T(\theta) Z \dot{x}_{k+1}(\theta) d\theta d\vartheta \\ & & \displaystyle + \int_{t-d_1}^{t} x_{k+1}^T(\theta) W x_{k+1}(\theta) d\theta \\ V_{23}(k,t) & = & y_k^T(t) P_2 y_k(t) \\ V_{24}(k,t) & = & \displaystyle\sum_{\theta=1}^{d_2} \sum_{m=k-\theta}^{k-1} \Delta\xi_m^T(t) Q \Delta\xi_m(t) \\ & & \displaystyle + \sum_{\theta=1}^{d_2} y_{k-\theta}^T(t) U y_{k-\theta}(t) \end{array}$$

and $P_1 > 0$,$P_2 > 0$,$Q > 0$,$Z > 0$,$W > 0$ and $U > 0$ are matrices to be determined. Also

$$\Delta V_2(k,t) = \dot{V}_{21}(k,t) + \dot{V}_{22}(k,t) + \Delta V_{23}(k,t) + \Delta V_{24}(k,t) \quad (11)$$

Now we can define delay-dependent stability along the pass differential linear repetitive processes described in (1).

Lemma III.3 *[6] A differential linear repetitive processes described by (1) is said to be delay-dependent*

stable along the pass (i.e. stable along the pass for any delay $d_1 \in (0, \overline{d_1}]$ and $d_2 \in (0, \overline{d_2}]$ provided

$$\Delta V_2(k,t) < 0 \tag{12}$$

To proceed, first note that

$$x_{k+1}(t-d_1) = x_{k+1}(t) - \int_{t-d_1}^{t} \dot{x}_{k+1}(\theta)d\theta \tag{13}$$

and

$$y_{k-d_2}(t) = y_k(t) - \sum_{\theta=1}^{d_2}(y_{k-\theta+1}(t) - y_{k-\theta}(t)) \tag{14}$$

and substituting these into (1) yields

$$\begin{aligned} \dot{x}_{k+1}(t) &= (A+A_{1d})x_{k+1}(t) + B_0 y_k(t) \\ &\quad + B_{1d}y_{k-d2}(t) \\ &\quad - A_{1d}\int_{t-d_1}^{t} \dot{x}_{k+1}(\theta)d\theta \\ y_{k+1}(t) &= Cx_{k+1}(t) + C_{1d}x_{k+1}(t-d_1) \\ &\quad + (D_0 + D_{1d})y_k(t) \\ &\quad D_{1d} - \sum_{\theta=1}^{d_2}\Delta\xi_{k-\theta}(t) \end{aligned} \tag{15}$$

where $\Delta\xi_{k-\theta}(t) = y_{k-\theta+1}(t) - y_{k-\theta}(t)$.

IV. LMI STABILITY CONDITIONS

In this section, develop LMI based (and hence computable) conditions for the stability along the pass definitions of the previous section.
Consider again the Lyapunov function (10).then

$$\begin{aligned} \dot{V}_{21}(k,t) &= \dot{x}_{k+1}^T(t)P_1x_{k+1}(t) + x_{k+1}^T(t)P_1\dot{x}_{k+1}(t) \\ &= 2x_{k+1}^T(t)P_1B_{1d}y_{k-d_2}(t) \\ &\quad + 2x_{k+1}^T(t)P_1B_0y_k(t) \\ &\quad + x_{k+1}^T(t)P_1(A+A_{1d})x_{k+1}(t) \\ &\quad - 2x_{k+1}^T(t)P_1A_{1d}\int_{t-d_1}^{t}\dot{x}_{k+1}(\theta)d\theta \end{aligned}$$

