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H_∞ performance analysis for 2D discrete state delayed systems

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***Abstract*— This paper deals with the problem of H_∞ performance analysis for 2D discrete state delayed system. A sufficient condition to have a H_∞ performance bound of this class of systems described by the Fornasini-Marchesini state space model is addressed *via* Lyapunov techniques. This stability criterion is established using Leibniz-Newton formula with additional free weighting matrices. A numerical example is introduced to show the efficiency of the proposed criterion.**

keywords: 2D state delayed systems, H_∞ disturbance attenuation, delay-dependent, LMI.

I. INTRODUCTION

The discrete two dimensional (2D) systems [9], which are physical systems and dynamics depending on two independent integer variable i and j, have received considerable attention due to their theoretical and practical interest particularly in image and signal processing, coding/decoding, filtering, etc. (for an overview, see [1], [10], [5]). Many control problems such as H_∞ filter for $2D$ uncertain nonlinear state delayed systems are proposed in [15], [14], H_∞ control for 2D state delayed systems has been investigated in [17], and the mixed H_2/H_∞ filtering for 2D state delayed systems with nonlinear fraction transformation uncertainties is reported in [4]. These papers present delay-independent criteria. While delay-dependent conditions for 2D systems are dealt with, for instance, in [3], [4] where they investigated the delay-dependent H_∞ and robust H_∞ filtering based on a frequency approach and for uncertain state delayed systems in Fornasini-Marchesini model. In [16], the problem of delay-dependent H_∞ control for 2D discrete state delay systems in the second Fornasini-Marchesini model is considered. The problem of robust H_∞ filtering for uncertain 2D systems described by Roesser model is investigated in [6]. Moreover, the H_∞ problem is adressed for 2D systems with stochastic perturbations in [7] and for 2D discrete Markovian jump systems described by Roesser model in [13]. In [11], the authors dealt with delay-independent and delay-dependent H_∞ filtering for 2D discrete state delayed systems. In general, the delay-dependent results are less conservative than the delay-independent ones. Therefore, it is natural to try to derive similar results on the same problems of 2D systems with state delays using delay-dependent approach. Our work is inspired from [17] which is concerned with the problem of H_∞ control for 2D discrete state delayed systems and from ([8], [18]) where some delay-dependent stability criteria, for one and two dimensional continuous systems, are derived using the Leibniz-Newton formula and free weighting matrices are used to express this relationship between the current and delayed states leading to linear matrix inequalities. The free weighting matrix approach is used to reduce the conservativeness of the delay-dependent condition.

The purpose of the problem under investigation is to analyze the H_∞ performance in order to guarantee the asymptotic stability of the system and to achieve a prescribed H_∞ performance level. The paper is organized as follows. In section II, we introduce the mathematical background needed to address the problem. In section III, we introduce our main result: a delay-dependent condition is derived to ensure a H_∞ performance bound of the system using Lyapunov techniques. Furthermore, the sufficient condition is expressed in terms of LMIs (linear matrix inequalities, see [2]). Finally section IV presents an illustrative example to show the effectiveness of the proposed criterion.

Notations:

Throughout the paper we will use the following notations:

- $\|x(i,j)\|_W$ will denote the norm of the vector $x(i,j)$ defined as
$$\|x(i,j)\|_W^2 = x(i,j)^T W x(i,j)$$
where W is nonnegative matrix.
- $\|x\|_2$ will denote the norm 2 of the vector $x(i,j)$ defined as
$$\|x\|_2 = \sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|x(i,j)\|^2}$$
- A matrix added to its symmetric will be called
$$\mathrm{sym}\{A\} = A^T + A$$
- $D(r)$ denotes the set defined by
$$D(r) = \{(i,j) : i+j = r, i \geq 0, j \geq 0\} \qquad (1)$$

