2D Inverse Problem in a Distributed Parameter System

S. RACHIDI, R. CHANNA, A. KARAMA

Abstract— **In this paper, we present a numerical solution for an inverse heat problem to estimate 2D space-wise coefficient in a parabolic system describing heat transfer. The inverse problem is recast into an optimization problem solved with a conjugate gradient method. This approach use the nonlinear optimization formulation by minimizing a cost function taking into account both the measured outputs of the system and the outputs calculated by means of the model. The solution is performed using the free finite element software FreeFem. Numerical example is carried out to check the validity of the proposed method.**

I. INTRODUCTION

Inverse problems have been studied intensively in the establishment of mathematical theories and various engineering disciplines. It has been received great attention from many investigators due to practical importance and mathematical interest. This approach is well adapted to characterize materials under experimental conditions which reproduce as close as possible some processing conditions which are difficult or even impossible to investigate with conventional techniques.

Many researchers have been interested in the inverse problem in estimating parameters, boundary conduction or heat source therm. One of the important applications of the inverse problem is the parameters estimation in thermal systems. For example, thermal characterization combining experimental and mathematical studies has been studied by Jarny and al. [7] to determine thermophysical properties of materials. Kim and al. [9] used the inverse problem technique to estimate temperature dependent thermal conductivity in a transient non linear heat conduction medium without internal measurements for one dimensional heat conduction. The authors consider that the thermal conductivity is assumed to vary linearly with respect to the temperature. The estimation heat conductivity problem becomes a classical inverse problem to estimate unknown coefficients. Huang and al. [6] used the direct integration method together with Levenberg-Markardt method in estimating the temperature dependant thermal conductivity and heat capacity. The inverse problem technique has also used by [1];[2] in estimating an internal shape of material by exploiting measurements performed on its boundary. The authors used the domain derivative technique in order to construct an efficient numerical algorithm to recover the shape of the domain.

The purpose of this paper is to apply the inverse problem technique to estimate a 2D space dependent conductivity in non linear parabolic system. The reminder of this paper is organized as follow:

In the second section, we present the model equations for a heat transfer system. The resolution of the equations is carried out using a finite elements method. The third section is devoted to the formulation of the identification problem. We assume that some observations on the system are known, then we formulate the identification problem in tow steps: In the first, we define the criterion that will be minimized and we introduce an adjoint state and define a Lagrangian associated with the identification problem. In the second step we discretize the problem and present the estimation algorithm. In the last section we present a numerical example for the proposed algorithm.

II. THE MODEL EQUATIONS

Let us consider a system described by the following parabolic partial differential equation:

$$\left\{\begin{array}{l} \rho c\dfrac{\partial u}{\partial t} = \nabla.(q(x)\nabla u) + f(x,t) \\ u(x,0) = u_0(x) \quad x \in \mathbf{R}^n,\ t \in [0,T] \\ \frac{\partial u}{\partial x}|_{\partial\Omega\times[0,T]} = 0 \end{array}\right. \tag{1}$$

This model can describe a large kind of systems. For example, underground water exploration [4], petroleum reservoirs [8], [12], dispersion in rivers [11]. In this paper, we consider a thermal system described by

R. Channa, A. Karama are with Cadi Ayyad University, Faculty of Sciences Semlalia, P.O. Box 2390, Marrakech, Morocco, `{channa, karama}@uca.ma`

equations (1) where ρ, c and $q(x)$ are respectively the density, the specific heat capacity and the space-wise heat conductivity that will be estimated. $u(x,t)$ is the solution of equations model, Ω can be any bounded domain in $R^n; n = 1,2,3$, $u_0(x)$ is the initial condition and $f(x,t)$ is a known source term.

The problem that we consider here, is to estimate the parameter $q(x)$ from observation given by the model solution with a known parameter $q(x)$ and a source term $f(x,t)$. To do this, we formulate the inverse problems as an optimization problem using an Output Least Square Method (OLSM)[3]. It consists in minimizing a cost function taking into account both the observations and the outputs calculated by means of the model equations. The minimization is carried out with a Lagrangian method by introducing an adjoint state and an adjoint equation. The gradient of the criterion is performed by calculating the lagrangian derivative with respect to the parameters $q(x)$. The optimization problem is then resolved using a conjugate gradient method [10].