Also set $a(.) = x_{k+1}(t)$, $b(.) = \dot{x}_{k+1}(t)$, $N = P_1A_{1d}$ and then apply result of lemma II.1 to yield

$$\begin{aligned} \dot{V}_{21}(k,t) &\leq 2x_{k+1}^T(t)P_1B_{1d}y_{k-d_2}(t) \\ &\quad + 2x_{k+1}^T(t)P_1B_0y_k(t) \\ &\quad + 2x_{k+1}^T(t)P_1(A+A_{1d})x_{k+1}(t) \\ &\quad + d_1x_{k+1}^T(t)Xx_{k+1}(t) \\ &\quad + \int_{t-d_1}^{t}\dot{x}_{k+1}^T(\theta)Z\dot{x}_{k+1}(\theta)d\theta \\ &\quad + 2x_{k+1}^T(t)(Y - P_1A_{1d})\int_{t-d_1}^{t}\dot{x}_{k+1}(\theta)d\theta \\ &\leq 2x_{k+1}^T(t)(P_1A + A^TP_1 + d_1X \\ &\quad + Y + Y^T)x_{k+1}(t) \\ &\quad + 2x_{k+1}^T(t)P_1B_0y_k(t) \\ &\quad + \int_{t-d_1}^{t}\dot{x}_{k+1}^T(\theta)Z\dot{x}_{k+1}(\theta)d\theta \\ &\quad + 2x_{k+1}^T(t)(P_1A_{1d} - Y)x_{k+1}(t-d_1) \\ &\quad + 2x_{k+1}^T(t)P_1B_{1d}y_{k-d_2}(t) \end{aligned}$$

Also

$$\begin{aligned} \dot{V}_{22}(k,t) &= d_1\dot{x}_{k+1}^T(t)Z\dot{x}_{k+1}(t) \\ &\quad - \int_{t-d_1}^{t}\dot{x}_{k+1}^T(\theta)Z\dot{x}_{k+1}(\theta)d\theta \\ &\quad + x_{k+1}^T(t)Wx_{k+1}(t) \\ &\quad - x_{k+1}^T(t-d_1)Wx_{k+1}(t-d_1) \end{aligned}$$

and for convenience of notation introduce
$\varphi_k(t) = [x_{k+1}^T(t)\ \ x_{k+1}^T(t-d_1)\ \ y_k^T(t)]^T$
$\Lambda = [C\ \ C_{1d}\ \ (D_0 + D_{1d})]$
Then using (15)

$$\begin{aligned} \Delta V_{23}(k,t) &= y_{k+1}^T(t)P_2y_{k+1}(t) - y_k^T(t)P_2y_k(t) \\ &= \varphi_k^T(t)\Lambda^TP_2\Lambda\varphi_k(t) - y_k^T(t)P_2y_k(t) \\ &\quad - 2\varphi_k^T(t)\Lambda^TP_2D_{1d}\sum_{\theta=1}^{d_2}\Delta\xi_{k-\theta}(t) \\ &\quad + (y_k(t) - y_{k-d_2}(t))^TD_{1d}^TP_2 \\ &\quad \times D_{1d}(y_k(t) - y_{k-d_2}(t)) \end{aligned}$$

Now consider the term

$$-2\varphi_k^T(t)\Lambda^TP_2D_{1d}\sum_{\theta=1}^{d_2}\Delta\xi_{k-\theta}(t)$$

and set $a(.) = \Lambda\varphi_k(t)$, $b(.) = \Delta\xi_{k-\theta}(t)$ and $M = P_2D_{1d}$. Then an obvious application of Lemma II.2 gives

$$\begin{aligned} &-2\varphi_k^T(t)\Lambda^TP_2D_{1d}\textstyle\sum_{\theta=1}^{d_2}\Delta\xi_{k-\theta}(t) \\ &\leq \textstyle\sum_{\theta=1}^{d_2}\begin{bmatrix}\Lambda\varphi_k(t)\\ \Delta\xi_{k-\theta}(t)\end{bmatrix}^T\begin{bmatrix}R & S-M\\ S^T-M^T & Q\end{bmatrix} \\ &\times\begin{bmatrix}\Lambda\varphi_k(t)\\ \Delta\xi_{k-\theta}(t)\end{bmatrix} \end{aligned} \tag{16}$$

$$
\begin{aligned}
&= d_2\varphi_k^T(t)\Lambda^T R\Lambda\varphi_k(t)\\
&\quad -(y_k(t)-y_{k-d_2}(t))^T[S-M]^T\Lambda\varphi_k(t)\\
&\quad +\varphi_k^T(t)\Lambda^T[S-M](y_k(t)-y_{k-d_2}(t))\\
&\quad +\sum_{\theta=1}^{d_2}\Delta\xi_{k-\theta}^T(t)Q\Delta\xi_{k-\theta}(t)
\end{aligned}
$$