Some formula will be used in the paper, in particular the Leibniz-Newton formula which is given by

$$x(i-d_1, j+1) = x(i, j+1) - \sum_{\theta=-d_1}^{-1} \Delta_i x(i+\theta, j+1)$$

with

$$\Delta_i x(i+\theta, j+1) = x(i+\theta+1, j+1) - x(i+\theta, j+1)$$

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II. PROBLEM FORMULATION

Consider the following state delayed 2D system described by Fornasini-Marchesini model with state delays :

$$\begin{aligned} x(i+1,j+1) &= A_1x(i,j+1)+A_2x(i+1,j) \\ &\quad + A_{1d}x(i-d_1,j+1)+A_{2d}x(i+1,j-d_2) \\ &\quad + B_1\omega(i,j+1)+B_2\omega(i+1,j) \\ z(i,j) &= Hx(i,j)+L\omega(i,j) \end{aligned} \tag{2}$$

where $x(i,j)$ is the local state vector in $\mathbb{R}^n$ with $i,j\in\mathbb{N}$, $z(i,j)\in\mathbb{R}^p$ is the controlled output, $\omega(i,j)\in\mathbb{R}^q$ is the disturbance input which belongs to $l_2^n\{[0,\infty),[0,\infty)\}$. d_l $(l=1,2)$ is an unknown positive integer representing the delay along the direction l and the matrices A_l, A_{ld}, B_l, H and L are constant matrices with appropriate dimensions. We also assume a set of initial conditions on a compact support, i.e., there exists a positive integer U, such that

$$\begin{aligned} x(i_0,j) &= 0, \quad \forall j\geq U, \quad i_0=-d_1,-d_1+1,...,0 \\ x(i,j_0) &= 0, \quad \forall i\geq U, \quad j_0=-d_2,-d_2+1,...,0 \end{aligned} \tag{3}$$

Denote

$$X_r = sup\{\|x(i,j)\| : i+j=r, \quad i,j\in\mathbb{R}\}$$

We first give the definition of asymptotic stability for (2) with $\omega(i,j)=0$ and the initial conditions (3).

Definition 1: Consider the 2D discrete state delayed system (2) is asymptotically stable if $lim_{r\to\infty}X_r = 0$ with $\omega(i,j)=0$ and the initial condition (3).

Definition 2: Consider the 2D discrete system (2) with initial condition (3).

Given a scalar $\gamma>0$, and symmetric positive definite weighting matrices Q_h, Q_v, W_h, W_v, S_h, $S_v\in\mathbb{R}^{n\times n}$, the 2D state delayed system (2) is said to have an H_∞ disturbance attenuation bound γ if it is asymptotically stable and satisfies

$$J=\sup_{0\neq\omega\in l_2}\frac{\|\bar{z}\|_2^2}{\|\bar{\omega}\|_2^2+D_h(d_1,j)+D_v(i,d_2)}\prec\gamma^2 \tag{4}$$

where

$$\|\bar{\omega}\|_2^2=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left\|\begin{bmatrix}\omega(i,j+1)\\ \omega(i+1,j)\end{bmatrix}\right\|^2$$

$$\|\bar{z}\|_2^2=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left\|\begin{bmatrix}z(i,j+1)\\ z(i+1,j)\end{bmatrix}\right\|^2$$

$$\begin{aligned} D_h(d_1,j) &= \sum_{j=0}^{\infty}\left\{\|x(0,j+1)\|_{Q_h}^2+\sum_{l=-d_1}^{-1}\|x(l,j+1)\|_{W_h}^2\right. \\ &\quad \left.+\sum_{\theta=-d_1+1}^{0}\sum_{s=\theta+1}^{0}\|\Delta_i x(i+s,j+1)\|_{S_h}^2\right\} \\ D_v(i,d_2) &= \sum_{i=0}^{\infty}\left\{\|x(i+1,0)\|_{Q_v}^2+\sum_{l=-d_2}^{-1}\|x(i+1,l)\|_{W_v}^2\right. \\ &\quad \left.+\sum_{\theta=-d_2+1}^{0}\sum_{s=\theta+1}^{0}\|\Delta_j x(i+1,j+s)\|_{S_v}^2\right\} \end{aligned}$$

with

$$\begin{aligned} \Delta_i x(i+s,j+1) &= x(i+s+1,j+1)-x(i+s,j+1), \\ \Delta_j x(i+1,j+s) &= x(i+1,j+s+1)-x(i+1,j+s) \end{aligned}$$

Remark 1: Definition 2 is borrowed from [17] but adapted to our case as we use a different Lyapunov function.