III. THE IDENTIFICATION PROBLEM

We consider the problem in which some observations $u_d(x,t)$ are performed on the system in the domain Ω and we try to recover parameter $q(x)$ by minimizing the cost function that characterizes the difference between the observations $u_d(x,t)$ and the model solution $u(x,t;q)$. This cost functional is defined by:

$$J(q) = \int_0^T \int_\Omega \mid u(x,t;q) - u_d(x,t) \mid^2 dxdt \qquad (2)$$

where $u(x,t;q)$ is the solution of equations (1). The following variational formulation can be associated with system (1):

$$\begin{cases} \Big(\dfrac{\partial u}{\partial t}, \upsilon\Big) = \Big(\dfrac{\partial}{\partial x}(q(x)\dfrac{\partial u}{\partial x}), \upsilon\Big) + (f,\upsilon), \\ \qquad\qquad \forall \upsilon \in \mathbf{V} = H^1(\Omega) \\ u(x,0) = u_0(x);\ x \in \Omega \end{cases} \qquad (3)$$

The identification problem of the parameter $q(x)$ can be set as:

$$(\mathbf{P1}) : \qquad \text{Find } \ q^* \in A_{ad}, \ \text{ such that} \\ J(q^*) \leq \ J(q), \ \ \forall \ \ q \in A_{ad}$$

where A_{ad} is an admissible set of functions containing some information on the parameter q, for example:

$$A_{ad} = \{q \in \mathbf{R}/0 < q_{min} < q < q_{max} < \infty\}$$

where q_{min} and q_{max} are given.

A. *The Lagrangian method*

To solve the problem $(\mathbf{P1})$, we shall use the lagrangian method which consists in introducing the state equations (1) as a constraint in criterion (2). We introduce an adjoint state $p(x,t)$ and we define a Lagrangian $L : V \times V \times A_{ad} \to \mathbf{R}$ associated with the identification problem by:

$$L(u,p,q) = J(q) + \\ \int_0^T \Big(\Big(\frac{\partial u}{\partial t} - (\frac{\partial}{\partial x} q(x) \frac{\partial u}{\partial x} - f, p\Big)\Big) dt \qquad (4)$$

The variation of lagrangian $L(u,p,q)$ is given by :

$$\delta L = \frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial q}\delta q.$$

where

$$\frac{\partial L}{\partial q}\delta q = \int_0^T \int_\Omega \delta q \nabla u . \nabla P dxdt.$$

$$\frac{\partial L}{\partial u}\delta u = 2\int_0^T (u - u_d, \delta u) dxdt \\ + \int_0^T \{(\delta \partial_t u, p) + \xi_q(\delta u, p)\} dt$$

and $\xi_q(\delta u, p)$ is a bilinear form defined by:

$$\xi_q(\delta u, p) = \int_\Omega q \nabla \delta u . \nabla p dx.$$

The stationarity of $L(u,p,q)$ with respect to u, leads to:

$$\frac{\partial L}{\partial u}.\delta u = 0 \quad \forall \ \delta u \in V.$$

This condition can be written as :

$$\int_0^T (u - u_d, \delta u) dt + \int_0^T \Big((\partial_t p, \delta u) \\ + \xi_q(p, \delta u)\Big) dt + (p, \ \delta u) = 0.$$

We obtain the variational formulation for adjoint state :

$$\begin{cases} \dfrac{\partial p}{\partial t} = -\nabla\Big(q(x)\nabla p\Big) + \Big(u(x,t) - u_d(x,t)\Big) = 0, \\ p|_{\partial\Omega\times]0,T[} = 0; \quad x \in \Omega \\ p(x,T) = 0 \quad t \in]0,T[\end{cases} \tag{5}$$

When $u(x,t)$ and $p(x,t)$ are solutions of equations (1) and (5) respectively, the gradient of the cost function is obtained by differentiating the Lagrangian $L(u,p,q)$ with respect to parameter $q(x)$. We obtain:

$$\frac{\partial J}{\partial q} = 2\int_0^T \nabla u.\nabla p dt \tag{6}$$

B. *The discretized identification problem*

In order to identify $q(x)$ through $(\mathbf{P1})$, we discretize the model equations using a finite difference method in time involving an implicit scheme and a finite element method of first order in space. To do this, we divide the time interval $[0,T]$ into M equal subintervals $[t_n, t_{n+1}]$ of length τ ; $t_n = n\tau$ with $\tau = T/M$. In order to approximate $q(x)$ and $u(x,t)$, we introduce two finite element spaces V_h and A_{adh} respectively with basis: $(\psi_1, \psi_2, ...\psi_N)$ and $(\varphi_1, \varphi_2, ...\varphi_p)$. The discretized solution u_h^n is then obtained recursively by solving the discretized system :