Hence

$$
\Delta V_{23}(k,t)\le\sum_{\theta=1}^{d_2}\Delta\xi_{k-\theta}^T(t)Q\Delta\xi_{k-\theta}(t)+\chi^T(k,t)\Theta\chi(k,t)
$$

where

$$
\chi(k,t)=[x_{k+1}^T(t)\;\; y_k^T(t)\;\; x_{k+1}^T(t-d_1)\;\; y_{k-d_2}^T(t)]^T
$$

and

$$
\Theta=\begin{bmatrix}\Upsilon_{11}&\Upsilon_{12}&\Upsilon_{13}&\Upsilon_{14}\\ \Upsilon_{21}&\Upsilon_{22}&\Upsilon_{23}&\Upsilon_{24}\\ \Upsilon_{31}&\Upsilon_{32}&\Upsilon_{33}&\Upsilon_{34}\\ \Upsilon_{41}&\Upsilon_{42}&\Upsilon_{43}&\Upsilon_{44}\end{bmatrix}
$$

$\Upsilon_{11}=C^T(P_2+d_2R)C$
$\Upsilon_{12}=C^T(P_2+d_2R)(D_0+D_{1d})+C^T(S-P_2D_{1d})$
$\Upsilon_{13}=C^T(P_2+d_2R)C_{1d}$
$\Upsilon_{14}=-C^T(S-P_2D_{1d})$
$\Upsilon_{21}=(D_0+D_{1d})^T(P_2+d_2R)C+(S-P_2D_{1d})^TC$
$\Upsilon_{22}=(D_0+D_{1d})^T(P_2+d_2R)(D_0+D_{1d})+D_{1d}^TP_2D_{1d}$

$+(S-P_2D_{1d})^T(D_0+D_{1d})+(D_0+D_{1d})^T(S-P_2D_{1d})-P_2$

$\Upsilon_{23}=(D_0+D_{1d})^T(P_2+d_2R)C_{1d}+(S-P_2D_{1d})^TC_{1d}$
$\Upsilon_{24}=-(D_0+D_{1d})^T(S-P_2D_{1d})-D_{1d}^TP_2D_{1d}$
$\Upsilon_{31}=C_{1d}^T(P_2+d_2R)C$
$\Upsilon_{32}=C_{1d}^T(P_2+d_2R)(D_0+D_{1d})+C_{1d}^T(S-P_2D_{1d})$
$\Upsilon_{33}=C_{1d}^T(P_2+d_2R)C_{1d}^T$
$\Upsilon_{34}=-C_{1d}^T(S-P_2D_{1d})$
$\Upsilon_{41}=-(S-P_2D_{1d})^TC$
$\Upsilon_{42}=-(S-P_2D_{1d})^T(D_0+D_{1d})-D_{1d}^TP_2D_{1d}$
$\Upsilon_{34}=-(S-P_2D_{1d})^TC_{1d}$
$\Upsilon_{44}=D_{1d}^TP_2D_{1d}$
Also

$$
\begin{aligned}
\Delta V_{24}(k,t) &= \sum_{\theta=1}^{d_2}\sum_{m=-\theta+1}^{0}\Delta\xi_{k+m}^T(t)Q\Delta\xi_{k+m}(t)\\
&\quad -\sum_{\theta=1}^{d_2}\sum_{m=-\theta}^{-1}\Delta\xi_{k+m}^T(t)Q\Delta\xi_{k+m}(t)\\
&\quad +\sum_{\theta=0}^{d_2-1}y_{k-\theta}^T(t)Uy_{k-\theta}(t)\\
&\quad -\sum_{\theta=1}^{d_2}y_{k-\theta}^T(t)Uy_{k-\theta}(t)\\
&= d_2\Delta\xi_k^T(t)Q\Delta\xi_k(t)\\
&\quad -\sum_{\theta=1}^{d_2}\Delta\xi_{k-\theta}^T(t)Q\Delta\xi_{k-\theta}(t)\\
&\quad +y_k^T(t)Uy_k(t)-y_k^T(t)Uy_k(t) \qquad (17)
\end{aligned}
$$