Define $V(x)=V_h(x)+V_v(x)$ and define $\Delta V(x(i+1,j+1))\equiv\Delta V(i+1,j+1)\equiv\Delta V$ as the increment of V along the trajectories of (2) by:

$$\begin{aligned} \Delta V(i+1,j+1) &\triangleq V_h(i+1,j+1)+V_v(i+1,j+1) \\ &\quad - V_h(i,j+1)-V_v(i+1,j) \end{aligned} \tag{5}$$

III. MAIN RESULTS

In this section, we investigate the H_∞ control problem for 2D discrete state delayed system (2), using Leibniz-Newton formula. The relationship between the terms in the formula is taken into account by adding free weighting matrices, given on the left side of this equation [1], to the increment of Lyapunov functional ΔV.

$$\begin{aligned} &2\left[S_0x(i,j+1)+S_1x(i-d_1,j+1)\right]^T\times \\ &\left\{x(i,j+1)-x(i-d_1,j+1)-\sum_{\theta=-d_1}^{-1}\Delta_i x(i+\theta,j+1))\right\} \\ &=0 \end{aligned} \tag{6}$$

In the previous equation, the free weighting matrices S_0 and S_1 indicate the relationship between the terms in the Leibniz-Newton formula. We add also the following expression to the increment of Lyapunov functional ΔV.

$$\begin{aligned} &2\{E_0x(i+1,j+1)+E_1x(i,j+1)+E_2x(i+1,j) \\ &+E_3x(i-d_1,j+1)+E_4x(i+1,j-d_2) \\ &+E_5\omega(i,j+1)+E_6\omega(i+1,j)\}^T\times \\ &\{x(i+1,j+1)-A_1x(i,j+1)-A_2x(i+1,j) \\ &-A_{1d}x(i-d_1,j+1)-A_{2d}x(i+1,j-d_2) \\ &-B_1\omega(i,j+1)-B_2\omega(i+1,j)\}=0 \end{aligned} \tag{7}$$

As is shown in the following theorem, the free weighting matrices E_i $(i=1,...,6)$, S_0, S_1, T_0 and T_1 can easily be determined by solving the corresponding linear matrix inequalities.

Theorem 1: A 2D state-delayed system (2) is asymptotically stable and has H_∞ disturbance attenuation level bound γ if there exist symmetric positive definite matrices P_h, P_v, R_h, R_v, N_h and N_v satisfying $P_h\prec\gamma^2Q_h$, $P_v\prec\gamma^2Q_v$, $R_h\prec\gamma^2W_h$, $R_v\prec\gamma^2W_v$, $N_h\prec\gamma^2S_h$ and $N_v\prec\gamma^2S_v$ and some free weighting matrices E_i $(i=1,...,6)$, S_0, S_1, T_0

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[1] We have given the equation only for horizontal direction since the expression for vertical direction is similar with T_0 and T_1 as free weighting matrices.

and T_1 with appropriate dimensions such that

$$\begin{aligned}&\begin{bmatrix}\phi_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{12}^T & \alpha_{22} & \alpha_{23}\\ \alpha_{13}^T & \alpha_{23}^T & -\alpha_{33}\end{bmatrix}\\ &+\text{sym}\left\{\begin{bmatrix}E_0^T\\ E_{1,6}^T\\ 0_{2n\times n}\end{bmatrix}\begin{bmatrix}I & -b & 0_{n\times 2n}\end{bmatrix}\right\}\prec 0\end{aligned} \tag{8}$$

with

$$\begin{aligned}E_{1,6}&=\begin{bmatrix}E_{1,4} & E_5 & E_6\end{bmatrix},E_{1,4}=\begin{bmatrix}E_1 & E_2 & E_3 & E_4\end{bmatrix}\\ b&=\begin{bmatrix}b_0 & B_1 & B_2\end{bmatrix},b_0=\begin{bmatrix}A_1 & A_2 & A_{d1} & A_{d2}\end{bmatrix}\end{aligned}$$

$$\alpha_{22}=\begin{bmatrix}(\alpha_{22})_1 & (\alpha_{22})_2\\ * & (\alpha_{22})_3\end{bmatrix},\alpha_{33}=\begin{bmatrix}d_1N_h & 0\\ 0 & d_2N_v\end{bmatrix},$$