$$\begin{cases} \Big(\dfrac{u_h^{n+1} - u_h^n}{\tau}, v_h\Big) + \Big(q\nabla u_h^{n+1}, \nabla v_h\Big) = (f_h^{n+1}, v_h) \\ \qquad\qquad \forall v_h \in V_h, n = 0, ..., (M-1) \\ u_h^0 = u_{0h} \end{cases} \tag{7}$$

where $u_h^n = u(x_i, t^n)$; $f_h^n = f(x, t^n)$ and

$$V_h = \Big\{ v_h \in \mathcal{C}^0(\overline{\Omega}) \;/\; V_h \,|_{\mathrm{I}_i} \in P^1(\mathrm{I}_i), i = 1, \ldots, N \Big\}$$

where $\mathcal{C}^0(\overline{\Omega})$ is the set of continuous functions on $\overline{\Omega}$ and $P^1(\mathrm{I}_i)$ is the set of polynomials P on the interval I_i of degree less than or equal to one. The admissible set of functions is approximated by the r-dimensional space $\mathbf{A}_{ad_h}$ as:

$$\mathbf{A}_{ad_h} = \Big\{ (q_1, q_2 ... q_p) \in \mathbf{R}^p / q_h(x) = \sum_{j=1}^{p} q_j \varphi_j(x) \in \mathbf{A}_{ad} \cap \mathbf{V}_h, q_i = q(x_i) \Big\}.$$

We also discretize the cost functional $J(q)$:

$$J_h(q) = \sum_{n=1}^{M} \sum_{i=1}^{N} \mid u_h^n(x_i; q) - u_d^n(x_i) \mid^2 \tag{8}$$

The approximated identification problem becomes:

$$(\mathbf{P1h}) : \quad \text{Find } q_h^* \in A_{adh}, \text{ such that } J(q_h^*) \leq J(q_h), \ \forall \ q_h \in A_{adh},$$

In order to solve $(P1h)$ by Lagrangien method, we also need a discrete state adjoint $P_h^n \in V_h$ witch can be obtained by solving backward the system:

$$\begin{cases} \Big(\dfrac{P_h^{n-1} - P_h^n}{\tau}, v_h\Big) + \Big(q\nabla P_h^{n-1}, \nabla v_h\Big) \\ \qquad = (u_h^{n-1}, v_h), \forall v_h \in V_h, n = M, ..., 0 \\ P_h^M = 0 \end{cases} \tag{9}$$

Taking into account of the expression of the gradient obtained in the previous section, the components of the gradient as :

$$\frac{\partial J_h}{\partial q_j} = 2\sum_{n=1}^{M} \int_\Omega \nabla u_h^n.\nabla P_h^{n-1} \tag{10}$$

IV. MINIMIZATION

To solve the nonlinear optimization problem $(P1h)$, we use the output Least square method (OLSM) [3]. The minimization of the criterion (8) will be performed according to the conjugate gradient method. The algorithm selects the successive direction vectors as a conjugate version of the successive gradients obtained as the method progresses. Thus, the directions are not specified beforehand, but rather are determined sequentially at each step of the iteration. At step k one evaluates the current negative gradient vector and adds to it a linear combination of the previous direction vectors to obtain a new conjugate direction vector along which to move.

The basic steps of the estimation algorithm used in this section can be summarized as follows:

Step 1. Solve equation (7) with a known parameter $q(x)$ and get the observations $u_d(x,t)$.

Step 2. Initialize the parameter $q(x)$.

Step 3. Solve the direct system (7).

Step 4. Compute the criterion $J_h(q)$, if $J_h(q) \leq \epsilon$,(ϵ given) go to step 9

Step 5. Solve the adjoint system (9)

Step 6. Compute the components of the gradient $\dfrac{\partial J_h}{\partial q_j}$ with (10)
Step 7. Compute the new component q_j
Step 8. Go to step 3
Step 9. Stop.

This algorithm is implemented with the free finite element software FreeFem [5]. Systems (7) and (9) are solved here using finite element method in space since it is more suitable to deal with the non homogeneous heat equation where all the domain D must be meshed.