and further manipulations lead to

$$
\begin{aligned}
\Delta V_{24}(k,t) &= d_2\Xi^TQ\Xi-\sum_{\theta=1}^{d_2}\Delta\xi_{k-\theta}^T(t)Q\Delta\xi_{k-\theta}(t)\\
&\quad +y_k^T(t)Uy_k(t)-y_k^T(t)Uy_k(t) \qquad (18)
\end{aligned}
$$

where $\Xi=Cx_{k+1)}(t)+C_{1d}x_{k+1}(t-d_1)+(D_0-I)y_k(t)+D_{1d}y_{k-d_2}(t)$. Now we have the following result that gives a condition for delay-dependent stability along the pass. This theorem constitute the corrected results of theorem 2 in [6], which continent some mistakes.

Theorem 1 *(correct theorem 2 in [6]) A differential linear repetitive processes described by (1) is stable along the pass for any delay* $0<d_1\le\overline{d_1}$ *and* $0<d_2\le\overline{d_2}$ *if there exist matrices* $P_1>0$, $P_2>0$, $Q>0$, $Z>0$, $U>0$, $W>0$ *and* R, S, X, Y *such that the following LMIs are feasible*

$$
\begin{bmatrix}\Lambda_{11}&\Lambda_{12}&\Lambda_{13}&\Lambda_{14}&\Lambda_{15}&\Lambda_{16}&\Lambda_{17}\\ *&\Lambda_{22}&\Lambda_{23}&\Lambda_{24}&\Lambda_{25}&\Lambda_{26}&\Lambda_{27}\\ *&*&\Lambda_{33}&\Lambda_{34}&\Lambda_{35}&\Lambda_{36}&\Lambda_{37}\\ *&*&*&\Lambda_{44}&\Lambda_{45}&\Lambda_{46}&0\\ *&*&*&*&\Lambda_{55}&0&0\\ *&*&*&*&0&\Lambda_{66}&0\\ *&*&*&0&0&0&\Lambda_{77}\end{bmatrix}<0 \qquad (19)
$$

$$
\begin{bmatrix}X&Y\\ Y^T&Z\end{bmatrix}\ge 0 \qquad (20)
$$

$$
\begin{bmatrix}R&S\\ S^T&Q\end{bmatrix}\ge 0 \qquad (21)
$$

where
$\Lambda_{11}=A^TP_1+P_1A+d_1X+Y+Y^T+W$
$\Lambda_{12}=C^T(S-P_2D_{1d})+P_1B_0$
$\Lambda_{13}=P_1A_{1d}-Y$
$\Lambda_{14}=C^T(-S+P_2D_{1d})+P_1B_{1d}$
$\Lambda_{15}=d_2C^TQ$, $\Lambda_{16}=d_1A^TZ$

$\Lambda_{17} = C^T(P_2 + d_2R)$
$\Lambda_{22} = (S - P_2D_{1d})^T(D_0 + D_{1d}) + (D_0 + D_{1d})^T(S - P_2D_{1d}) - P_2 + U + D_{1d}^T P_2 D_{1d}$
$\Lambda_{23} = (S - P_2D_{1d})^T C_{1d}$
$\Lambda_{24} = -D_{1d}^T P_2 D_{1d} - (D_0 + D_{1d})^T(S - P_2D_{1d})$
$\Lambda_{25} = d_2(D_0 - I)^T Q, \Lambda_{26} = d_1 B_0^T Z$
$\Lambda_{27} = (D_0 + D_{1d})^T(P_2 + d_2R)$
$\Lambda_{33} = -W$, $\Lambda_{34} = -(C_{1d})^T(S - P_2D_{1d})$
$\Lambda_{35} = d_2 C_{1d}^T Q$, $\Lambda_{36} = d_1 A_{1d}^T Z$
$\Lambda_{37} = (C_{1d})^T(P_2 + d_2R)$, $\Lambda_{44} = D_{1d}^T P_2 D_{1d} - U$
$\Lambda_{45} = d_2 D_{1d}^T Q, \Lambda_{46} = d_1 B_{1d}^T Z$
$\Lambda_{55} = -d_2 Q$, $\Lambda_{66} = -d_1 Z, \Lambda_{77} = -(P_2 + d_2R)^T$