$$\begin{aligned}(\alpha_{22})_1&=\begin{bmatrix}\phi_{22}+H^TH & 0 & 0 & 0\\ * & \phi_{33}+H^TH & 0 & 0\\ * & * & \phi_{44} & 0\\ * & * & * & \phi_{55}\end{bmatrix},\\ (\alpha_{22})_2&=\begin{bmatrix}H^TH & 0\\ 0 & H^TH\\ 0 & 0\\ 0 & 0\end{bmatrix},\\ (\alpha_{22})_3&=\begin{bmatrix}L^TL-\gamma^2I & 0\\ 0 & L^TL-\gamma^2I\end{bmatrix},\end{aligned}$$

$$\begin{aligned}\alpha_{12}&=\begin{bmatrix}\phi_{12} & \phi_{13} & 0 & 0 & 0 & 0\end{bmatrix},\alpha_{13}=\begin{bmatrix}0 & 0\end{bmatrix},\\ \alpha_{23}&=\begin{bmatrix}(\alpha_{23})_1\\ 0_{2n\times 2n}\end{bmatrix},(\alpha_{23})_1=\begin{bmatrix}-d_1S_0^T & 0\\ 0 & -d_2T_0^T\\ -d_1S_1^T & 0\\ 0 & -d_2T_1^T\end{bmatrix},\end{aligned}$$

$$\begin{aligned}\phi_{11}&=P_h+P_v+d_1N_h+d_2N_v,\\ \phi_{22}&=-P_h+R_h+d_1N_h+\text{sym}\{S_0\},\\ \phi_{33}&=-P_v+R_v+d_2N_v+\text{sym}\{T_0\},\\ \phi_{44}&=-R_h-\text{sym}\{S_1\},\\ \phi_{55}&=-R_v-\text{sym}\{T_1\},\\ \phi_{12}&=d_1N_h,\\ \phi_{13}&=d_2N_v\end{aligned}$$

Proof: Let us consider the following Lyapunov candidate function:

$$V(i,j)=V_h(i,j)+V_v(i,j)$$

where

$$\begin{aligned}V_h(i,j)&=V_{1h}(i,j)+V_{2h}(i,j)+V_{3h}(i,j)\\ &=\|x(i,j)\|_{P_h}^2+\sum_{l=-d_1}^{-1}\|x(i+l,j)\|_{R_h}^2\\ &+\sum_{\theta=-d_1+1}^{0}\sum_{s=\theta+1}^{0}\|\Delta_ix(i+s,j)\|_{N_h}^2\end{aligned} \tag{9}$$

$$\begin{aligned}V_v(i,j)&=V_{1v}(i,j)+V_{2v}(i,j)+V_{3v}(i,j)\\ &=\|x(i,j)\|_{P_v}^2+\sum_{l=-d_2}^{-1}\|x(i,j+l)\|_{R_v}^2\\ &+\sum_{\theta=-d_2+1}^{0}\sum_{s=\theta+1}^{0}\|\Delta_jx(i,j+s)\|_{N_v}^2\end{aligned} \tag{10}$$

where $P_h\succ 0$, $R_h\succ 0$, $N_h\succ 0$, $P_v\succ 0$, $R_v\succ 0$ and $N_v\succ 0$ are given.

The increment along any trajectory of system (2) is given by (5) and can be rewritten as such

$$\Delta V(i+1,j+1)=\Delta V_h(i+1,j+1)+\Delta V_v(i+1,j+1)$$

where

$$\begin{aligned}\Delta V_h&=\Delta V_{1h}+\Delta V_{2h}+\Delta V_{3h}\\ \Delta V_v&=\Delta V_{1v}+\Delta V_{2v}+\Delta V_{3v}\end{aligned}$$

Because of the similarities between the 2 dimensions, we will develop the calculations only in the horizontal dimension. Let us first calculate ΔV_{1h}, ΔV_{2h} and ΔV_{3h}.

$$\begin{aligned}\Delta V_{1h}&=\|x(i+1,j+1)\|_{P_h}^2-\|x(i,j+1)\|_{P_h}^2\\ &=x^T(i+1,j+1)P_hx(i+1,j+1)\\ &-x^T(i,j+1)P_hx(i,j+1)\\ \Delta V_{2h}&=\|x(i,j+1)\|_{R_h}^2-\|x(i-d_1,j+1)\|_{R_h}^2\\ &=x^T(i,j+1)R_hx(i,j+1)\\ &-x^T(i-d_1,j+1)R_hx(i-d_1,j+1)\end{aligned}$$