V. NUMERICAL EXAMPLE

In this section, we present a numerical example for the parameter identification. We consider a domain D defined by:

$$D = \{(x,y) \in \mathbf{R}^2 / \;\; 0.25 < x^2 + y^2 < 4\}$$

associated with the following internal Γ_i and external Γ_e boundaries:

$$\Gamma_i = \{(x,y) \in \mathbf{R}^2 / \;\; x^2 + y^2 = 0.25\}$$

$$\Gamma_e = \{(x,y) \in \mathbf{R}^2 / \;\; x^2 + y^2 = 4\}$$

the global domain Ω is: $\Omega = D \cup \Gamma_e \cup \Gamma_i$, with $\partial\Omega = \Gamma_i \cup \Gamma_e$

The mesh of domain D is generated by dividing the internal boundary Γ_i and exterior boundary Γ_e in N equal segments as shown in figure1.

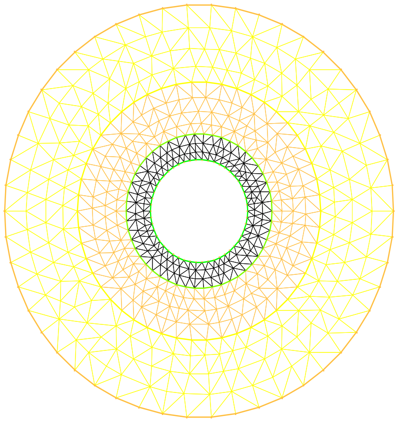

Figure 1 : Domain mesh Ω.

To simulate the observations $u_d(x,y;t)$, we choose an initial condition $u_0(x,y) = x^2 + y^2$. and an exact space-wise parameter defined on the domain Ω by:

$$q(x,y) = \begin{cases} q_1 = 3 & (x,y) \in \Omega_1 \\ q_2 = 4.5 & (x,y) \in \Omega_2 \\ q_3 = 5.5 & (x,y) \in \Omega_3 \end{cases}$$

where Ω_1, Ω_2 ,Ω_2 are sub-domains of Ω defined by:

$$\begin{array}{l} \Omega_1 = \{(x,y) \in \mathbf{R}^2 / \;\; 0.5 \leq \sqrt{x^2+y^2} \prec 0.75\} \\ \Omega_2 = \{(x,y) \in \mathbf{R}^2 / \;\; 0.75 \leq \sqrt{x^2+y^2} \prec 1.5\} \\ \Omega_3 = \{(x,y) \in \mathbf{R}^2 / \;\; 1.5 \leq \sqrt{x^2+y^2} \leq 2\} \end{array}$$

The observations $u_d(x,t)$ are generated numerically by solving the direct system (7) using FreeFem with $\delta t = 0.02$, N = 50 and the simulation time is $T = 1$. The identification algorithm is initialized on the domainΩ by the parameter: $q_{ini}(x,y) = 1$. The source term f(x,y;t) is calculated from equation (1)such that:

$$f(x,y) = \begin{cases} 1 - 4q_1 & (x,y) \in \Omega_1 \\ 1 - 4q_2 & (x,y) \in \Omega_2 \\ 1 - 4q_3 & (x,y) \in \Omega_3 \end{cases}$$

The exact and estimated parameters are presented respectively in figures 2 and 3 and show clearly the convergence of the proposed estimation algorithm. The value of ϵ in the estimation algorithm is taken equal to 10^{-6} .

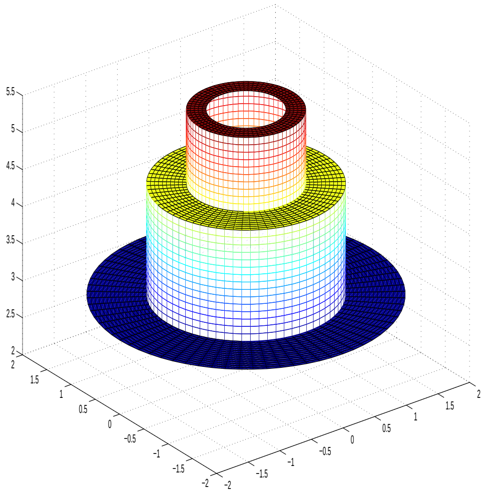

Figure 2 : 2D Exact parameter.