Proof: Assume that there exist matrices $P_1 > 0$, $P_2 > 0$, $Q > 0$, $Z > 0$, $U > 0$, $W > 0$ and R, S, X, Y such that the LMIs (19), (20) and (21) are feasible. Then the Lyapunov function (10) is positive definite and the corresponding increment is given by (11). Next, apply the Schur's complement to the inequality (19) and proceed as per the proof of the last result to establish that

$$\Delta V_2(k,t) < 0$$

as required ■

Theorem 2 *A differential linear repetitive processes described by (1) is stable along the pass for any delay* $0 < d_1 \leq \overline{d_1}$ *and* $0 < d_2 \leq \overline{d_2}$ *if there exist matrices* $P_1 > 0$, $P_2 > 0$, $Q > 0$, $Z > 0$, $U > 0$, $W > 0$ *and* Y_1, T_1, Y_2, T_2 *such that the following LMI are feasible*

$$\Theta_1 < 0$$

Where

$$\Theta_1 = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & \Psi_{15} & 0 & \Psi_{17} & \Phi_{18} & \Psi_{19} \\ * & \Psi_{22} & 0 & 0 & \Psi_{25} & 0 & \Psi_{27} & \Psi_{28} & \Psi_{29} \\ * & * & \Phi_{33} & \Psi_{34} & 0 & \Psi_{36} & \Psi_{37} & \Psi_{38} & \Psi_{39} \\ * & * & * & \Psi_{44} & 0 & \Psi_{46} & \Psi_{47} & \Psi_{48} & \Psi_{49} \\ * & * & * & * & \Psi_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Psi_{66} & 0 & 0 & 0 \\ * & * & * & * & * & * & \Psi_{77} & 0 & 0 \\ * & * & * & * & * & * & * & \Psi_{88} & 0 \\ * & * & * & * & * & * & * & * & \Psi_{99} \end{bmatrix} \quad (22)$$

where
$\Psi_{11} = A^T P_1 + P_1 A + Y_1 + Y_1^T + W$
$\Psi_{12} = P_1 A_{1d} + T_1^T - Y_1$, $\Psi_{13} = P_1 B_0$
$\Psi_{14} = P_1 B_{1d}$, $\Psi_{15} = -d_1 Y_1$, $\Psi_{17} = A^T d_1 Z$
$\Phi_{18} = C^T P_2$, $\Psi_{19} = d_2 C^T Q$, $\Psi_{22} = -T_1^T - T_1 - W$
$\Psi_{25} = -d_1 T_1$, $\Psi_{27} = A_{1d}^T d_1 Z$, $\Psi_{28} = C_{1d}^T P_2$
$\Psi_{29} = d_2 C_{1d}^T Q$, $\Phi_{33} = Y_2^T + Y_2 + U - P_2$, $\Psi_{34} = T_2^T - Y_2$
$\Psi_{36} = -d_2 Y_2$, $\Psi_{37} = B_0^T d_1 Z$, $\Phi_{38} = D_0^T P_2$
$\Phi_{39} = d_2(D_0 - I)^T Q$, $\Psi_{44} = -T_2^T - T_2 - U$
$\Psi_{46} = -d_2 T_2$, $\Psi_{47} = B_{1d}^T d_1 Z$, $\Phi_{48} = D_{1d}^T P_2$
$\Phi_{49} = d_2 D_{1d}^T Q$, $\Psi_{55} = -d_1 Z$, $\Psi_{66} = -d_2 Q$
$\Psi_{77} = -d_1 Z$, $\Psi_{88} = -P_2$, $\Psi_{99} = -d_2 Q$