$$\begin{aligned}\Delta V_{3h}&=d_1\|\Delta_ix(i+1,j+1)\|_{N_h}^2\\ &-\sum_{\theta=-d_1+1}^{0}\|\Delta_ix(i+\theta,j+1)\|_{N_h}^2\\ &=d_1\Delta_i^Tx(i+1,j+1)N_h\Delta_ix(i+1,j+1)\\ &-\sum_{\theta=-d_1+1}^{0}\Delta_i^Tx(i+\theta,j+1)N_h\Delta_ix(i+\theta,j+1)\\ &=d_1x^T(i+1,j+1)N_hx(i+1,j+1)\\ &-d_1x^T(i+1,j+1)N_hx(i,j+1)\\ &-d_1x^T(i,j+1)N_hx(i+1,j+1)\\ &+d_1x^T(i,j+1)N_hx(i,j+1)\\ &-\sum_{\theta=-d_1+1}^{0}\Delta_i^Tx(i+\theta,j+1)N_h\Delta_ix(i+\theta,j+1)\end{aligned}$$

Then, we add (6) and (7) with $\omega(i,j)=0$ to ΔV_h. The same work can be done in the vertical dimension which leads to the following expression for ΔV:

$$\begin{aligned}
\Delta V = {} & \Delta V_{1h} + \Delta V_{2h} + \Delta V_{3h} + \Delta V_{1v} + \Delta V_{2v} + \Delta V_{3v} \\
& + 2 \{ E_0 x(i+1,j+1) + E_1 x(i,j+1) + E_2 x(i+1,j) \\
& + E_3 x(i-d_1,j+1) + E_4 x(i+1,j-d_2) \}^T \times \\
& \{ x(i+1,j+1) - A_1 x(i,j+1) - A_2 x(i+1,j) \\
& - A_{1d} x(i-d_1,j+1) - A_{2d} x(i+1,j-d_2) \} \\
& + 2 \left[S_0 x(i,j+1) + S_1 x(i-d_1,j+1) \right]^T \times \\
& \left\{ x(i,j+1) - x(i-d_1,j+1) \right. \\
& \left. - \sum_{\theta=-d_1}^{-1} \Delta_i x(i+\theta,j+1)) \right\} \\
& + 2 \left[T_0 x(i+1,j) + T_1 x(i+1,j-d_2) \right]^T \times \\
& \left\{ x(i+1,j) - x(i+1,j-d_2) \right. \\
& \left. - \sum_{s=-d_2}^{-1} \Delta_j x(i+1,j+s) \right\}
\end{aligned}$$

Let

$$\begin{aligned}
\xi = & [x^T(i+1,j+1) \quad x^T(i,j+1) \quad x^T(i+1,j) \\
& x^T(i-d_1,j+1) \quad x^T(i+1,j-d_2) \\
& \Delta_i^T x(i+\theta,j+1) \quad \Delta_j^T x(i+1,j+s)]^T
\end{aligned}$$

The increment $\Delta V(i+1,j+1)$ along any trajectory of system (2) with $(\omega(i,j)=0)$ satisfies

$$\Delta V = \sum_{\theta=-d_1}^{-1} \sum_{s=-d_2}^{-1} \xi^T \Omega_0 \xi \tag{11}$$

with

$$\begin{aligned}
\Omega_0 = & \begin{bmatrix} \phi_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12}^T & (\alpha_{22})_1 & (\alpha_{23})_1 \\ \alpha_{13}^T & (\alpha_{23})_1^T & -\alpha_{33} \end{bmatrix} \\
& + \mathrm{sym} \left\{ \begin{bmatrix} E_0^T \\ E_{1,4}^T \\ 0_{2n\times n} \end{bmatrix} \begin{bmatrix} I & -b_0 & 0_{n\times 2n} \end{bmatrix} \right\}
\end{aligned}$$

with ϕ_{11}, α_{ij} $(i,j=1,2,3)$, $(\alpha_{22})_1$, $(\alpha_{23})_1$, $E_{1,4}$ and b_0 are given in Theorem 1.