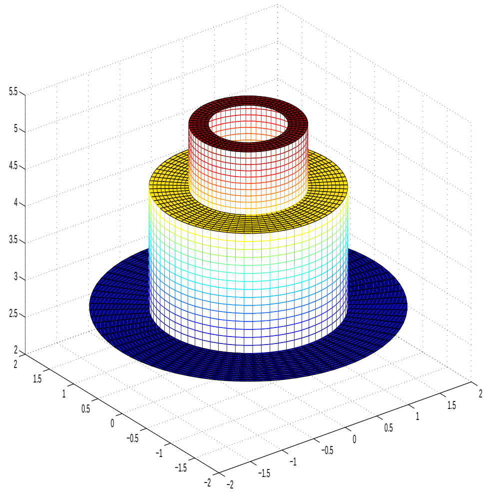

Figure 3 : 2D Estimated parameter.

We denote by q_* the parameter obtained by solving the problem (P). To check the quality of estimation, we introduce the error of estimation defined by $\dfrac{\|q - q_*\|}{\|q\|}$. In figure 4 we present the evolution of the relative error between exact and estimated parameters. We note that the algorithm convegence depends on the parameter initialization. For this example we obtain the convergence after 450 iterations.

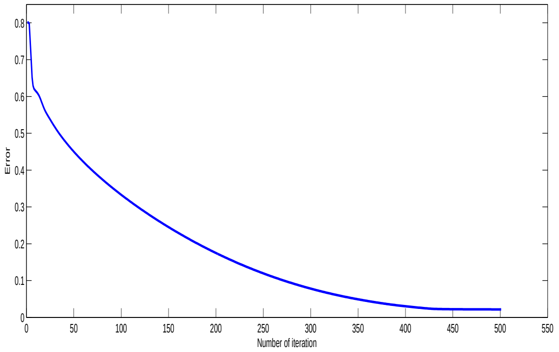

Figure 4 : Evolution of the estimation error

VI. CONCLUSION

In this paper, a numerical solution for an inverse problem to estimate a piecewise dependent parameter in a parabolic system is presented. The proposed algorithm is implemented with a finite element method using a FreeFem software. The obtained results show that the exact and the identified parameters are identical.

REFERENCES

[1] Chaji,K., El Bagdouri,M.2007 Identification of an internal material boundary. *Inverse Problems in science and Enginneerin. Vol.* 00, N^o 00, pp 1-11.

[2] Chapko,R.,Kress,R. and Yoon,J. 1998 On the numerical solution of an inverse boundary value problem for the heat equation Inverse Problems 14 853-67.

[3] Chavent,G. 1979 Identification of distributed parameter system: about the output least square method, its implementation and identifiability. Porc of the IFAC sympoosium on identification and system parameter estimation, Pergamon Press.

[4] Chavent,G.,Jaffe,J. 1986 Mthematical models and finite elements for reseviors simulation. "'Studies in mathematics and its applications"', Vol 17, North- holland.

[5] Hecht F., Pironneau O., and Ohtsuka K 2004 FreeFem++ manual, version 1.42.

[6] Huang,C.H. and Ozisik,M.N. A Direct Integration Approach for Simultaneously Estimating Temperature Dependent Thermal Conductivity and Heat Capacity, Numerical Heat Transfer, Part A,Vol. 20, No. 1, pp.95- 110, (1991).

[7] Jarny,Y.C. 2002 Inverse heat transfert problems and thermal characterization of materials. 4^{th} International Conference on Inverse Problems in Engineering Rio de Janeiro, Brazil.

[8] Kravaris,C.,Seinfeld,J. H. 1986 identification of spatially varying parameter in distributed parameter systems by discrete regularization, J. Math. Anal. Appl. 119, 128-152.

[9] Kim,S., Cho,H. 2004 Determination of thermal conductivity in a non linear heat conduction problem without interior measurements, Inverse problems,Design and Optimization Symposium, Rio de Janeiro, Brazil.

[10] Minoux,M. 1982 *Programmation mathématique, théorie et algorithme*. Dunod, Paris.

[11] Yu,W.H. 1987 on identification of a diffusion coefficient in a river water quality model, Acta. Math. Sinnica.(N.S.) 10 (Chinese).

[12] Yu,W.H.,Seinfeld,J.H. 1988 Identification of parabolic distributed paramter systems by rgularization with differential operators, J. Math. Anal. Appl. 132, 365-387.