Proof:
Using a similar line as in the proof of theorem 1 in [7] (1D).
First evaluate $\dot{V}_{21}(k,t)$ (using (13))to yield

$$\begin{aligned} \dot{V}_{21}(k,t) &= \dot{x}_{k+1}^T(t) P_1 x_{k+1}(t) + x_{k+1}^T(t) P_1 \dot{x}_{k+1}(t) \\ &\quad + 2x_{k+1}^T(t) Y_1 [x_{k+1}(t) - x_{k+1}(t-d_1)] \\ &\quad + 2x_{k+1}^T(t-d_1) T_1 [x_{k+1}(t) - x_{k+1}(t-d_1)] \\ &\quad - 2x_{k+1}^T(t) Y_1 \int_{t-d_1}^{t} \dot{x}_{k+1}(\theta_1) d\theta_1 \\ &\quad - 2x_{k+1}^T(t-d_1) T_1 \int_{t-d_1}^{t} \dot{x}_{k+1}(\theta_1) d\theta_1 \end{aligned}$$

Hence

$$\dot{V}_{21}(k,t) = \frac{1}{d_1 d_2} \int_{t-d_1}^{t} \sum_{\theta_2=1}^{d_2} \chi_1^T(k,t) \Theta_1 \chi_1(k,t) d\theta_1 \quad (23)$$

Where

$$\chi_1 = [x_{k+1}(t) \quad x_{k+1}(t-d_1) \quad y_k(t) \quad y_{k-d_2}(t)$$

$$\dot{x}_{k+1}(\theta_1) \quad \triangle \xi_{k-\theta_2}(t)]$$

and
$\Theta_1 =$

$$\begin{bmatrix} \Xi_1 & P_1 A_{1d} - Y_1 + T_1^T & P_1 B & P_1 B_{1d} & -d_1 Y_1 & 0 \\ * & -T_1^T - T_1 & 0 & 0 & -d_1 T_1 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \end{bmatrix}$$

$\Xi_1 = P_1 A + A^T P_1 + Y_1 + Y_1^T$
Also

$$\begin{aligned} \dot{V}_{22}(k,t) &= \int_{t-d_1}^{t} [\dot{x}_{k+1}(t) Z \dot{x}_{k+1}(t) \\ &\quad - \dot{x}_{k+1}(\theta_1) Z \dot{x}_{k+1}(\theta_1)] d\theta_1 \\ &\quad + x_{k+1}(t) W x_{k+1}(t) \\ &\quad - x_{k+1}(t-d_1) W x_{k+1}(t-d_1) \end{aligned}$$

$$\begin{aligned} \dot{V}_{22}(k,t) &= \frac{1}{d_1 d_2} \int_{t-d_1}^{t} \sum_{\theta_2=1}^{d_2} [\chi_1^T(k,t) \Lambda_1^T d_1 Z \Lambda_1 \chi_1(k,t) \\ &\quad - \dot{x}_{k+1}(\theta_1) d_1 Z \dot{x}_{k+1}(\theta_1) + x_{k+1}(t) W x_{k+1}(t) \\ &\quad - x_{k+1}(t-d_1) W x_{k+1}(t-d_1)] d\theta_1 \quad (24) \end{aligned}$$

Where
$\Lambda_1 = [A \;\; A_{1d} \;\; B \;\; B_{1d} \;\; 0 \;\; 0]$
Using a similar line as in the proof of theorem 1 in [8] (1D).

Then using (14)

$$\begin{aligned}\Delta V_{23}(k,t) &= y_{k+1}^T(t)P_2y_{k+1}(t)\\ &\quad -y_k^T(t)P_2y_k(t)\\ &\quad +2y_k^T(t)Y_2[y_k(t)-y_{k-d_2}(t)]\\ &\quad +2y_{k-d_2}^T(t)T_2[y_k(t)-y_{k-d_2}(t)]\\ &\quad -2y_k^T(t)Y_2\sum_{\theta_2=1}^{d_2}\Delta\xi_{k-\theta_2}(t)\\ &\quad -2y_{k-d_2}^T(t)T_2\sum_{\theta_2=1}^{d_2}\Delta\xi_{k-\theta_2}(t)\end{aligned}$$