It follows from (8) that $\Delta V \le 0$, i.e.,

$$\begin{aligned}
V_h(i+1,j+1) + V_v(i+1,j+1) \le & V_h(i,j+1) \\
& + V_v(i+1,j)
\end{aligned} \tag{12}$$

For any integers $r \ge max\{U\}$, it follows from (12) and the initial conditions (3) that

$$\begin{aligned}
\sum_{i+j\in D(r)} V(i,j) & = \sum_{i+j\in D(r)} [V_h(i,j) + V_v(i,j)] \\
& = V_h(r,0) + V_h(r-1,1) + V_h(r-2,2) \\
& + \ldots + V_h(1,r-1) + V_h(0,r) \\
& + V_v(r,0) + V_v(r-1,1) + V_v(r-2,2) \\
& + \ldots + V_v(1,r-1) + V_v(0,r) \\
& \ge V_h(r+1,0) + V_h(r,1) + V_h(r-1,2) \\
& + \ldots + V_h(1,r) + V_h(0,r+1) \\
& + V_v(r+1,0) + V_v(r,1) + V_v(r-1,2) \\
& + \ldots + V_v(1,r) + V_v(0,r+1) \\
& - V_h(-1,r+1) - V_v(r+1,-1) \\
& = \sum_{i+j\in D(r+1)} V(i,j)
\end{aligned} \tag{13}$$

where the equality sign holds when

$$\sum_{i+j\in D(r)} V(i,j) = 0$$

This implies that the whole energies stored at the points $\{(i,j) : i+j = r+1\}$ is strictly less than those at the points $\{(i,j) : i+j = r\}$ unless all $x(i,j)=0$. Thus, we obtain

$$lim_{r\to\infty} \sum_{i+j\in D(r)} V(i,j) = 0 \tag{14}$$

It follows that

$$lim_{r\to\infty} V(i,j) = 0, lim_{r\to\infty} \|x(i,j)\| = 0$$

Consequently, we conclude from Definition 1 that the system (2) is asymptotically stable.
To estabish the H_∞ performance of system (2) for $\omega(i,j) \in l_2\{[0,\infty),[0,\infty)\}$, we consider the following expression

$$\begin{aligned}
& \Delta V(i+1,j+1) + \begin{bmatrix} z(i,j+1) \\ z(i+1,j) \end{bmatrix}^T \begin{bmatrix} z(i,j+1) \\ z(i+1,j) \end{bmatrix} \\
& - \gamma^2 \begin{bmatrix} \omega(i,j+1) \\ \omega(i+1,j) \end{bmatrix}^T \begin{bmatrix} \omega(i,j+1) \\ \omega(i+1,j) \end{bmatrix}
\end{aligned}$$

Consider the following expression:

$$\begin{aligned}
\Delta V = {} & \Delta V_{1h} + \Delta V_{2h} + \Delta V_{3h} + \Delta V_{1v} + \Delta V_{2v} + \Delta V_{3v} \\
& + 2\left\{E_0 x(i+1,j+1) + E_1 x(i,j+1) + E_2 x(i+1,j) \right.\\
& + E_3 x(i-d_1,j+1) + E_4 x(i+1,j-d_2) \\
& \left. + E_5 \omega(i,j+1) + E_6 \omega(i+1,j)\right\}^T \times \\
& \left\{x(i+1,j+1) - A_1 x(i,j+1) - A_2 x(i+1,j) \right.\\
& - A_{1d} x(i-d_1,j+1) - A_{2d} x(i+1,j-d_2) \\
& \left. - B_1 \omega(i,j+1) - B_2 \omega(i+1,j)\right\} \\
& + 2\left[S_0 x(i,j+1) + S_1 x(i-d_1,j+1)\right]^T \\
& \times \left\{x(i,j+1) - x(i-d_1,j+1) \right.\\
& \left. - \sum_{\theta=-d_1}^{-1} \Delta_i x(i+\theta,j+1))\right\} \\
& + 2\left[T_0 x(i+1,j) + T_1 x(i+1,j-d_2)\right]^T \\
& \times \left\{x(i+1,j) - x(i+1,j-d_2) \right.\\
& \left. - \sum_{s=-d_2}^{-1} \Delta_j x(i+1,j+s)\right\}
\end{aligned}$$

$$\Delta V + z^T z - \gamma^2 \omega^T \omega = \sum_{\theta=-d_1}^{-1} \sum_{s=-d_2}^{-1} \frac{1}{d_1 d_2} X^T \Omega X$$

where Ω is given by (8) and

$$\begin{aligned}
X = {} & [x^T(i+1,j+1) \quad x^T(i,j+1) \quad x^T(i+1,j) \\
& x^T(i-d_1,j+1) \quad x^T(i+1,j-d_2) \\
& \omega^T(i,j+1) \quad \omega^T(i+1,j) \\
& \Delta_i^T x(i+\theta,j+1) \quad \Delta_j^T x(i+1,j+s)]^T
\end{aligned}$$

Based on the same proof given by [17], we have

$$\begin{aligned}
\|z\|_2^2 \prec {} & \gamma^2 \|\omega\|_2^2 \qquad (15)\\
& + \gamma^2 \sum_{j=0}^{\infty} \left\{ \|x(0,j+1)\|_{Q_h}^2 + \sum_{l=-d_1}^{-1} \|x(l,j+1)\|_{W_h}^2 \right.\\
& \left. + \sum_{\theta=-d_1+1}^{0} \sum_{s=\theta+1}^{0} \|\Delta_i x(s,j+1)\|_{S_h}^2 \right\} \\
& + \gamma^2 \sum_{i=0}^{\infty} \left\{ \|x(i+1,0)\|_{Q_v}^2 + \sum_{l=-d_2}^{-1} \|x(i+1,l)\|_{W_v}^2 \right.\\
& \left. + \sum_{\theta=-d_2+1}^{0} \sum_{s=\theta+1}^{0} \|\Delta_j x(i+1,s)\|_{S_v}^2 \right\}
\end{aligned}$$

Therefore, it follows from Definition 2 that the result of the theorem holds.

In the case where the initial condition is known to be zero, the conditions $P_h \prec \gamma^2 Q_h$, $P_v \prec \gamma^2 Q_v$, $R_h \prec \gamma^2 W_h$, $R_v \prec \gamma^2 W_v$, $N_h \prec \gamma^2 S_h$ and $N_v \prec \gamma^2 S_v$ in Theorem 1 are no longer needed.
It follows from (15) that

$$\|z\|_2^2 \prec \gamma^2 \|\omega\|_2^2$$

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left\| \begin{bmatrix} z(i,j+1) \\ z(i+1,j) \end{bmatrix} \right\|^2 \prec \gamma^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left\| \begin{bmatrix} \omega(i,j+1) \\ \omega(i+1,j) \end{bmatrix} \right\|^2 \qquad (16)$$

Then, we have

$$2\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|z(i,j)\|^2 - \sum_{i=0}^{\infty} \|z(i,0)\|^2 - \sum_{j=0}^{\infty} \|z(0,j)\|^2 \prec \qquad (17)$$

$$2\gamma^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|\omega(i,j)\|^2 - \gamma^2 \sum_{i=0}^{\infty} \|\omega(i,0)\|^2 - \gamma^2 \sum_{j=0}^{\infty} \|\omega(0,j)\|^2$$

By considering the zero initial conditions $x(i,0) = x(0,j) = 0$. Then, from system (2), we have that $z(i,0) = L\omega(i,0)$ and $z(0,j) = L\omega(0,j)$. Thus, it follows from (17) that

$$\begin{aligned}
& 2\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|z(i,j)\|^2 - 2\gamma^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \|\omega(i,j)\|^2 \prec \\
& \sum_{i=0}^{\infty} \omega(i,0)^T \left(L^T L - \gamma^2 I\right) \omega(i,0) \\
& + \sum_{j=0}^{\infty} \omega(0,j)^T \left(L^T L - \gamma^2 I\right) \omega(0,j)
\end{aligned}$$

To show that $L^T L - \gamma^2 I$ is strictly negative, notice that applying Schur complement to inequality (8), we have

$$\begin{aligned}
& \begin{bmatrix} \phi_{11} & \alpha_{12} \\ \alpha_{12}^T & \alpha_{22} \end{bmatrix} + \begin{bmatrix} \alpha_{13} \\ \alpha_{23} \end{bmatrix} \alpha_{33}^{-1} \left[\alpha_{13}^T \alpha_{23}^T\right] \\
& + \text{sym}\left\{ \begin{bmatrix} E_0^T \\ E_{1,6}^T \end{bmatrix} \begin{bmatrix} I & -b \end{bmatrix} \right\} \prec 0
\end{aligned}$$