$$\begin{aligned}\Delta V_{23}(k,t) &= \frac{1}{d_1d_2}\int_{t-d_1}^{t}\sum_{\theta_2=1}^{d_2}[\chi_1^T(k,t)\Lambda_2^TP_2\Lambda_2\chi_1(k,t)\\ &\quad -y_k^T(t)P_2y_k(t)+2y_k^T(t)Y_2[y_k(t)-y_{k-d_2}(t)]\\ &\quad +2y_{k-d_2}^T(t)T_2[y_k(t)-y_{k-d_2}(t)]\\ &\quad -2y_k^T(t)d_2Y_2\Delta\xi_{k-\theta_2}(t)\\ &\quad -2y_{k-d_2}^T(t)d_2T_2\Delta\xi_{k-\theta_2}(t)]d\theta_1\end{aligned}\tag{25}$$

Where
$\Lambda_2=[C\ \ C_{1d}\ \ D_0\ \ D_{1d}\ \ 0\ \ 0]$
Also

$$\begin{aligned}\Delta V_{24}(k,t) &= \sum_{\theta_2=1}^{d_2}\sum_{m=-\theta_2+1}^{0}\Delta\xi_{k+m}^T(t)Q\Delta\xi_{k+m}(t)\\ &\quad -\sum_{\theta_2=1}^{d_2}\sum_{m=-\theta_2}^{-1}\Delta\xi_{k+m}^T(t)Q\Delta\xi_{k+m}(t)\\ &\quad +\sum_{\theta_2=0}^{d_2-1}y_{k-\theta_2}^T(t)Uy_{k-\theta_2}(t)\\ &\quad -\sum_{\theta_2=1}^{d_2}y_{k-\theta_2}^T(t)Uy_{k-\theta_2}(t)\\ &= \frac{1}{d_2}\sum_{\theta_2=1}^{d_2}[\Delta\xi_k^T(t)d_2Q\Delta\xi_k(t)\\ &\quad -\Delta\xi_{k-\theta_2}^T(t)d_2Q\Delta\xi_{k-\theta_2}(t)\\ &\quad +y_k^T(t)Uy_k(t)-y_{k-d_2}^T(t)Uy_{k-d_2}(t)]\end{aligned}$$

Then using (14)

$$\begin{aligned}\Delta V_{24}(k,t) &= \frac{1}{d_1d_2}\int_{t-d_1}^{t}\sum_{\theta_2=1}^{d_2}[\chi_1^T(k,t)\Lambda_3^Td_2Q\Lambda_3\chi_1(k,t)\\ &\quad -\Delta\xi_{k-\theta_2}^T(t)d_2Q\Delta\xi_{k-\theta_2}(t)\\ &\quad +y_k^T(t)Uy_k(t)-y_{k-d_2}^T(t)Uy_{k-d_2}(t)]d\theta_1\end{aligned}\tag{26}$$

Where
$\Lambda_3=[C\ \ C_{1d}\ \ D_0-I\ \ D_{1d}\ \ 0\ \ 0]$

It then follows from (23)-(26) that

$$\Delta V_2(k,t)=\frac{1}{d_1d_2}\int_{t-d_1}^{t}\sum_{\theta_2=1}^{d_2}[\chi_1^T(k,t)\Theta_2\chi_1(k,t)]d\theta_1$$

Where

$$\Theta_2=\begin{bmatrix}\Xi_{11}&\Xi_{12}&\Xi_{13}&\Xi_{14}&-d_1Y_1&0\\ *&\Xi_{22}&\Xi_{23}&\Xi_{24}&-d_1T_1&0\\ *&*&\Xi_{33}&\Xi_{34}&0&-d_2Y_2\\ *&*&*&\Xi_{44}&0&-d_2T_2\\ *&*&*&*&-d_1Z&0\\ *&*&*&*&*&-d_2Q\end{bmatrix}$$