Based on Corollary 2.3.9 given by [12], the previous inequality is equivalent to

$$\begin{bmatrix} b \\ I \end{bmatrix}^T \left(\begin{bmatrix} \phi_{11} & \alpha_{12} \\ \alpha_{12}^T & \alpha_{22} \end{bmatrix} + \begin{bmatrix} \alpha_{13} \\ \alpha_{23} \end{bmatrix} \alpha_{33}^{-1} \left[\alpha_{13}^T \alpha_{23}^T\right] \right) \begin{bmatrix} b \\ I \end{bmatrix} \prec 0 \qquad (18)$$

Equation (18) can be written as

$$b^T \phi_{11} b + \text{sym}\left\{b^T \alpha_{12}\right\} + \alpha_{23} \alpha_{33}^{-1} \alpha_{23}^T + \alpha_{22} \prec 0$$

Then, we have

$$\begin{bmatrix} * & * \\ * & \begin{bmatrix} \phi_{55} & 0 \\ 0 & \phi_{66} \end{bmatrix} \end{bmatrix} \prec 0$$

with

$$\begin{aligned}
\phi_{55} &= B_1^T \phi_{11} B_1 + L^T L - \gamma^2 I \\
\phi_{66} &= B_2^T \phi_{11} B_2 + L^T L - \gamma^2 I
\end{aligned}$$

Thus, for all nonzero $\omega(i,j)$, we have

$$\|z\|_2 \prec \gamma \|\omega\|_2 \qquad (19)$$

■

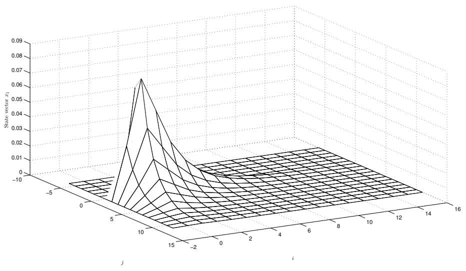

Fig. 1. The state evolution of the first component of $x(i, j)$

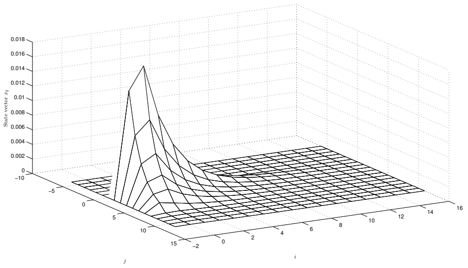

Fig. 2. The state evolution of the second component of $x(i, j)$

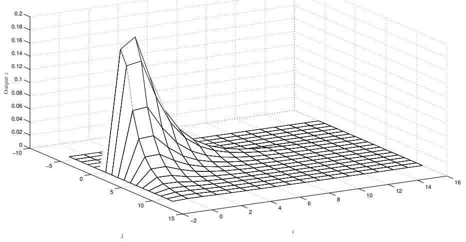

Fig. 3. The state evolution of the controlled output $z(i, j)$

IV. EXAMPLES

Let us illustrate our previous result with an example taken from [17] using the optimal H_∞ controller. Consider the stabilized system with the following coefficient matrices

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0.25 & 0.65 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A_{1d} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$A_{2d} = \begin{bmatrix} 0 & 0 \\ 0 & -0.04 \end{bmatrix}, B_1 = \begin{bmatrix} 0.2 \\ 0.04 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 \\ 0.04 \end{bmatrix},$$
$$H = \begin{bmatrix} 1 & 1 \end{bmatrix}, L = 0.5$$

and the disturbance input given by

$$\omega(i, j) = \exp -0.5(i + j - d_1 - d_2)$$

The figures Fig. 1, Fig. 2 and Fig. 3 are obtained using the zero initial conditions $x(i, 0) = x(0, j) = 0$.

Using our criterion given by Theorem 8, we obtain for $d_1 = 1$ and $d_2 = 6$, the delay-dependent H_∞ disturbance attenuation bound $\gamma = 0.9646$ which is smaller than the delay-independent one given by [17].

In this example, LMI (8) is feasible and we observe that the trajectories of the system in the given figures are asymptotically stable.

V. CONCLUSION

To conclude, let us highlight the general contribution of this paper. We developed a delay-dependent sufficient condition for 2D discrete systems with state delays to have a specific H_∞ disturbance attenuation bound. An LMI-based method has been proposed that involves the use of some free weighting matrices to express the relationship between the terms in the Leibniz-Newton formula and the system matrices. A numerical example is provided to illustrate the effectiveness of the proposed criterion.

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