$\Xi_{11}=P_1A+A^TP_1+Y_1+Y_1^T+W+A^Td_1ZA+C^TP_2C+C^Td_2QC$
$\Xi_{12}=P_1A_{1d}-Y_1+T_1^T+A^Td_1ZA_{1d}+C^TP_2C_{1d}+C^Td_2QC_{1d}$
$\Xi_{13}=P_1B_0+A^Td_1ZB_0+C^TP_2D_0+C^Td_2Q(D_0-I)$
$\Xi_{14}=P_1B_{1d}+A^Td_1ZB_{1d}+C^TP_2D_{1d}+C^Td_2QD_{1d}$
$\Xi_{22}=-T_1-T_1^T-W+A_{1d}^Td_1ZA_{1d}+C_{1d}^TP_2C_{1d}+C_{1d}^Td_2QC_{1d}$
$\Xi_{23}=A_{1d}^Td_1ZB_0+C_{1d}^TP_2D_0+C_{1d}^Td_2Q(D_0-I)$
$\Xi_{24}=A_{1d}^Td_1ZB_{1d}+C_{1d}^TP_2D_{1d}+C_{1d}^Td_2QD_{1d}$
$\Xi_{33}=Y_2+Y_2^T-P_2+U+B_0^Td_1ZB_0+D_0^TP_2D_0+(D_0-I)^Td_2Q(D_0-I)$
$\Xi_{34}=B_0^Td_1ZB_{1d}+D_0^TP_2D_{1d}+(D_0-I)^Td_2QD_{1d}-Y_2+T_2^T$
$\Xi_{44}=-T_2-T_2^T-U+B_{1d}^Td_1ZB_{1d}+D_{1d}^TP_2D_{1d}+D_{1d}^Td_2QD_{1d}$

A differential linear repetitive processes described by (1) is stable along the pass for any delay $0<d_1\le\overline{d_1}$ and $0<d_2\le\overline{d_2}$ if there exist matrices $P_1>0$, $P_2>0$, $Q>0$, $Z>0$, $U>0$, $W>0$ and Y_1, T_1, Y_2, T_2 such that the following LMI are feasible

$$\Theta_2<0\tag{27}$$

Using the Schur complement to (27) yields (22). This completes the proof. ■

V. Illustrative example

In this section, one example is provided to illustrate the effectiveness of the results in the previous section. Consider Differential Linear Repetitive Processes time-Delays system (1) with the following parameters:

$$\begin{pmatrix}A&B_0\\C&D_0\end{pmatrix}=\begin{pmatrix}-0.6&0.6\\0.35&0.7\end{pmatrix}$$

And

$$\begin{pmatrix}A_{1d}&B_{1d}\\C_{1d}&D_{1d}\end{pmatrix}=\begin{pmatrix}-0.1&0\\-0.2&-0.1\end{pmatrix}$$

Our purpose is to find the maximum continuous delay $\overline{d_1}$ for fixed value of discrete delay d_2 and the maximum discrete delay $\overline{d_2}$ for fixed value of continuous delay d_1 such that the Differential Linear Repetitive Processes time-delays system is stable along the pass. computational results are shown in table 1 and table 2, which summarizes the obtained maximum allowable delays by using the previously published method

Theorem 2 in [6](corrected in this paper: Theorem 1) and our method(Theorem 2).

Table 1

Maximum allowable continuous delay comparison

d_2	$\overline{d_1}$(Theorem 1)	$\overline{d_1}$(Theorem 2)
1	2.6167	3.0747
2	2.3429	2.8479
4	1.8263	2.3245
6	1.4359	1.8555
9	1.0965	1.5559
13	0.8945	1.4093

Table 2

Maximum allowable discrete delay comparison

d_1	$\overline{d_2}$(Theorem 1)	$\overline{d_2}$(Theorem 2)
1	10	∞
2	3	5
2.5	1	3
3	infeasible	1

The results compared in Table 1,2. It can be seen that the delay-dependent stability conditions given by Theorem 2 are less conservative than this previously reported in the literature (see Theorem 1)

VI. Conclusions

In this paper the stability theory for differential linear repetitive processes has been extended to include cases where there are delays in both directions of information propagation. The delay-dependent case have been considered. An improved delay-dependent stability condition has also been proposed(Theorem 2) for differential linear repetitive processes delay system. Numerical example have demonstrated the effectiveness of the proposed method.

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